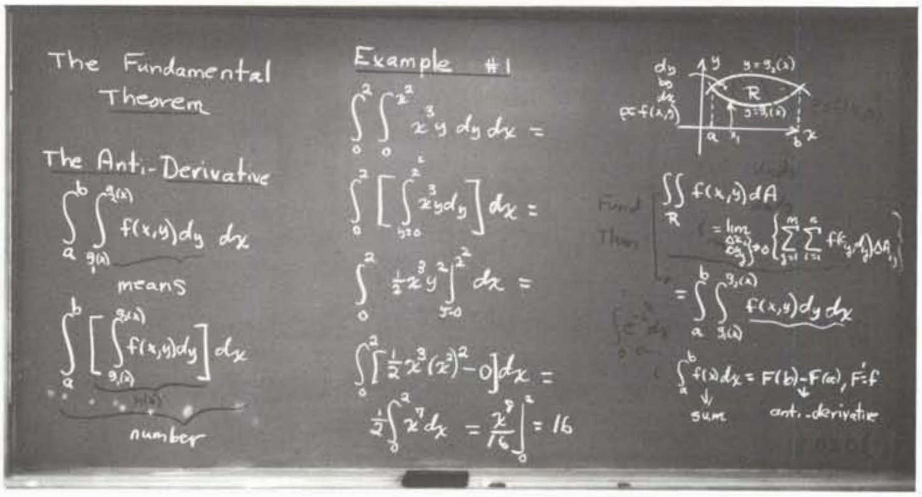
## Unit 2: The Double Sum as an Iterated Integral

1. Overview

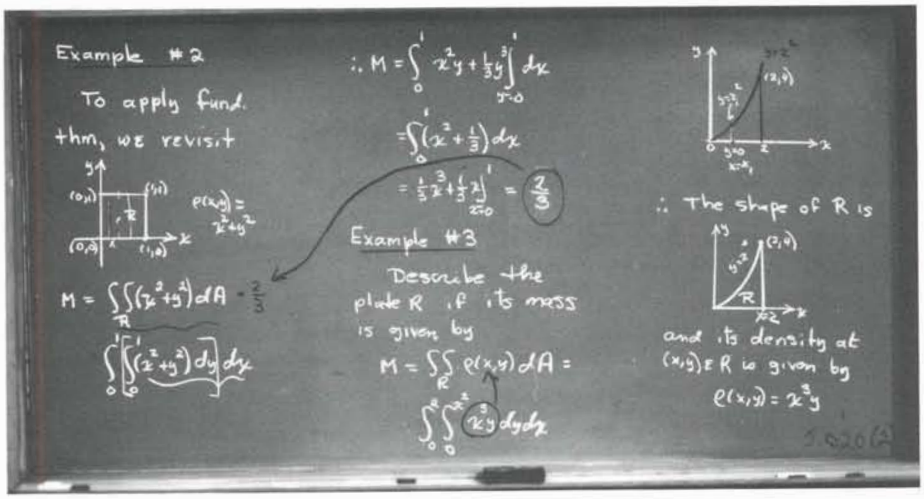
In Part 1 of our course, we showed that there was a relationship between a certain infinite sum known as the definite integral and the inverse (anti-) derivative evaluated between two numbers. In this unit, we show that a similar relationship exists when we talk about functions of several variables. While the basic underlying philosophy remains the same as in the case of a single variable, certain subtleties exist in the case of more than one variable that did not exist in the case of only one variable. From a geometric point of view, the subtleties arise from the fact that for one variable the domain of the independent variable is a <u>line</u> while for two independent variables it is a <u>plane</u>; and clearly the geometry of 2-space is more complex than that of 1-space.

2. Lecture 5.020

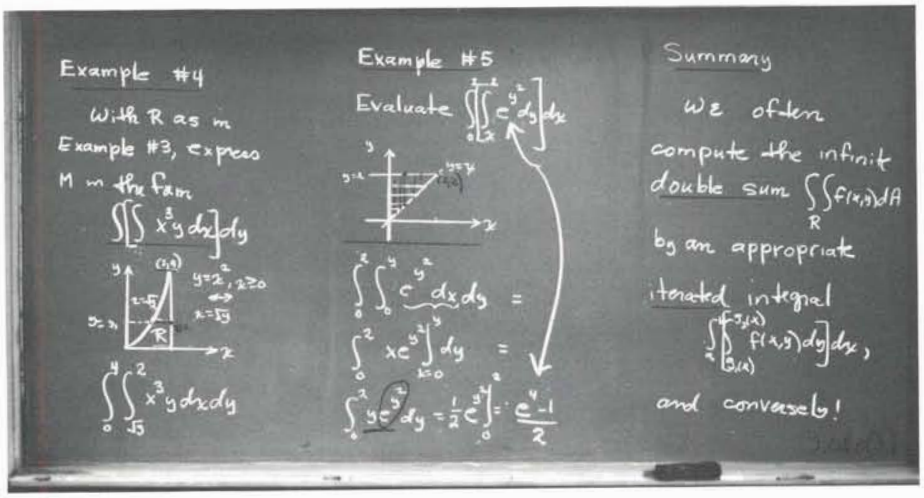



a.

Lecture 5.020 continued




b.




c.

3. Read Thomas: Sections 16.1, 16.2, and 16.3.

(Note: In a way, 16.3 is more important than 16.2, even though it is not our tendency to stress applications in this course. The point is that we would rarely, if ever, find an area by double integration as opposed to using single integration. Indeed, many uses of the double integral to find area are artificial. For example, one might use the fact that

$$f(x) = \int_0^{f(x)} dy$$

and then write

$$\int_a^b f(x)dx$$

as

$$\int_a^b \left(\int_0^{f(x)} dy\right) dx$$

but clearly such a rewriting does not simplify any computation that must be performed. What is true, however, is that when for physical reasons we must restrict the changes in <u>each</u> independent variable to be small (such as in finding mass when there is a density distribution that varies from point to point), then we must use double integration. In Section 16.3, the author indicates several real-life cases in which one would have to use multiple integration. Once the need is noted, the technique is the same as that presented in Section 16.2; and it is the general technique which we emphasize in this unit.)

3. Exercises:

5.2.1(L)

a. Show that the expression

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5.2.1(L) continued

$$\int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right]dx$$

is well-defined in terms of the usual notion of the definite integral, provided $\alpha(x)$ and $\beta(x)$ are <u>differentiable</u> functions of x, and $f(x,y)$ is a <u>continuous</u> function of y for each value of x where $a \leqslant x \leqslant b$.

b. Assuming that $f(x,y) \geqslant 0$, $\alpha(x) \leqslant \beta(x)$, and that $a \leqslant x \leqslant b$, show how

$$\int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right]dx$$

may be viewed as denoting a volume, and describe the solid which has this (value) as its volume.

c. Suppose the region R may be written as

$\{(x,y): a \leqslant x \leqslant b, \alpha(x) \leqslant y \leqslant \beta(x)\}$

and

$\{(x,y): \gamma(y) \leqslant x \leqslant \delta(y), c \leqslant y \leqslant d\}$.

Show by a suitable geometric interpretation that

$$\int_a^b \left[\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right]dx = \int_c^d \left[\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\right]dy.$$

5.2.2

a. Evaluate

$$\int_0^3 \left[\int_0^{x^3} x^2 y\,dy\right]dx.$$

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5.2.2 continued

b. Describe the region R defined by the limits of integration given in part (a). Then write R as a set of points in the plane using two different descriptions of R.

c. Compute the integral in part (a) with the order of integration reversed.

d. Give two different models which are represented by the answer to part (a) [or (c)].

5.2.3

Suppose R is the region bounded above by $y = \beta(x)$, below by $y = \alpha(x)$, on the left by $x = a$, and on the right by $x = b$. In terms of the meaning of

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dydx,$$

explain why the area of R, $A_R$, may be viewed as

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} dydx.$$

5.2.4(L)

a. Describe the domain of g if

$$g(x) = \int_0^1 \frac{(x - y)dy}{(x + y)^3}.$$

b. Compute g(x) explicitly for each $x \varepsilon$ dom g.

c. Evaluate

$$\int_0^1 \left[\int_0^1 \frac{(x - y)dy}{(x + y)^3}\right]dx.$$

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5.2.4(L) continued

d. Evaluate

$$\int_0^1 \left[\int_0^1 \frac{(x - y)\,dx}{(x + y)^3}\right] dy.$$

e. Compare (c) and (d) to conclude that

$$\int_0^1 \int_0^1 \frac{(x - y)}{(x + y)^3}\,dydx \neq \int_0^1 \int_0^1 \frac{(x - y)}{(x + y)^3}\,dxdy$$

and explain why this result does not contradict the fundamental theorem.

5.2.5(L)

Interchange the order of integration to compute

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dydx.$$

5.2.6

Show that

$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy}\,dy,$$

and use this fact to evaluate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx \qquad (a,b > 0).$$

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