Unit 6: The Chain Rule, Part 2

3.6.1(L)

a. We have

$$w = f(x,y) \tag{1}$$

and letting $x = r\cos\theta$ and $y = r\sin\theta$, we also have that

$$w = f(r\cos\theta,\ r\sin\theta)$$

$$= g(r,\theta). \tag{2}$$

Now, by the chain rule

$$w_r = w_x x_r + w_y y_r. \tag{3}$$

[Again, to emphasize f and g, equation (3) should be viewed as

$$g_r = f_x x_r + f_y y_r$$

since g is used to express w in terms of r and θ while f is used to express w in terms of x and y.]

From $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\left.\begin{aligned} x_r &= \cos\theta,\ y_r = \sin\theta \\ x_\theta &= -r\sin\theta,\ y_\theta = r\cos\theta \end{aligned}\right\} \tag{4}$$

Substituting (4) into (3), we obtain

$$w_r = w_x\cos\theta + w_y\sin\theta. \tag{5}$$

To find w_{rr}, we now need only take the partial derivative of equation (5) with respect to r. This, of course, assumes that θ is our other independent variable. Thus, looking at the right side of equation (5), we see that both $\sin\theta$ and $\cos\theta$ may be

3.6.1(L) continued

viewed as constants when we take the partial derivative with respect to r. The "trickier" aspects involve differentiating w_x and w_y with respect to r. The key is that both w_x and w_y are themselves bona fide functions of x and y, so that the chain rule also applies, just as it did to w. In still other words, we may think of $w_x(x,y)$ as being denoted by, say, h(x,y). Then to differentiate w_x with respect to r, we need only differentiate h with respect to r, but from the chain rule we know that

$$h_r = h_x x_r + h_y y_r \tag{6}$$

or

$$h_r = h_x \cos\theta + h_y \sin\theta. \tag{6'}$$

Since $h = w_x$, it should be easy to see that $h_x = w_{xx}$ and that $h_y = \left(w_x\right)_y = w_{xy}$.

With this notation in mind, equation (6') may be read as

$$\left(w_x\right)_r = w_{xx}\cos\theta + w_{xy}\sin\theta. \tag{7}$$

If the substitution $h(x,y) = w_x(x,y)$ seems a bit artificial, the following explanation may seem more acceptable.

In the expression

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \tag{8}$$

think of f as being a "place holder" in the sense that equation (8) really says

$$\frac{\partial(\ )}{\partial r} = \frac{\partial(\ )}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial(\ )}{\partial y}\frac{\partial y}{\partial r}. \tag{8'}$$

If we now "fill in" the parentheses in (8') by $\frac{\partial w}{\partial x}$, we obtain:

$$\frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial r} = \frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial y}\frac{\partial y}{\partial r}. \tag{9}$$

3.6.1(L) continued

If we next let $\frac{\partial(\frac{\partial w}{\partial x})}{\partial x} = w_{xx}$, $\frac{\partial(\frac{\partial w}{\partial x})}{\partial y} = w_{xy}$, $\frac{\partial x}{\partial r} = \cos\theta$, and $\frac{\partial y}{\partial r} = \sin\theta$, we see that equation (9) is the same as equation (7).

At any rate, if we return to equation (5), we have

$$w_{rr} = \frac{\partial(w_r)}{\partial r}$$

$$= \frac{\partial(w_x \cos\theta + w_y \sin\theta)}{\partial r}$$

$$= \frac{\partial(w_x \cos\theta)}{\partial r} + \frac{\partial(w_y \sin\theta)}{\partial r}$$

$$= \frac{\partial(w_x)}{\partial r}\cos\theta + \frac{\partial(w_y)}{\partial r}\sin\theta$$

(recalling that $\sin\theta$ and $\cos\theta$ are <u>constants</u> when we differentiate with respect to r).

From either equation (7) or equation (9), we know that

$$\frac{\partial(w_x)}{\partial r} = \frac{\partial^2 w}{\partial x^2}\cos\theta + \frac{\partial^2 w}{\partial x\partial y}\sin\theta.$$

By similar reasoning

$$\frac{\partial(w_y)}{\partial r} = \frac{\partial(w_y)}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial(w_y)}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial^2 w}{\partial y\partial x}^{*}\frac{\partial x}{\partial r} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial r}$$

$$= \frac{\partial^2 w}{\partial y\partial x}\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin\theta.$$

*Conceptually, $\frac{\partial^2 w}{\partial x\partial y}$ and $\frac{\partial^2 w}{\partial y\partial x}$ are very different. In <u>most</u> cases, they happen to be equal, but this need not be the case. For a more complete discussion, see Exercise 3.6.2.

3.6.1(L) continued

Therefore,

$$\frac{\partial^2 w}{\partial r^2} = \left(\frac{\partial^2 w}{\partial x^2}\cos\theta + \frac{\partial^2 w}{\partial x\partial y}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 w}{\partial y\partial x}\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin\theta\right)\sin\theta$$

$$= \frac{\partial^2 w}{\partial x^2}\cos^2\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta + \left(\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 w}{\partial y\partial x}\right)\sin\theta\cos\theta. \qquad (10)$$

[Again, notice that we are not presupposing that $\frac{\partial^2 w}{\partial x\partial y} \equiv \frac{\partial^2 w}{\partial y\partial x}$. If it happens that $\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 w}{\partial y\partial x}$, then (10) may be written as

$$\frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 w}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 w}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta.]$$

b. If $w = f(x,y) = x^2y^3$, then

$$w = (r\cos\theta)^2(r\sin\theta)^3$$

$$= r^5\sin^3\theta\cos^2\theta = g(r,\theta)$$

$$\frac{\partial w}{\partial r} = 5r^4\sin^3\theta\cos^2\theta$$

$$\frac{\partial^2 w}{\partial r^2} = 20r^3\sin^3\theta\cos^2\theta. \qquad (11)$$

On the other hand

$$\frac{\partial w}{\partial x} = 2xy^3;\ \frac{\partial w}{\partial y} = 3x^2y^2$$

$$\frac{\partial^2 w}{\partial x^2} = 2y^3;\ \frac{\partial^2 w}{\partial y^2} = 6x^2y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3) = 6xy^2;\ \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2.$$

3.6.1(L) continued

[Note that $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$ in this example.]

So equation (10) becomes in this case

$$\frac{\partial^2 w}{\partial r^2} = 2y^3\cos^2\theta + 6x^2y\ \sin^2\theta + (6xy^2 + 6xy^2)\sin\theta\ \cos\theta$$

$$= 2y^3\cos^2\theta + 6x^2y\ \sin^2\theta + 12xy^2\sin\theta\ \cos\theta$$

$$= 2(r\ \sin\theta)^3\cos^2\theta + 6(r\ \cos\theta)^2(r\ \sin\theta)\sin^2\theta + 12(r\ \cos\theta)$$

$$(r\ \sin\theta)^2\sin\theta\ \cos\theta$$

$$= 20r^3\sin^3\theta\ \cos^2\theta. \qquad (12)$$

A comparison of equations (11) and (12) shows us that we obtain the same value for $\frac{\partial^2 w}{\partial r^2}$ using either method.

3.6.2

There are several particular results which we wish to emphasize in this exercise. First of all, let us observe that our definition of f is such that f is continuous at each point (x,y) in the plane; and that in particular, f is continuous in a neighborhood of (0,0). Since f is defined in terms of a product and quotient of polynomials in x and y, it should be intuitively clear that the only possible trouble spots are places at which our denominator is 0. Since our denominator is $x^2 + y^2$, we see that we are in trouble only if $x^2 + y^2 = 0$, and this can happen if and only if both x and y equal zero. In other words, f should be continuous in the neighborhood of each point in the plane, except possibly for the point (0,0).

To check whether f is continuous at (0,0), we must show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0). \qquad (1)$$

3.6.2 continued

Keeping in mind the crucial fact that $\lim_{(x,y)\to(0,0)} f(x,y)$ must be independent of the path (otherwise the limit doesn't exist), it is probably wise to switch to polar coordinates since then $(x,y)\to(0,0)$ is equivalent to $r\to 0$, regardless of the value of θ.

At any rate, if we switch to polar coordinates, then $(x,y) \neq (0,0)$ implies

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

$$= \frac{(r \cos \theta)(r \sin \theta)(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2}$$

$$= \frac{r^4\sin \theta \cos \theta \ (\cos^2\theta - \sin^2\theta)}{r^2}$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0}\left[\frac{r^4\sin \theta \cos \theta \ (\cos^2\theta - \sin^2\theta)}{r^2}\right]$$

$$= \lim_{r\to 0} [r^2\sin \theta \cos \theta \ (\cos^2\theta - \sin^2\theta)] \text{ (since } r\neq 0)$$

$$= 0 \sin \theta \cos \theta \ (\cos^2\theta - \sin^2\theta)$$

$$= 0 \qquad (2)$$

and since $f(0,0)$ is defined to be 0, we have from equation (2) that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

so that f is continuous at $(0,0)$.

Our next endeavor is to show that f_x exists at each point (x,y) in the plane and that f_x is continuous at $(0,0)$. Again, from the definition of f, the trouble spot involves the computation of

3.6.2 continued

$f_x(0,0)$ [since f has a "special" definition at (0,0)]. If $(x,y) \neq (0,0)$, $f_x(x,y)$ is still cumbersome to compute, but the computation is straightforward.

Namely, if $(x,y) \neq (0,0)$, then

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} = \frac{x^3y - xy^3}{x^2 + y^2}. \tag{3}$$

Hence, by the quotient rule, (3) yields

$$f_x(x,y) = \frac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2}$$

$$= \frac{3x^4y + 2x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2 + y^2)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$$

$$= \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}. \tag{4}$$

The right side of (4) is well-defined provided $(x^2 + y^2)^2 \neq 0$ and since our assumption in deriving equation (4) was that $(x,y) \neq (0,0)$, it is clear that

$(x^2 + y^2)^2 \neq 0.$

Before we compute $f_x(0,0)$, let us observe that the continuity of f_x at (0,0) will involve showing that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0). \tag{5}$$

3.6.2 continued

Thus, we might as well compute $\lim_{(x,y)\to(0,0)} f_x(x,y)$ from equation (4). Again, introducing polar coordinates, we have

$$f_x(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$= \frac{r \sin\theta\,[r^4\cos^4\theta + 4r^2\cos^2\theta\, r^2\sin^2\theta - r^4\sin^4\theta]}{r^4}$$

$$= \frac{r^5}{r^4} \sin\theta\,(\cos^4\theta + 4\sin^2\theta\cos^2\theta - \sin^4\theta)$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{r\to 0} [r \sin\theta\,(\cos^4\theta + 4\sin^2\theta\cos^2\theta - \sin^4\theta)]$$

$$= 0. \tag{6}$$

To compute $f_x(0,0)$, it is perhaps safest to return to the basic definition

$$f_x(a,b) = \lim_{\Delta x\to 0}\left[\frac{f(a + \Delta x,b) - f(a,b)}{\Delta x}\right]$$

so that

$$f_x(0,0) = \lim_{\Delta x\to 0}\left[\frac{f(\Delta x,0) - f(0,0)}{\Delta x}\right]. \tag{7}$$

Since $\lim_{\Delta x\to 0}$ implies $\Delta x \neq 0$, we may use equation (3) to conclude that

$$f(\Delta x,0) = \frac{(\Delta x)(0)(\overline{\Delta x}^2 - 0^2)}{\overline{\Delta x}^2 + 0^2} = \frac{0}{\overline{\Delta x}^2} = 0.$$

Moreover,

$f(0,0) = 0$

3.6.2 continued

by definition.

Hence, (7) may be rewritten as

$$f_x(0,0) = \lim_{\Delta x \to 0} \left[\frac{0 - 0}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} 0$$

$$= 0. \tag{8}$$

Combining (8) with (6), we have

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

so that f_x is continuous at (0,0).

A similar treatment shows that f_y exists and is continuous at (0,0). In more detail, we differentiate (3) with respect to y to obtain

$$f_y(x,y) = \frac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)2y}{(x^2 + y^2)^2}$$

$$= \frac{x^5 - 2x^3y^2 - 3xy^4 - 2x^3y^2 + 2xy^4}{(x^2 + y^2)^2}$$

$$= \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}. \tag{9}$$

From (9), it follows that $\lim_{(x,y)\to(0,0)} f_y(x,y) = 0$.

[We could carry out the details as we did before. A quicker observation is that if we express the right side of (9) in polar coordinates, the numerator has r^5 as a factor and the denominator has r^4 as a factor so that the quotient has the form $r\,g(\theta)$, whence the limit as $r\to 0$ is 0.]

3.6.2 continued

Then

$$f_y(0,0) = \lim_{\Delta y \to 0} \left[\frac{f(0,\Delta y) - f(0,0)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{0 - 0}{\Delta y}\right]$$

$$= 0. \qquad (10)$$

Hence,

$$\lim_{(x,y)\to(0,0)} f_y(x,y) = f_y(0,0).$$

Summarizing our results to this point, we have shown that f, f_x, and f_y exist and are continuous at (0,0).

Let us next investigate $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

To begin with, $f_{xy}(0,0)$ means

$$\frac{\partial}{\partial y}[f_x(x,y)]\Bigg|_{(0,0)} .$$

For convenience, if we let $h(x,y) = f_x(x,y)$, we have that

$$f_{xy} = h_y$$

Therefore,

$$f_{xy}(0,0) = h_y(0,0)$$

$$= \lim_{\Delta y \to 0} \left[\frac{h(0,\Delta y) - h(0,0)}{\Delta y}\right]$$

and since $h = f_x$, it follows that

$$f_{xy}(0,0) = \lim_{\Delta y \to 0} \left[\frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}\right]. \qquad (11)$$

3.6.2 continued

From (8), $f_x(0,0) = 0$, while from (4) [since $\Delta y \neq 0$]

$$f_x(0,\Delta y) = \frac{\Delta y(0^4 + 40^2\,\overline{\Delta y}^2 - \overline{\Delta y}^4)}{\overline{\Delta y}^4} = -\frac{\overline{\Delta y}^5}{\overline{\Delta y}^4} = -\Delta y.$$

Hence, equation (11) may be rewritten as

$$f_{xy}(0,0) = \lim_{\Delta y \to 0}\left[\frac{-\Delta y - 0}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[-\frac{\Delta y}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}[-1]$$

$$= -1. \qquad (12)$$

Similarly,

$$f_{yx}(0,0) = k_x(0,0)$$

where

$$k = f_y.$$

Hence,

$$f_{yx}(0,0) = \lim_{\Delta x \to 0}\left[\frac{k(\Delta x,0) - k(0,0)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{f_y(\Delta x,0) - f_y(0,0)}{\Delta x}\right]. \qquad (13)$$

Again, from (10), $f_y(0,0) = 0$, while from (9)

$$f_y(\Delta x,0) = \frac{\Delta x(\overline{\Delta x}^4 - 4\overline{\Delta x}^2 0^2 - 0^4)}{\overline{\Delta x}^4}$$

$$= \Delta x.$$

3.6.2 continued

Hence, equation (13) may be rewritten as

$$f_{yx}(0,0) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= 1. \qquad (14)$$

Comparing (12) and (14), we see that

$f_{xy}(0,0) \neq f_{yx}(0,0)$

since $f_{xy}(0,0) = -1$ while $f_{yx}(0,0) = 1$.

Now, if the theorem stated in the introduction to this exercise is correct, it must mean that f_{xy} is not continuous at (0,0), for if it were, the theorem guarantees that in this event $f_{yx}(0,0)$ exists and is equal to $f_{xy}(0,0)$.

Thus, to round out this exercise, we should compute f_{xy} and show that f_{xy} is not continuous at (0,0).

From (4), if $(x,y) \neq (0,0)$, we may differentiate f_x with respect to y to obtain

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left[\frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}\right]$$

$$= \frac{(x^2+y^2)^2(x^4+12x^2y^2-5y^4) - (x^4y+4x^2y^3-y^5)2(x^2+y^2)2y}{(x^2 + y^2)^4}$$

$$= \frac{(x^2+y^2)[(x^2+y^2)(x^4+12x^2y^2-5y^4) - 4y(x^4y+4x^2y^3-y^5)]}{(x^2 + y^2)^4}$$

or, since $(x,y) \neq (0,0)$ [so that $x^2 + y^2 \neq 0$],

3.6.2 continued

$$f_{xy}(x,y) = \frac{x^6+12x^4y^2-5x^2y^4+x^4y^2+12x^2y^4-5y^6-4x^4y^2-16x^2y^4+4y^6}{(x^2+y^2)^3}$$

Therefore,

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} \qquad (15)$$

From (15),

$$f_{xy}(x,0) = \frac{x^6}{x^6} = 1 \quad (x \neq 0) \qquad (16)$$

while

$$f_{xy}(0,y) = -\frac{y^6}{y^6} = -1 \quad (y \neq 0). \qquad (17)$$

From (16), we see that

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) = 1$$

if $(x,y)\to(0,0)$ along the line $y = 0$. While from (17), we see that

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) = -1$$

if $(x,y)\to(0,0)$ along the line $x = 0$.

Thus, while $f_{xy}(0,0)$ exists, $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist since the value of the limit depends on the path along which $(x,y)\to(0,0)$.

As a final note on this exercise, observe that we have just shown by example that the fact that f, f_x, and f_y all exist and are continuous at (a,b) [in this case, $(0,0)$] is not enough to guarantee that f_{xy} or f_{yx} will also be continuous at (a,b). In

3.6.2 continued

other words, the fact that f_x and f_y are separately continuous does not guarantee that the mixed partials f_{xy} or f_{yx} will be continuous.

The converse is also true. That is, if we know that f_{yx}, say, exists, we cannot conclude that both f_x and f_y exist. As a trivial (and not too exciting) example, suppose $f(x,y) = g(x)$ where g is a function of x which is not differentiable. Then, since $g'(x)$ doesn't exist and $g'(x) = f_x(x,y)$, it follows that f_x does not exist. On the other hand, the fact that $f(x,y) = g(x)$ means that f is independent of y and this in turn means that $f_y(x,y)$ not only exists but that it is identically zero, since f depends on x alone. Therefore, f_y is identically zero, and accordingly

$$(f_y)_x = \frac{\partial}{\partial x}(f_y) = \frac{\partial(0)}{\partial x} \equiv 0.$$

In other words, in this example, we have shown that f_{yx} exists even though f_x does not exist.

The main point through all of this is to learn to be "respectful" to the subtleties of taking partial derivatives and in particular to learn not to jump to "obvious" conclusions which happen to be false.

3.6.3

a. $w_\theta = w_x x_\theta + w_y y_\theta$. (1)

Now, since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$x_\theta = -r\sin\theta$ and $y_\theta = r\cos\theta$. (2)

Putting (2) into (1) yields

$w_\theta = -w_x r\sin\theta + w_y r\cos\theta$. (3)

Therefore,

3.6.3 continued

$$w_{\theta\theta} = \frac{\partial}{\partial\theta}[-w_x r \sin\theta + w_y r \cos\theta]$$

$$= \frac{\partial}{\partial\theta}[-w_x r \sin\theta] + \frac{\partial}{\partial\theta}[w_y r \cos\theta] \qquad (4)$$

Since r is constant when we differentiate with respect to θ, equation (4) may be rewritten as

$$w_{\theta\theta} = -r\frac{\partial}{\partial\theta}[w_x \sin\theta] + r\frac{\partial}{\partial\theta}[w_y \cos\theta]. \qquad (5)$$

We must next remember that both w_x and w_y are, by the chain rule, functions of r and θ. Hence, evaluating either $\frac{\partial}{\partial\theta}[w_x \sin\theta]$ or $\frac{\partial}{\partial\theta}[w_y \cos\theta]$ requires the use of the product rule.
More specifically,

$$\frac{\partial}{\partial\theta}[w_x \sin\theta] = w_x \cos\theta + \frac{\partial(w_x)}{\partial\theta}\sin\theta \qquad (6)$$

and

$$\frac{\partial}{\partial\theta}[w_y \cos\theta] = -w_y \sin\theta + \frac{\partial(w_y)}{\partial\theta}\cos\theta. \qquad (7)$$

Now

$$\frac{\partial(w_x)}{\partial\theta} = \frac{\partial(w_x)}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial(w_x)}{\partial y}\frac{\partial y}{\partial\theta}$$

$$= -\frac{\partial^2 w}{\partial x^2} r \sin\theta + \frac{\partial^2 w}{\partial x \partial y} r \cos\theta \qquad (8)$$

and

$$\frac{\partial(w_y)}{\partial\theta} = \frac{\partial(w_y)}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial(w_y)}{\partial y}\frac{\partial y}{\partial\theta}$$

$$= -\frac{\partial^2 w}{\partial y \partial x} r \sin\theta + \frac{\partial^2 w}{\partial y^2} r \cos\theta. \qquad (9)$$

3.6.3 continued

Substituting (8) into (6) and (9) into (7), we have

$$\frac{\partial}{\partial \theta}[w_x \sin \theta] = w_x \cos \theta - w_{xx} r \sin^2\theta + w_{xy} r \sin \theta \cos \theta \qquad (10)$$

and

$$\frac{\partial}{\partial \theta}[w_y \cos \theta] = -w_y \sin \theta - w_{yx} r \sin \theta \cos \theta + w_{yy} r \cos^2\theta. \qquad (11)$$

Finally, we substitute the results of (10) and (11) into equation (5) to obtain

$$w_{\theta\theta} = [-rw_x \cos \theta + w_{xx} r^2 \sin^2\theta - w_{xy} r^2 \sin \theta \cos \theta]$$

$$+ [-rw_y \sin \theta - w_{yx} r^2 \sin \theta \cos \theta + w_{yy} r^2 \cos^2\theta]. \qquad (12)$$

Since w, w_x, w_y, w_{xy} exist and are continuous, it follows that $w_{yx} = w_{xy}$, and we have

$$w_{\theta\theta} = w_{xx} r^2 \sin^2\theta - 2w_{xy} r^2 \sin \theta \cos \theta + w_{yy} r^2 \cos^2\theta - w_x r \cos \theta$$

$$- w_y r \sin \theta. \qquad (13)$$

b. From Exercise 3.6.1,

$$w_{rr} = w_{xx} \cos^2\theta + 2w_{xy} \sin \theta \cos \theta + w_{yy} \sin^2\theta.$$

Therefore,

$$r^2 w_{rr} = w_{xx} r^2 \cos^2\theta + 2w_{xy} r^2 \sin \theta \cos \theta + w_{yy} r^2 \sin^2\theta. \qquad (14)$$

Adding equations (13) and (14) yields

$$r^2 w_{rr} + w_{\theta\theta} = w_{xx} r^2 + w_{yy} r^2 - r(w_x \cos \theta + w_y \sin \theta). \qquad (15)$$

3.6.3 continued

c. Since $\cos\theta = x_r$ and $\sin\theta = y_r$,

$$w_x \cos\theta + w_y \sin\theta = w_x x_r + w_y y_r = w_r.$$

Therefore,

$$r^2 w_{rr} + w_{\theta\theta} = r^2(w_{xx} + w_{yy}) - r\, w_r$$

Therefore,

$$r^2(w_{xx} + w_{yy}) = r^2 w_{rr} + w_{\theta\theta} + r\, w_r$$

Therefore, if $r \neq 0$, then

$$w_{xx} + w_{yy} = w_{rr} + \frac{1}{r^2} w_{\theta\theta} + \frac{1}{r} w_r. \tag{16}$$

3.6.4(L)

Aside from supplying us with additional drill, another major aim of this exercise is to generalize our results beyond polar coordinates. In this exercise, we assume only that w is a continuously differentiable function of the two independent variables u and v, and that u and v are differentiable functions of x and y. In the special case of polar coordinates $u = r$ and $v = \theta$.

a. At any rate, by use of the chain rule, we have that

$$w_x = w_u u_x + w_v v_x. \tag{1}$$

Applying the chain rule to (1), we have:

$$(w_x)_x = (w_u u_x + w_v v_x)_x$$

$$= (w_u u_x)_x + (w_v v_x)_x$$

$$= [w_u u_{xx} + (w_u)_x u_x] + [w_v v_{xx} + (w_v)_x v_x]. \tag{2}$$

3.6.4(L) continued

Now, by the chain rule,

$$(w_u)_x = (w_u)_u u_x + (w_u)_v v_x$$

$$= w_{uu}u_x + w_{uv}v_x \tag{3}$$

and

$$(w_v)_x = (w_v)_u u_x + (w_v)_v v_x$$

$$= w_{vu}u_x + w_{vv}v_x. \tag{4}$$

Putting the results of (3) and (4) into (2), we obtain

$$w_{xx} = [w_u u_{xx} + (w_{uu}u_x + w_{uv}v_x)u_x]$$

$$+ [w_v v_{xx} + (w_{vu}u_x + w_{vv}v_x)v_x]. \tag{5}$$

Since the conditions of this exercise guarantee that $w_{uv} = w_{vu}$, we may collect terms in (5) to obtain

$$w_{xx} = w_{uu}u_x{}^2 + w_{vv}v_x{}^2 + 2w_{uv}(u_x v_x) + w_u u_{xx} + w_v v_{xx}. \tag{6}$$

By reversing the roles of x and y, we may deduce from equation (6) that

$$w_{yy} = w_{uu}u_y{}^2 + w_{vv}v_y{}^2 + 2w_{uv}u_y v_y + w_u u_{yy} + w_v v_{yy}. \tag{7}$$

Of course, had we wished, we could have derived (7) without reference to (6). That is, we have

$$w_y = w_u u_y + w_v v_y.$$

Hence,

3.6.4(L) continued

$$w_{yy} = (w_y)_y = (w_u u_y + w_v v_y)_y$$

$$= (w_u u_y)_y + (w_v v_y)_y$$

$$= [w_u u_{yy} + (w_u)_y u_y] + [w_v v_{yy} + (w_v)_y v_y]$$

$$= [w_u u_{yy} + (w_{uu} u_y + w_{uv} v_y) u_y] + [w_v v_{yy} + (w_{vu} u_y + w_{vv} v_y) v_y].$$

At any rate, we need now only add equations (6) and (7) to obtain

$$w_{xx} + w_{yy} = w_{uu}(u_x{}^2 + u_y{}^2) + w_{vv}(v_x{}^2 + v_y{}^2) + 2w_{uv}(u_x v_x + u_y v_y)$$

$$+ w_u(u_{xx} + u_{yy}) + w_v(v_{xx} + v_{yy}). \qquad (8)$$

Notice from equation (8) that in general, $w_{xx} + w_{yy}$ involves <u>five</u> terms; i.e., w_{uu}, w_{vv}, w_{uv}, w_u, and w_v when we make our change of variables.

b. In the special case of polar coordinates, we have $u = r$ and $v = \theta$. From $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$, we obtain

$$2r\, r_x = 2x.$$

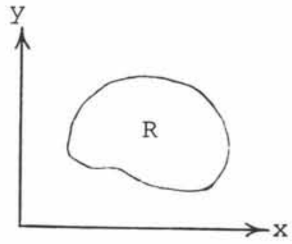
Therefore,

$$r_x = \frac{x}{r} = \cos \theta$$

<u>provided $r \neq 0$</u>.

(If $r = 0$, $\frac{x}{r}$ is undefined. To avoid this dilemma, we assume that the region in which our function $w = f(r,\theta)$ is defined does <u>not</u> include $r = 0$. For example, if R denotes the domain of f, we could have

3.6.4(L) continued



The case $r = 0$ will be discussed in more detail in a later unit.)

Similarly,

$$2r\ r_y = 2y.$$

Therefore,

$$r_y = \frac{y}{r} = \sin\theta,\ r \neq 0.$$

Finally, $\tan\theta = \frac{y}{x}$ implies

$$\sec^2\theta\ \frac{\partial\theta}{\partial x} = -\frac{y}{x^2},$$

therefore

$$\theta_x = -\frac{y}{x^2}\cos^2\theta = -\frac{r\sin\theta}{r^2\cos^2\theta}\cos^2\theta = -\frac{\sin\theta}{r}$$

and

$$\sec^2\theta\ \frac{\partial\theta}{\partial y} = \frac{1}{x},$$

therefore

$$\theta_y = \frac{\cos^2\theta}{x} = \frac{\cos^2\theta}{r\cos\theta} = \frac{\cos\theta}{r}.$$

Therefore,

3.6.4(L) continued

$$\left.\begin{aligned} u_x &= r_x = \cos\theta \\ u_y &= r_y = \sin\theta \end{aligned}\right\} u_x^{\;2} + u_y^{\;2} = \cos^2\theta + \sin^2\theta = 1 \qquad (9)$$

$$\left.\begin{aligned} v_x &= \theta_x = -\frac{\sin\theta}{r} \\ v_y &= \theta_y = \frac{\cos\theta}{r} \end{aligned}\right\} v_x^{\;2} + v_y^{\;2} = \frac{\sin^2\theta}{r^2} + \frac{\cos^2\theta}{r^2} = \frac{1}{r^2} \qquad (10)$$

$$\left.\begin{aligned} u_x v_x &= \cos\theta\left(-\frac{\sin\theta}{r}\right) \\ u_y v_y &= \sin\theta\left(\frac{\cos\theta}{r}\right) \end{aligned}\right\} u_x v_x + u_y v_y = 0 \qquad (11)$$

$$u_x = \cos\theta \rightarrow u_{xx} = (\cos\theta)_x = -\sin\theta\,\frac{\partial\theta}{\partial x}$$

$$= \frac{\sin^2\theta}{r}$$

$$u_y = \sin\theta \rightarrow u_{yy} = (\sin\theta)_y = \cos\theta\,\frac{\partial\theta}{\partial y}$$

$$= \frac{\cos^2\theta}{r}$$

Therefore

$$u_{xx} + u_{yy} = \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} = \frac{1}{r} \qquad (12)$$

$$v_x = -\frac{\sin\theta}{r} \rightarrow v_{xx} = \left(-\frac{\sin\theta}{r}\right)_x = \frac{-r\cos\theta\,\frac{\partial\theta}{\partial x} + \sin\theta\,\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{[-r\cos\theta]\left[-\frac{\sin\theta}{r}\right] + \sin\theta\cos\theta}{r^2} = \frac{2\sin\theta\cos\theta}{r^2}$$

$$v_y = \frac{\cos\theta}{r} \rightarrow v_{yy} = \left(\frac{\cos\theta}{r}\right)_y = \frac{-r\sin\theta\,\frac{\partial\theta}{\partial y} - \cos\theta\,\frac{\partial r}{\partial y}}{r^2}$$

Therefore,

3.6.4(L) continued

$$v_{yy} = \frac{-r \sin\theta \left(\frac{\cos\theta}{r}\right) - \cos\theta \sin\theta}{r^2} = -\frac{2 \sin\theta \cos\theta}{r^2}$$

Therefore,

$$v_{xx} + v_{yy} = \frac{2 \sin\theta \cos\theta}{r^2} - \frac{2 \sin\theta \cos\theta}{r^2} \equiv 0. \qquad (13)$$

Substituting the results of (9), (10), (11), (12), and (13) into (8) yields

$$w_{xx} + w_{yy} = w_{uu} + \frac{1}{r^2} w_{vv} + 0\ w_{uv} + \frac{1}{r} w_u + 0\ w_v$$

$$= w_{rr} + \frac{1}{r^2} w_{\theta\theta} + \frac{1}{r} w_r \qquad (14)$$

and we see that equation (14) checks with equation (16) of Exercise 3.6.3.

c. Without reference to equation (8), we have

$$w = e^{x^2+y^2} \cos(x^2 - y^2)$$

Therefore,

$$w_x = 2xe^{x^2+y^2} \cos(x^2 - y^2) - 2xe^{x^2+y^2} \sin(x^2 - y^2)$$

$$= 2xe^{x^2+y^2} [\cos(x^2 - y^2) - \sin(x^2 - y^2)]$$

Therefore,

$$w_{xx} = 2e^{x^2+y^2} [\cos(x^2 - y^2) - \sin(x^2 - y^2)]$$

$$+ 2x\ (2xe^{x^2+y^2})\ [\cos(x^2 - y^2) - \sin(x^2 - y^2)]$$

$$+ 2xe^{x^2+y^2} [-2x \sin(x^2 - y^2) - 2x \cos(x^2 - y^2)]$$

3.6.4(L) continued

Therefore,

$$w_{xx} = 2e^{x^2+y^2}\Big\{[\cos(x^2 - y^2) - \sin(x^2 - y^2)] + 2x^2[\cos(x^2 - y^2) - \sin(x^2 - y^2)] + x[-2x \sin(x^2 - y^2) - 2x \cos(x^2 - y^2)]\Big\}$$

$$= 2e^{x^2+y^2}\Big\{\cos(x^2 - y^2) - \sin(x^2 - y^2) - 4x^2 \sin(x^2 - y^2)\Big\} \quad (15)$$

Similarly,

$$w_y = 2ye^{x^2+y^2} \cos(x^2 - y^2) + 2ye^{x^2+y^2} \sin(x^2 - y^2)$$

$$= 2ye^{x^2+y^2} [\cos(x^2 - y^2) + \sin(x^2 - y^2)]$$

$$w_{yy} = 2e^{x^2+y^2} [\cos(x^2 - y^2) + \sin(x^2 - y^2)] + 2y(2ye^{x^2+y^2})[\cos(x^2 - y^2) + \sin(x^2 - y^2)] + 2ye^{x^2+y^2} [2y \sin(x^2 - y^2) - 2y \cos(x^2 - y^2)]$$

Therefore,

$$w_{yy} = 2e^{x^2+y^2}\Big\{[\cos(x^2 - y^2) + \sin(x^2 - y^2)] + 2y^2[\cos(x^2 - y^2) + \sin(x^2 - y^2)] + y[2y \sin(x^2 - y^2) - 2y \cos(x^2 - y^2)]\Big\}$$

$$= 2e^{x^2+y^2}\Big\{\cos(x^2 - y^2) + \sin(x^2 - y^2) + 4y^2 \sin(x^2 - y^2)\Big\} \quad (16)$$

Adding (15) and (16) yields

$$w_{xx} + w_{yy} = 2e^{x^2+y^2}\Big\{2 \cos(x^2 - y^2) - 4(x^2 - y^2) \sin(x^2 - y^2)\Big\}$$

$$= 4e^{x^2+y^2} \cos(x^2 - y^2) - 8e^{x^2+y^2}(x^2 - y^2)\sin(x^2 - y^2). \quad (17)$$

3.6.4(L) continued

d. The cumbersome computation of part (c) can be simplified by use of equation (8) if we make the rather natural substitution

$u = x^2 + y^2$

and

$v = x^2 - y^2$.

This substitution yields

$$w = e^u \cos v \tag{18}$$

and, in this form, it is easy to differentiate with respect to either u or v since our factors have u and v separated. More specifically:

$w_u = e^u \cos v$

$w_{uu} = e^u \cos v$

$w_v = -e^u \sin v$

$w_{vv} = -e^u \cos v$

$w_{uv} = -e^u \sin v$

We also have that

$u_x = 2x$, $u_{xx} = 2$, $v_x = 2x$, $v_{xx} = 2$,

$u_y = 2y$, $u_{yy} = 2$, $v_y = -2y$, $v_{yy} = -2$

Hence,

3.6.4(L) continued

$${u_x}^2 + {u_y}^2 = 4x^2 + 4y^2 = 4(x^2 + y^2) = 4u$$

$${v_x}^2 + {v_y}^2 = 4x^2 + 4y^2 = 4u$$

$$u_x v_x + u_y v_y = 4x^2 - 4y^2 = 4(x^2 - y^2) = 4v$$

$$u_{xx} + u_{yy} = 2 + 2 = 4$$

$$v_{xx} + v_{yy} = 2 - 2 = 0$$

With these particular results, equation (8) becomes

$$w_{xx} + w_{yy} = 4uw_{uu} + 4uw_{vv} + 2w_{uv}(4v) + 4w_u + 0w_v$$

$$= 4ue^u \cos v + 4u(-e^u \cos v) + 8v(-e^u \sin v) + 4e^u \cos v$$

$$= 4e^u \cos v - 8ve^u \sin v \qquad (19)$$

and, if we now replace u by $x^2 + y^2$ and v by $x^2 - y^2$, we see that equation (19) is the same as equation (17).

While the next remark is subjective, it is our feeling that the second method, using equation (8), it less cumbersome and easier to keep track of than is the first method [part (c)].

3.6.5(L)

From a computational point of view, this exercise is a rather simple application of the previous exercise. What we did want to emphasize in this exercise, beyond any computational consequences, is the role of this unit in the solution of partial differential equations.

It turns out that in many applications of mathematics to physics and engineering we become involved with partial differential equations of the second order. Three "well-known" examples are:

(1) The Wave Equation

$$w_{tt} = a^2 w_{xx}$$

3.6.5(L) continued

(2) The Heat Transfer Equation

$$w_t = a^2 w_{xx}$$

(3) Laplace's Equation

$$w_{xx} + w_{yy} = 0$$

(If w denotes temperature, Laplace's equation is known as the steady state equation.)

In solving such equations, we are usually given certain boundary conditions, meaning we are told what the solution must look like along certain curves (boundaries). Very often, it is convenient to introduce other coordinate systems in order to express the boundary conditions in as helpful a way as possible.

For example, we might be called upon to solve Laplace's equation in a situation where we have circular symmetry. That is, suppose we know that $w_{xx} + w_{yy} = 0$ and also that w depends only on the distance of the point from the origin. As an illustration, w might denote the magnitude of the force at a point in a central force field in which the force is proportional to the distance from the origin to the point. In such a case, it would be natural to introduce polar coordinates and view w as the special case of a function of r and θ in which the function is independent of θ.

In terms of more formal language, we are saying that in this case $w = f(r,\theta) = h(r)$. What part (a) of this exercise asks us to do is express $w_{xx} + w_{yy}$ in terms of h(r) and r.

a. We already know that in general if $w = f(r,\theta)$ then

$$w_{xx} + w_{yy} = w_{rr} + \frac{w_{\theta\theta}}{r^2} + \frac{1}{r} w_r. \tag{1}$$

Since $w = h(r)$, it follows that w_θ must be identically zero since h(r) is independent of θ; while $w_r = h'(r)$ since w is a function of the single variable r, thus making the partial derivative and the ordinary derivative the same.

3.6.5 (L) continued

With these remarks in mind, equation (1) simplifies to

$$w_{xx} + w_{yy} = h''(r) + \frac{1}{r} h'(r). \qquad (2)$$

b. Thus, in this particular situation (i.e. knowing that w depends only on r), Laplace's equation

$$w_{xx} + w_{yy} = 0$$

may be expressed in polar form as

$$h''(r) + \frac{1}{r} h'(r) = 0. \qquad (3)$$

Since h' and h" appear in (3) but not h, the substitution g = h' reduces (3) to a first order differential equation in which the variables (g and r) may be separated. Once this is done, equation (3) may be solved by the method of anti-derivatives discussed in Part 1 of this course. More specifically, letting g = h', equation (3) becomes

$$g'(r) + \frac{1}{r} g(r) = 0 \qquad (4)$$

and in differential notation, equation (4) may be written as

$$\frac{dg}{dr} + \frac{g}{r} = 0,$$

whereupon

$$\frac{dg}{g} = -\frac{dr}{r}$$

so that

$$\begin{aligned} \ln|g| &= -\ln|r| + c_1 \\ &= \ln \frac{1}{|r|} + \ln c_2 \\ &= \ln \frac{c_2}{|r|}. \end{aligned}$$

3.6.5(L) continued

Therefore,

$$g(r) = \frac{c}{r}$$

and since g' = h, it follows that

$$h'(r) = \frac{c}{r}$$

so that

$$h(r) = c \ln|r| + k. \tag{5}$$

If we restrict r to mean magnitude (without direction), then $r \geqslant 0$, so that (5) may be written as

$$h(r) = c \ln r + k \tag{6}$$

where c and k are arbitrary constants. Recalling that $r = \sqrt{x^2 + y^2}$, we may now rewrite equation (6) in Cartesian coordinates as

$$w = h(r) = h\left(\sqrt{x^2 + y^2}\right) = c \ln \sqrt{x^2 + y^2} + k. \tag{7}$$

Equation (7) gives us the following interesting information. Suppose that r denotes the <u>distance</u> from the origin to a point in the plane and that w is a function of r alone which satisfies Laplace's equation $w_{xx} + w_{yy} = 0$. Then w <u>must</u> have the form

$$w(x,y) = c \ln \sqrt{x^2 + y^2} + k,$$

where c and k are arbitrary constants.

As a computational check, let us take the special case c = 2 (which eliminates the radical sign) and k = 0. We then obtain that

$$w(x,y) = 2 \ln \sqrt{x^2 + y^2} = \ln\left(\sqrt{x^2 + y^2}\right)^2$$

$$= \ln (x^2 + y^2). \tag{8}$$

3.6.5(L) continued

Then

$$w_x = \frac{2x}{x^2 + y^2}$$

$$w_{xx} = \frac{(x^2 + y^2)2 - 2x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} \quad (9)$$

Similarly,

$$w_y = \frac{2y}{x^2 + y^2}$$

$$w_{yy} = \frac{(x^2 + y^2)2 - 2y(2y)}{(x^2 + y^2)^2}$$

Therefore,

$$w_{yy} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \quad (10)$$

Adding equations (9) and (10), we have

$$w_{xx} + w_{yy} \equiv 0.$$

(Again, notice that our results are valid provided only that $r \neq 0$. In other words, we are again assuming that the domain of w does not include the origin. Physically, in situations such as "inverse square laws," the origin is often omitted. Namely, if the force is proportional to $\frac{1}{r}$ or $\frac{1}{r^2}$, then r must not equal 0.)

3.6.6(L)

a. We have

$$x = e^u \cos v,\ y = e^u \sin v. \tag{1}$$

Hence,

$$x_u = e^u \cos v,\ y_u = e^u \sin v \tag{2}$$

and

$$x_v = -e^u \sin v,\ y_v = e^u \cos v. \tag{3}$$

From the chain rule

$$w_u = w_x x_u + w_y y_u,$$

so that from (2),

$$\begin{aligned} w_u &= w_x e^u \cos v + w_y e^u \sin v \\ &= e^u (w_x \cos v + w_y \sin v). \end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned} w_{uv} &= \frac{\partial}{\partial v} (w_u) \\ &= \frac{\partial}{\partial v} [e^u (w_x \cos v + w_y \sin v)], \end{aligned}$$

and since e^u is constant when we differentiate with respect to v,

$$w_{uv} = e^u \frac{\partial}{\partial v} (w_x \cos v + w_y \sin v). \tag{5}$$

Since w_x and w_y are also functions of v (and u), we use the product rule on (5) to obtain

3.6.6(L) continued

$$w_{uv} = e^u\left[w_x \frac{\partial(\cos v)}{\partial v} + \frac{\partial(w_x)}{\partial v}\cos v + w_y \frac{\partial(\sin v)}{\partial v} + \frac{\partial(w_y)}{\partial v}\sin v\right]$$

$$= e^u\left[-w_x \sin v + \frac{\partial(w_x)}{\partial v}\cos v + w_y \cos v + \frac{\partial(w_y)}{\partial v}\sin v\right]. \qquad (6)$$

Applying the chain rule to $\frac{\partial(w_x)}{\partial v}$ yields

$$\frac{\partial(w_x)}{\partial v} = \frac{\partial(w_x)}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial(w_x)}{\partial y}\frac{\partial y}{\partial v},$$

so using (2),

$$\frac{\partial(w_x)}{\partial v} = w_{xx}(-e^u \sin v) + w_{xy}(e^u \cos v). \qquad (7)$$

Similarly,

$$\frac{\partial(w_y)}{\partial v} = \frac{\partial(w_y)}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial(w_y)}{\partial y}\frac{\partial y}{\partial v}$$

$$= w_{yx}(-e^u \sin v) + w_{yy}(e^u \cos v). \qquad (8)$$

Since $w_{yx} = w_{xy}$ in this exericse, the result obtained by substituting (7) and (8) into (6) is

$$w_{uv} = e^u\Big[-w_x \sin v + \{w_{xx}(-e^u \sin v) + w_{xy}(e^u \cos v)\}\cos v +$$

$$w_y \cos v + \{w_{xy}(-e^u \sin v) + w_{yy}(e^u \cos v)\}\sin v\Big]$$

$$= w_x(-e^u \sin v) + w_{xx}(-e^u \sin v)(e^u \cos v) + w_{xy}\Big[(e^u \cos v)^2$$

$$+ (-e^u \sin v)(e^u \sin v)\Big] + w_y\, e^u \cos v + w_{yy}(e^u \cos v)$$

$$(e^u \sin v). \qquad (9)$$

Finally, from (1), $e^u \cos v = x$ and $e^u \sin v = y$, hence

3.6.6(L) continued

$$w_{uv} = -yw_x - xy\ w_{xx} + (x^2 - y^2)w_{xy} + x\ w_y + xy\ w_{yy}$$

Therefore

$$w_{uv} = xy(w_{yy} - w_{xx}) + (x^2 - y^2)w_{xy} - y\ w_x + x\ w_y. \qquad (10)$$

b. From equation (10),

$$xy(w_{yy} - w_{xx}) + (x^2 - y^2)w_{xy} - y\ w_x + x\ w_y = 0$$

is equivalent to

$$w_{uv} = 0 \qquad (11)$$

where $x = e^u \cos v$, $y = e^u \sin v$.

The point is that equation (11) is particularly easy to solve. The technique hinges on the generalization of what we mean by a constant in terms of differentiation. Recall that in the case of a single real variable, f(x) denoted a constant if and only if f'(x) was identically 0 (that is, f'(x) = 0 for each value of x). In the case of several real variables, suppose $w = f(x_1,\ldots,x_n)$ where $x_1,\ldots,$ and x_n are independent variables. Suppose also that, say, w_{x_1} is identically 0. This says that if all the variables except x_1 are held constant then the rate of change of w with respect to x_1 is zero. That is, w is constant as far as x_1 is concerned. In another perspective, this says that w depends only on $x_2,\ldots,$ and x_n.

With this idea in mind, the fact that

$(w_u)_v$ is identically zero

means that w_u is constant as far as v is concerned. That is, w_u is a function of u alone. Say $w_u = f(u)$. Suppose now that F is any function such that $F'(u) = f(u)$. Then the fact that $w_u = f(u)$ means that $w = F(u)$ would be an acceptable solution to equation (11). Now in the case of a single variable, we may always add on a constant of integration. In the case of several real variables,

3.6.6(L) continued

we may add on a <u>function</u> of integration. That is, suppose $G(v)$ denotes any differentiable function of v. Then since u and v are independent variables, it follows that $G_u(v)$ is identically zero.

In other words, if $w_u = f(u)$ and $F'(u) = f(u)$, then $w = F(u) + G(v)$ where G is any differentiable function of v has the property that $w_u = f(u)$. Namely

$$w_u = F_u(u) + G_u(v) = F'(u) + 0 = f(u).$$

Again, applying this discussion to equation (11), we have that since w_{uv} is identically zero,

$$w = F(u) + G(v) \qquad (12)$$

where F and G are any differentiable functions of a single real variable.

The hardest part of this type of problem is that since x and y are given in terms of u and v, we may not always be able to solve explicitly for u and v in terms of x and y. (This will be discussed in more detail later in this block under the heading of the Inverse Function Theorem). In this particular case, however, it is not too difficult to express u and v in terms of x and y. Namely, from (1) we have that

$$x = e^u \cos v$$

$$y = e^u \sin v$$

Dividing the second equation by the first yields

$$\frac{y}{x} = \tan v$$

(where we must beware of any value of v for which $\cos v = 0$, since as usual we are not permitted to divide by 0).

If we now restrict our attention to principal values, we have that

$$v = \tan^{-1}\left(\frac{y}{x}\right). \qquad (13)$$

3.6.6(L) continued

To find u, we square both equations in (1) to obtain

$$x^2 = e^{2u} \cos^2 v$$

$$y^2 = e^{2u} \sin^2 v$$

and adding these two equations yields

$$x^2 + y^2 = e^{2u} (\cos^2 v + \sin^2 v) = e^{2u}$$

so that

$$2u = \ln (x^2 + y^2)$$

or

$$u = \frac{1}{2} \ln (x^2 + y^2)$$

or

$$u = \ln\sqrt{x^2 + y^2}. \qquad (14)$$

Putting the results of (13) and (14) into (12), we obtain

$$w = F\left(\ln \sqrt{x^2 + y^2}\right) + G\left[\tan^{-1}\left(\frac{y}{x}\right)\right] . \qquad (15)$$

The main observations here are that (i) we cannot always find u and v explicitly in terms of x and y, and (ii) given a differential equation such as (10), it is not often easy to find the change of variables that will convert the cumbersome equation into something as simple as (11).

c. Letting $F(u) = 2u$ and $G(v) = \tan v$, equation (15) becomes

$$w = 2 \ln \sqrt{x^2 + y^2} + \tan \left(\tan^{-1} \frac{y}{x}\right)$$

$$= \ln\left(\sqrt{x^2 + y^2}\right)^2 + \frac{y}{x}$$

$$= \ln(x^2 + y^2) + \frac{y}{x}.$$

3.6.6(L) continued

Check

$$w_x = \frac{2x}{x^2 + y^2} - \frac{y}{x^2}$$

$$w_y = \frac{2y}{x^2 + y^2} + \frac{1}{x}$$

$$w_{xy} = \frac{\partial\left[\frac{2x}{x^2 + y^2} - \frac{y}{x^2}\right]}{\partial y}$$

$$= \frac{-2x(2y)}{(x^2 + y^2)^2} - \frac{1}{x^2}$$

$$w_{xx} = \frac{\partial}{\partial x}\left[\frac{2x}{x^2 + y^2} - \frac{y}{x^2}\right] = \frac{(x^2 + y^2)2 - 2x(2x)}{(x^2 + y^2)^2} + \frac{2y}{x^3}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2y}{x^3}$$

$$w_{yy} = \frac{\partial\left[\frac{2y}{x^2 + y^2} + \frac{1}{x}\right]}{\partial y} = \frac{(x^2 + y^2)2 - 2y(2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

Therefore,

$$xy(w_{yy} - w_{xx}) + (x^2 - y^2)w_{xy} - yw_x + xw_y = xy\left[\frac{2x^2 - 2y^2}{(x^2 + y^2)^2}\right.$$

$$\left.-\left\{\frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2y}{x^3}\right\}\right] + (x^2 - y^2)\left[\frac{-4xy}{(x^2 + y^2)^2} - \frac{1}{x^2}\right] - y\left[\frac{2x}{x^2 + y^2} - \frac{y}{x^2}\right]$$

$$+ x\left[\frac{2y}{x^2 + y^2} + \frac{1}{x}\right] = \left[\frac{4(x^2 - y^2)}{(x^2 + y^2)^2}xy - \frac{2y^2}{x^2}\right] + \left[\frac{-4(x^2 - y^2)xy}{(x^2 + y^2)^2} - \frac{(x^2 - y^2)}{x^2}\right]$$

$$+ \left[\frac{-2xy}{x^2 + y^2} + \frac{y^2}{x^2}\right] + \left[\frac{2xy}{x^2 + y^2} + 1\right] = \frac{4(x^2 - y^2)xy}{(x^2 + y^2)^2} - \frac{2y^2}{x^2} - \frac{4(x^2 - y^2)xy}{(x^2 + y^2)^2}$$

$$- 1 + \frac{y^2}{x^2} - \frac{2xy}{x^2 + y^2} + \frac{y^2}{x^2} + \frac{2xy}{x^2 + y^2} + 1 = \left[\frac{4(x^2 - y^2)xy}{(x^2 + y^2)^2} - \frac{4(x^2 - y^2)xy}{(x^2 + y^2)^2}\right]$$

$$+ \left[\frac{-2y^2}{x^2} + \frac{y^2}{x^2} + \frac{y^2}{x^2}\right] + \left[\frac{-2xy}{x^2 + y^2} + \frac{2xy}{x^2 + y^2}\right] + [-1 + 1] = 0.$$

3.6.7

a. We have

$$x = u^2 - v^2 \tag{1}$$

and

$$y = 2uv. \tag{2}$$

Hence,

$$x_u = 2u,\ y_u = 2v \tag{3}$$

and

$$x_v = -2v,\ y_v = 2u. \tag{4}$$

By the chain rule,

$$\begin{aligned} w_u &= w_x x_u + w_y y_u \\ &= w_x(2u) + w_y(2v) \\ &= 2[u\,w_x + v\,w_y]. \end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned} w_{uv} &= (w_u)_v = \frac{\partial (w_u)}{\partial v} \\ &= 2\,\frac{\partial}{\partial v}\,[u\,w_x + v\,w_y] \\ &= 2\left[u\,\frac{\partial w_x}{\partial v} + v\,\frac{\partial w_y}{\partial v} + w_y\right] \\ &= 2\left[u\left\{\frac{\partial w_x}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial w_x}{\partial y}\,\frac{\partial y}{\partial v}\right\} + v\left\{\frac{\partial (w_y)}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial (w_y)}{\partial y}\,\frac{\partial y}{\partial v}\right\} + w_y\right] \\ &= 2ux_v w_{xx} + 2uy_v w_{xy} + 2vx_v w_{yx} + 2vy_v w_{yy} + 2w_y. \end{aligned} \tag{6}$$

3.6.7 continued

Using (4) and the fact that $w_{xy} = w_{yx}$, equation (6) becomes

$$w_{uv} = 2u(-2v)w_{xx} + 2u(2u)w_{xy} + 2v(-2v)w_{xy} + 2v(2u)w_{yy} + 2w_y$$

or

$$w_{uv} = -4uv\ w_{xx} + 4(u^2 - v^2)w_{xy} + 4uv\ w_{yy} + 2w_y. \qquad (7)$$

By (1), $u^2 - v^2 = x$ and $4uv = 2y$, so (7) becomes

$$w_{uv} = -2y\ w_{xx} + 4x\ w_{xy} + 2y\ w_{yy} + 2w_y$$

$$= 2[y\ w_{yy} + 2x\ w_{xy} - y\ w_{xx} + w_y]. \qquad (8)$$

b. Using (8), we have

$$y\ w_{yy} + 2x\ w_{xy} - y\ w_{xx} + w_y = \frac{1}{2}w_{uv}.$$

Hence,

$$y\ w_{yy} + 2x\ w_{xy} - y\ w_{xx} + w_y = 0 \qquad (9)$$

is equivalent to

$$w_{uv} = 0 \text{ where } x = u^2 - v^2 \text{ and } y = 2uv. \qquad (10)$$

The solution of (10) is

$$w = F(u) + G(v). \qquad (11)$$

Hence, the solution of (9) is

$$w = F(u) + G(v)$$

where u and v are defined implicitly in terms of x and y by

3.6.7 continued

$x = u^2 - v^2$

$y = 2uv$

We can solve for u and v in terms of x and y in this case as follows:

$x^2 = (u^2 - v^2)^2 = u^4 - 2u^2v^2 + v^4$

$y^2 = 4u^2v^2$

Therefore,

$x^2 + y^2 = u^4 + 2u^2v^2 + v^4 = (u^2 + v^2)^2$

Therefore,

$u^2 + v^2 = \sqrt{x^2 + y^2}$

and since

$u^2 - v^2 = x$

it follows that

$$2u^2 = \sqrt{x^2 + y^2} + x \qquad (12)$$

so that

$u^2 = \frac{1}{2}\sqrt{x^2 + y^2} + \frac{1}{2}x.$

Then, since $u^2 - v^2 = x$,

$v^2 = u^2 - x$

or

3.6.7 continued

$$v^2 = \frac{1}{2}\sqrt{x^2 + y^2} + \frac{1}{2}x - x$$

$$= \frac{1}{2}\sqrt{x^2 + y^2} - \frac{1}{2}x. \qquad (13)$$

From (12) and (13), we see that

$$u = \sqrt{\frac{1}{2}\sqrt{x^2 + y^2} + \frac{1}{2}x}$$

$$v = \sqrt{\frac{1}{2}\sqrt{x^2 + y^2} - \frac{1}{2}x}$$

so the required solution is

$$w = F\left(\sqrt{\frac{1}{2}\sqrt{x^2 + y^2} + \frac{1}{2}x}\right) + G\left(\sqrt{\frac{1}{2}\sqrt{x^2 + y^2} - \frac{1}{2}x}\right). \qquad (14)$$

(As an example, let $F(u) = u^2$ and $G(v) = v^2$. Then

$$F\left(\sqrt{\frac{1}{2}\sqrt{x^2 + y^2} + \frac{1}{2}x}\right) + G\left(\sqrt{\frac{1}{2}\sqrt{x^2 + y^2} - \frac{1}{2}x}\right)$$

$$= \frac{1}{2}\sqrt{x^2 + y^2} + \frac{1}{2}x + \frac{1}{2}\sqrt{x^2 + y^2} - \frac{1}{2}x$$

$$= \sqrt{x^2 + y^2}.$$

Therefore,

$$w = \sqrt{x^2 + y^2}$$

should be a solution of

$$y\, w_{yy} + 2x\, w_{xy} - y\, w_{xx} + w_y = 0.$$

Details are left to the interested student.)