
CALCULUS REVISITED
PART 2
A Self-Study Course

STUDY GUIDE
Block 4
Matrix Algebra

Herbert I. Gross
Senior Lecturer

Center for Advanced Engineering Study
Massachusetts Institute of
Technology

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Solutions

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Study Guide

BLOCK 4:

MATRIX ALGEBRA

Pretest

1. Solve the matrix equation $AX - BC = 0$ if

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}, \text{ and } 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. Find A^{-1} if

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 3 & 9 & 7 \end{bmatrix}$$

3. Consider the system of equations

$$\left. \begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 + 4x_4 &= b_2 \\ 3x_1 + 5x_2 + 2x_3 + x_4 &= b_3 \\ 3x_1 + 4x_2 + x_3 - x_4 &= b_4 \end{aligned} \right\}$$

How must b_3 and b_4 be related to b_1 and b_2 for this system to have solutions?

4. Use linear approximations to estimate the point (x,y) near $(3,2)$ for which

$$\begin{aligned} x^2 - y^2 &= 5.00052 \\ 2xy &= 12.00026 \end{aligned}$$

5. Let x be determined as a function of z by the pair of equations.

$$\left. \begin{aligned} x + y + z &= 0 \\ \frac{1}{3}x^3 + x - \frac{1}{3}y^3 - z^2y &= 0 \end{aligned} \right\}$$

Compute $\frac{dx}{dz}$.

6. Find the maximum and minimum values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the pair of constraints that $x^2 + 2y^2 + z^2 = 1$ and $x + y = 1$.

Unit 1: Linear Equations and Introduction to Matrices

1. Lecture 4.010

<p><u>Linearity Revisited</u></p> <p>Linear functions are "nice"!</p> $y = mx + b \leftrightarrow$ $x = \frac{y-b}{m}$ <p>$\therefore f(x) = mx + b \rightarrow$ f' exists</p> <p>"Most" functions are non-linear but</p>	<p><u>Key Point</u></p> <p>"Most" functions are "locally" linear</p> $f(a+\Delta x) - f(a) = f'(a)\Delta x + k\Delta x$ <p>where $\lim_{\Delta x \rightarrow 0} k = 0$</p> <p>provided f is (continuously) differentiable at $x=a$.</p>	<p>By "local" we mean:</p> <p>"Near" $x=a$</p> $\Delta f \approx f'(a)\Delta x$ <p>but near $x=b$</p> $\Delta f \approx f'(b)\Delta x$ <p>Since $f'(a)$ need not equal $f'(b)$, "$\Delta f \approx \Delta f_{\text{tan}}$" is a local property</p>
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a.

<p><u>Example</u></p> <p>$y = x^2$</p> <p>$(2,4) \rightarrow \frac{y-4}{x-2} = 4$</p> <p>$(1,1) \rightarrow \frac{y-1}{x-1} = 2$</p> <p>$\therefore$ Let $f(x) = x^2$ $g(x) = 4x - 4$ $h(x) = 2x - 1$</p>	<p>Near $x=2$, $x^2 \approx 4x - 4$ [but $4x - 4 \neq 2x - 1$]</p> <p><u>Summary</u></p> <p>If f is cont. diff. at $x=a$ then locally (i.e. near $x=a$) f behaves linearly, $f(x) \approx f(a) + f'(a)[x-a]$</p>
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b.

<p>Concept extends to n variables, but $n=2$ yields a good geometric insight.</p> <p>Example: $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$</p> <p>define $f: E^2 \rightarrow E^2$</p> <p>where $f(x,y) = (u,v)$.</p> <p>I.e. $(x,y) \xrightarrow{f} (x^2 - y^2, 2xy)$</p>	<p>Pictorially, f maps xy-plane into uv-plane</p> <p>Major Question: How does f behave near $(2,1)$? I.e., what is $f(2+\Delta x, 1+\Delta y)$?</p>	<p>$f(2+\Delta x, 1+\Delta y) = (3+\Delta u, 4+\Delta v)$</p> <p>$\Delta u_{\text{tan}} = 2x\Delta x - 2y\Delta y \Big _{(2,1)} = 4\Delta x - 2\Delta y$</p> <p>$\Delta v_{\text{tan}} = 2y\Delta x + 2x\Delta y \Big _{(2,1)} = 2\Delta x + 4\Delta y$</p> <p>$\therefore$ Near $(2,1)$</p> <p>$\Delta u \approx 4\Delta x - 2\Delta y$ $\Delta v \approx 2\Delta x + 4\Delta y$</p>
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c.

Lecture 4.010 continued

Key point
 If u is a cont. diff. function of x and y near (x_0, y_0) then:

$$\Delta u = u_x(x_0, y_0) \Delta x + u_y(x_0, y_0) \Delta y$$
 where $k_1, k_2 \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$

In n -variables, $w = f(x_1, \dots, x_n)$.
 Then if w is cont. diff. at $\underline{x} = \underline{a}$,
 $\Delta w \approx \Delta w_{lin}$, where

$$\Delta w_{lin} = f_x(\underline{a}) \Delta x_1 + \dots + f_n(\underline{a}) \Delta x_n$$

\therefore "Nice" functions are locally linear

This motivates linear systems

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

Solutions of such systems are "controlled" by the numbers a_{ij} ; $i, j = 1, \dots, n$

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d.

Example

$$\begin{cases} x + y = b_1 \\ x - y = b_2 \end{cases}$$

$$x = \frac{b_1 + b_2}{2}$$

$$y = \frac{b_1 - b_2}{2}$$

Solution depends on b_1 and b_2 numerically - but not structurally

Definition
 By an m by n matrix we mean a rectangular array of numbers - arranged in m rows and n columns

Example #1

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$
 is a 2 by 3 matrix

It "codes" the system

$$\begin{cases} z_1 = y_1 + y_2 + y_3 \\ z_2 = y_1 - y_2 + 2y_3 \end{cases}$$

Example #2

$$\begin{cases} y_1 = x_1 + 2x_2 + x_3 + x_4 \\ y_2 = 2x_1 - x_2 - x_3 + 3x_4 \\ y_3 = 3x_1 + x_2 + 2x_3 - x_4 \end{cases}$$

e.

Matrix of coefficients is now 3 by 4 - namely

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & -1 & 3 \\ 3 & 1 & 2 & -1 \end{bmatrix}$$

Example #3
 Express z_1 and z_2 in terms of x_1, x_2, x_3, x_4

We obtain

$$z_1 = 6x_1 + 2x_2 + 2x_3 + 3x_4$$

$$z_2 = 5x_1 + 5x_2 + 6x_3 - 4x_4$$

The chain rule motivates matrix "multiplication"

"Dot" i^{th} row of first with j^{th} column of second to obtain term in i^{th} row, j^{th} column of "product"

$$\begin{bmatrix} 6 & 2 & 2 & 3 \\ 5 & 5 & 6 & -4 \end{bmatrix}$$

(More generally, product of an m by n matrix and n by p matrix is an m by p matrix)

(# of columns here) = (# of rows here)

f.

2. Read Supplementary Notes, Chapter 6, Sections A, B, and C.

3. Exercises:

4.1.1

Compute the matrix product

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & 2 & 7 \\ 4 & 5 & 1 & 8 \\ 4 & 7 & 9 & 5 \end{pmatrix}$$

and use this result to show how to express z_1 and z_2 in terms of x_1 , x_2 , x_3 , and x_4 if

$$z_1 = y_1 + 2y_2 + 3y_3$$

$$z_2 = 3y_1 + y_2 + 2y_3$$

and

$$y_1 = 3x_1 + 6x_2 + 2x_3 + 7x_4$$

$$y_2 = 4x_1 + 5x_2 + x_3 + 8x_4$$

$$y_3 = 4x_1 + 7x_2 + 9x_3 + 5x_4$$

4.1.2

Compute

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 2 \\ 5 & 7 & 9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 2 \\ 5 & 7 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

Then interpret these two products in terms of systems of linear equations.

4.1.3

Compute

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

and try to generalize these results.

4.1.4

Compute

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 7 & 3 & -4 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 7 & 3 & -4 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

Interpret these products in terms of systems of linear equations.

4.1.5

Compute

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

and try to generalize this result.

4.1.6

a. Compute

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

(continued on next page)

4.1.6 continued

- b. (Optional) Let $A = (a_{ij})$ be an $n \times n$ matrix and let E_{ij} ($i \neq j$) denote the $n \times n$ matrix each of whose elements on the main diagonal and in the i^{th} row, j^{th} column are 1, and everywhere else are 0. Describe the products $E_{ij} A$.

4.1.7

- a. Compute

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- b. (Optional) With A and E_{ij} as in 4.1.6, compute $A E_{ij}$.

4.1.8

Compute

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

and generalize this result to show how we may multiply each element of a matrix by the same scalar.

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Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

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