CALCULUS REVISITED
PART 2
A Self-Study Course

STUDY GUIDE
Block 5
Multiple Integration

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Pretest

1. Compute the sum

$$\sum_{j=1}^{3} \sum_{i=1}^{4} (i + j).$$

2. Evaluate

$$\int_0^3 \int_0^{x^3} x^2 y \, dy \, dx.$$

3. By changing the order of integration, evaluate

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx.$$

4. Use polar coordinates to compute

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{-(x^2 + y^2)} \, dy \, dx.$$

5. Find the mass of the solid S if S is given by the equation $x^2 + y^2 + z^2 \leq 1$ and the density of S at each point is numerically equal to the distance of that point from the origin.

6. Find the surface area of the region S where S is that portion of $z = 1 - x^2 - y^2$ for which $z \geq 0$.

7. Find the volume of the region which is bounded between the two elliptic paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - y^2$.

8. Let c be the square defined by the equation $|x| + |y| = 1$. Compute

$$\oint_c y\,dx - x\,dy.$$

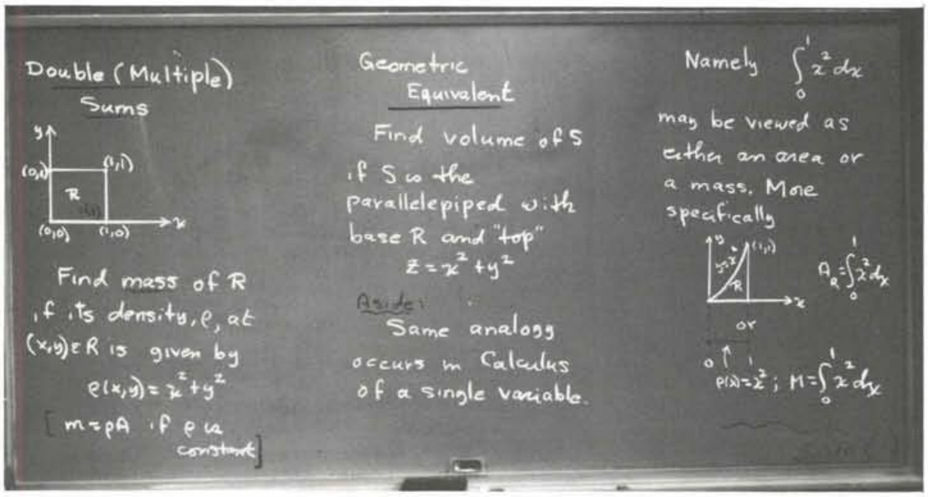
Unit 1: Double Sums

1. Overview

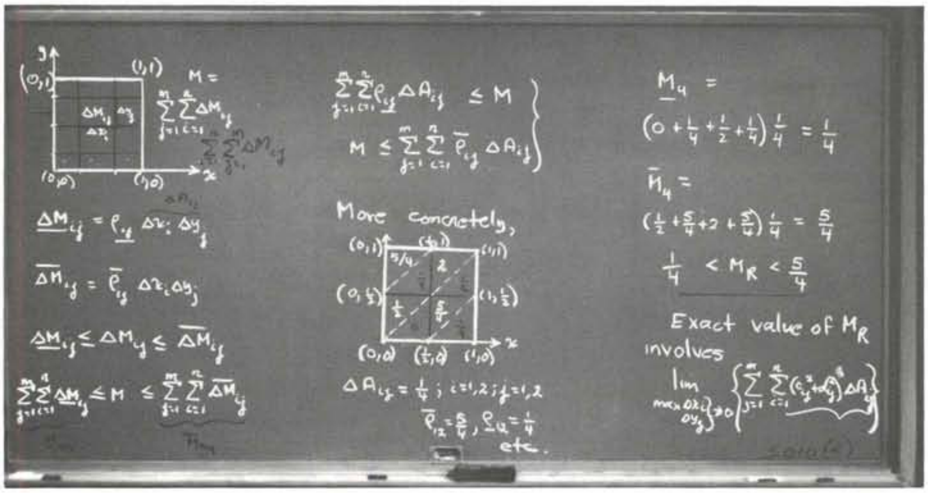
In the same way that certain types of infinite sums (i.e., definite integrals) can be identified with inverse differentiation (The First Fundamental Theorem of Integral Calculus), certain types of "multiple infinite sums" can be identified with inverse partial differentiation. In this unit, the lecture is primarily concerned with introducing the notion of a double infinite sum (the generalization to multiple infinite sums being rather straight-forward).

The lecture introduces the double sum in terms of finding the mass of a "thin plate" which has a density that varies from point to point on the plate. The exercises serve two purposes. On the one hand, the first few exercises are used to help you become more familiar with the notion of a double sum, and to help you get an idea of the computational properties of such sums. Exercise 5.1.4(L) is used to show the subleties that occur when we make the transition from double finite sums to double infinite sums. The problems are similar (but in some ways a bit more sophisticated) to what happened in the study of calculus of a single real variable when we went from adding arbitrarily large (but finite) numbers of terms to adding infinitely many terms; and we investigated how certain "trivial" properties for finite sums became "luxuries" for infinite sums. The remainder of the exercises "refine" and amplify the computational techniques which are introduced in the lecture.

2. Lecture 5.010



a.



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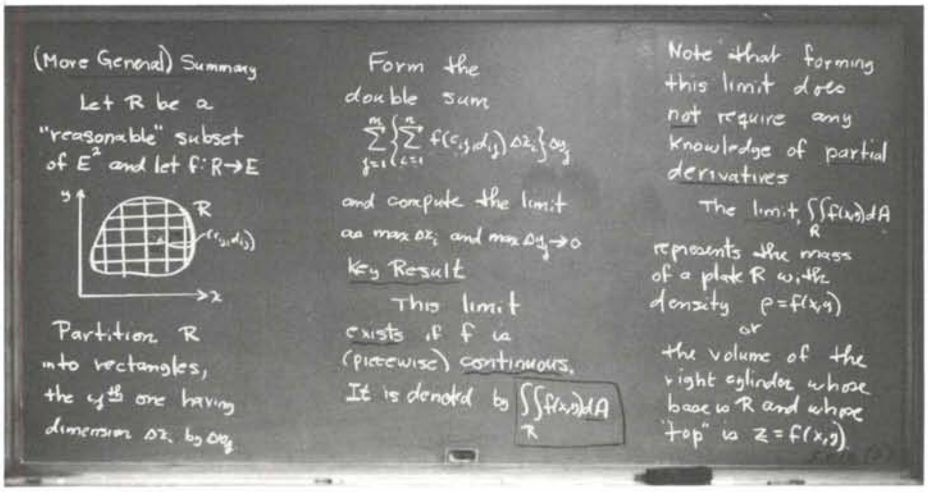

c.

3. Exercises:

5.1.1(L)

The expression

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}$$

is an abbreviation for

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} \right).$$

Similarly,

$$\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}$$

is an abbreviation for

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} \right).$$

Show that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \quad \text{and} \quad \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}$$

are equal, and discuss the nature of the terms that make up both sums.

5.1.2

Apply the results of Exercise 5.1.1(L) to compute each of the following double sums.

a. $\displaystyle\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij}$ and $\displaystyle\sum_{j=1}^{3} \sum_{i=1}^{2} a_{ij}$.

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5.1.2 continued

b. $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$ and $\sum_{j=1}^{3} \sum_{i=1}^{4} ij.$

c. $\sum_{j=1}^{3} \sum_{i=1}^{4} (i + j).$

5.1.3(L)

Verify each of the following.

a. $\sum_{i=1}^{n} \sum_{j=1}^{m} ca_{ij} = c\left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}\right)$

where c is a constant, independent of i and j.

b. $\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{m} b_j\right).$

c. Use part (b) of this exercise to check the result stated in part (b) of the previous exercise.

d. $\sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j) = m \sum_{i=1}^{n} a_i + n \sum_{j=1}^{m} b_j.$

e. Use the result of part (d) to check part (c) of the previous exercise.

5.1.4(L)

The symbol $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$, by definition, denotes $\sum_{i=1}^{\infty}\left\{\lim_{m\to\infty} \sum_{j=1}^{m} a_{ij}\right\} = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \lim_{m\to\infty} \sum_{j=1}^{m} a_{ij}\right]$. Similarly, $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ denotes

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5.1.4(L) continued

$\lim_{m\to\infty}\left[\sum_{j=1}^{m}\left\{\lim_{n\to\infty}\sum_{i=1}^{n}a_{ij}\right\}\right]$. We now define a_{ij}, for all positive integral values of i and j, by

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ -1, & \text{if } i = j - 1 \quad (j = 2,3,\ldots) \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$ and show, in particular, that in this example $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij} \neq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$.

5.1.5(L)

Consider the thin plate in the shape of a unit square with vertices at (0,0), (0,1), (1,0), and (1,1). Suppose that at each point (x,y) in the plate the density of the plate is given by $\rho(x,y) = x^2 + y^2$.

a. Mimic our procedure in the lecture to find upper and lower bounds for the mass of the plate as functions of m and n where m denotes the number of equal parts into which we partition the segment (0,0) to (0,1) and n, the number of parts into which (0,0) to (1,0) is partitioned.

b. Use the result of (a) with $n = m = 10^6$ to show that to five significant figures the mass of the plate is 0.66667.

c. Let m and n each approach infinity in part (a) and thus deduce the exact mass of the plate.

d. The solid S is the parallelepiped whose base is the unit square with vertices (0,0), (0,1), (1,0), and (1,1) and whose top is the surface $z = x^2 + y^2$. Find the volume of S.

5.1.6

Suppose R is the same thin-plate as in the previous exercise, but the density of the plate at the point (x,y) is now given by $\rho(x,y) = xy$. Find the mass of R.