CALCULUS REVISITED
PART 2
A Self-Study Course

STUDY GUIDE
Block 4
Matrix Algebra

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MATRIX ALGEBRA

Pretest

1. Solve the matrix equation AX - BC = 0 if

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}, \quad \text{and } 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. Find A^{-1} if

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 3 & 9 & 7 \end{bmatrix}$$

3. Consider the system of equations

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 + 4x_4 &= b_2 \\ 3x_1 + 5x_2 + 2x_3 + x_4 &= b_3 \\ 3x_1 + 4x_2 + x_3 - x_4 &= b_4 \end{aligned}\right\}$$

How must b_3 and b_4 be related to b_1 and b_2 for this system to have solutions?

4. Use linear approximations to estimate the point (x,y) near (3,2) for which

$$x^2 - y^2 = 5.00052$$
$$2xy = 12.00026$$

5. Let x be determined as a function of z by the pair of equations.

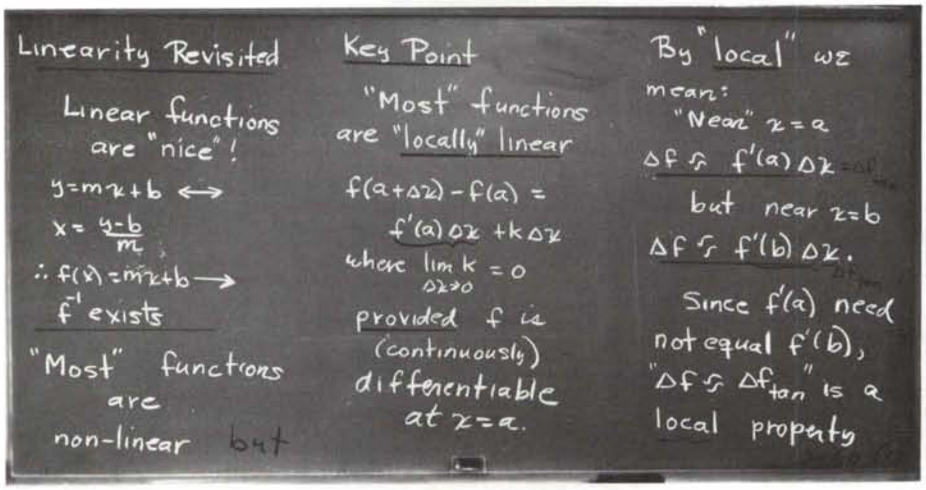
$$\left.\begin{aligned} x + y + z &= 0 \\ \tfrac{1}{3}x^3 + x - \tfrac{1}{3}y^3 - z^2y &= 0 \end{aligned}\right\}$$

Compute $\frac{dx}{dz}$.

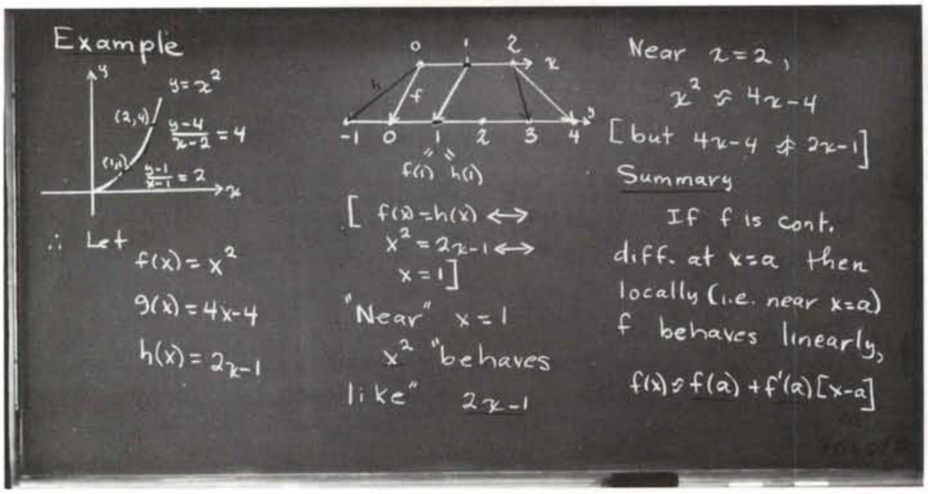
6. Find the maximum and minimum values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the pair of constraints that $x^2 + 2y^2 + z^2 = 1$ and $x + y = 1$.

Unit 1: Linear Equations and Introduction to Matrices

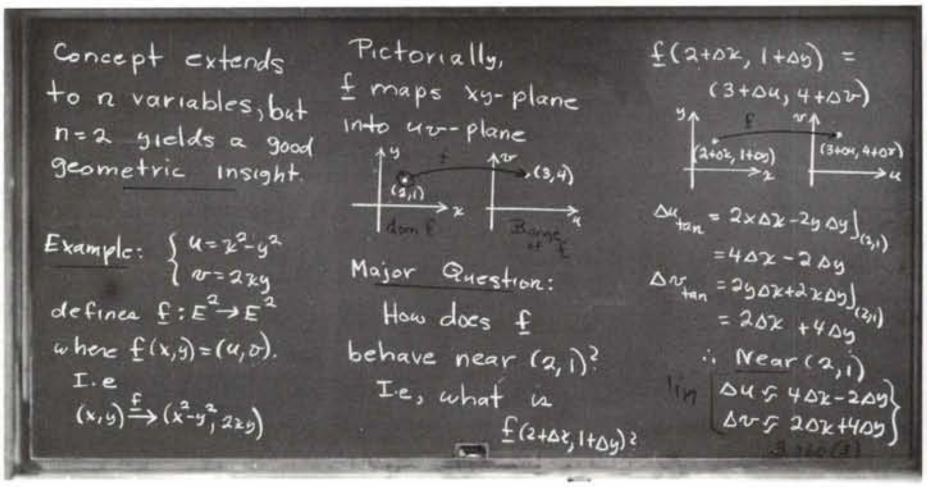
1. Lecture 4.010



a.

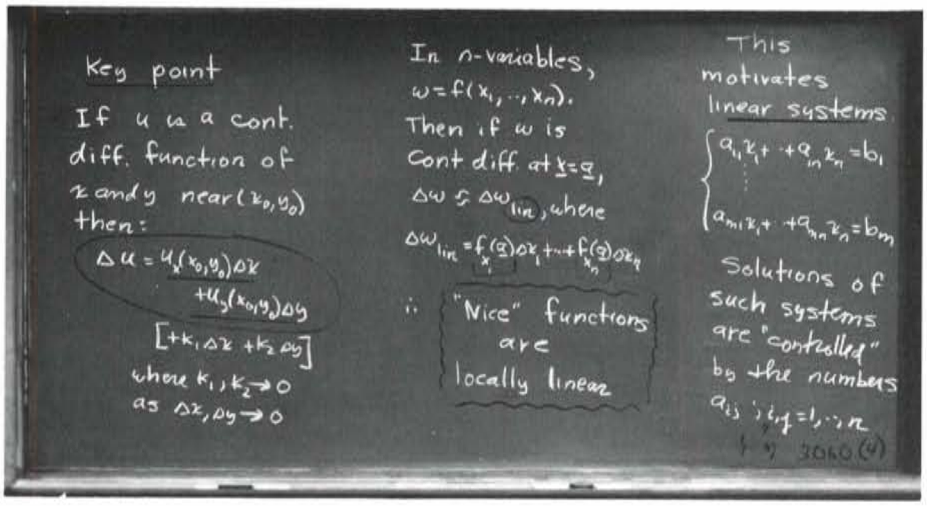


b.

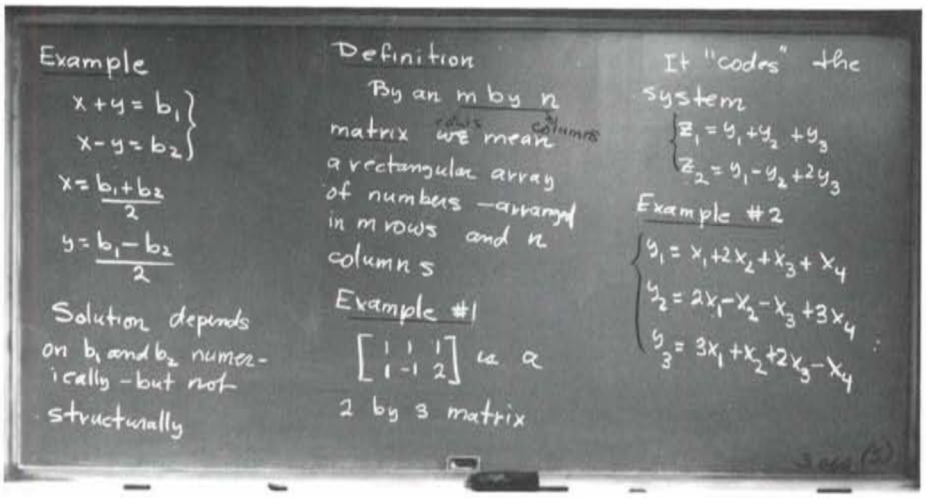


c.

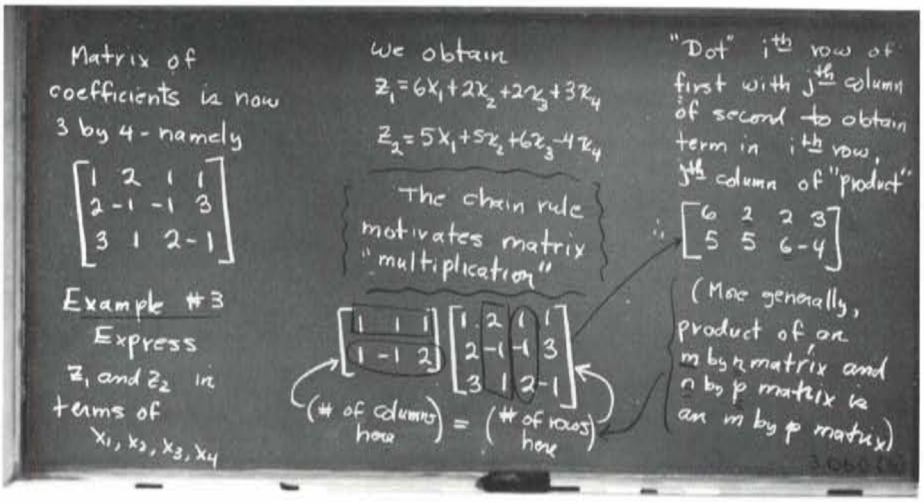
Lecture 4.010 continued



d.



e.



f.

2. Read Supplementary Notes, Chapter 6, Sections A, B, and C.

3. Exercises:

4.1.1

Compute the matrix product

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & 2 & 7 \\ 4 & 5 & 1 & 8 \\ 4 & 7 & 9 & 5 \end{pmatrix}$$

and use this result to show how to express z_1 and z_2 in terms of x_1, x_2, x_3, and x_4 if

$$z_1 = y_1 + 2y_2 + 3y_3$$

$$z_2 = 3y_1 + y_2 + 2y_3$$

and

$$y_1 = 3x_1 + 6x_2 + 2x_3 + 7x_4$$

$$y_2 = 4x_1 + 5x_2 + x_3 + 8x_4$$

$$y_3 = 4x_1 + 7x_2 + 9x_3 + 5x_4$$

4.1.2

Compute

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 2 \\ 5 & 7 & 9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 2 \\ 5 & 7 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

Then interpret these two products in terms of systems of linear equations.

4.1.3

Compute

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

and try to generalize these results.

4.1.4

Compute

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 7 & 3 & -4 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 7 & 3 & -4 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

Interpret these products in terms of systems of linear equations.

4.1.5

Compute

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

and try to generalize this result.

4.1.6

a. Compute

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

(continued on next page)

4.1.6 continued

b. (Optional) Let $A = (a_{ij})$ be an n x n matrix and let E_{ij} $(i \neq j)$ denote the n x n matrix each of whose elements on the main diagonal and in the i^{th} row, j^{th} column are 1, and everywhere else are 0. Describe the products $E_{ij} A$.

4.1.7

a. Compute

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b. (Optional) With A and E_{ij} as in 4.1.6, compute $A\,E_{ij}$.

4.1.8

Compute

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

and generalize this result to show how we may multiply each element of a matrix by the same scalar.