

Unit 5: The Chain Rule, Part 1

1. Lecture 3.040

<p><u>The Chain Rule</u></p> <p>Suppose <math>w = f(x, y, z)</math></p> <p>and <math>x = x(r, \Delta), y = y(r, \Delta), z = z(r, \Delta)</math></p> <p><math>\therefore w = g(r, \Delta)</math></p> <p><u>Problem</u></p> <p>To compute, say, <math>\frac{\partial w}{\partial r}</math></p> <p>knowing <math>\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}</math></p> <p><math>\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}</math> etc.</p>	<p><u>Claim</u></p> $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$ <p><u>Proof</u></p> $\Delta w = \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \frac{\partial w}{\partial z} \Delta z$ $= k_1 \Delta x + k_2 \Delta y + k_3 \Delta z$ <p>where <math>k_1, k_2, k_3 \rightarrow 0</math></p> <p>as <math>\Delta x, \Delta y, \Delta z \rightarrow 0</math></p>	<p><u>Example</u></p> $w = x^2 + y^2 + z^2$ $x = r + \Delta, y = r - \Delta, z = 2r$ $\therefore w = (r + \Delta)^2 + (r - \Delta)^2 + (2r)^2$ $= 6r^2 + 2\Delta^2$ $\therefore \frac{\partial w}{\partial r} = 12r$ $\frac{\partial w}{\partial x} = 2x, \frac{\partial w}{\partial y} = 2y, \frac{\partial w}{\partial z} = 2z$ $\frac{\partial x}{\partial r} = 1, \frac{\partial y}{\partial r} = 1, \frac{\partial z}{\partial r} = 2$ $\therefore \frac{\partial w}{\partial r} = 2x + 2y + 4z = 12r$
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a.

<p><u>More generally</u></p> <p>Let <math>w = f(x_1, \dots, x_n)</math></p> <p>where <math>f_{x_1}, \dots, f_{x_n}</math> are cont.</p> <p>Suppose <math>x_1 = g_1(y_1, \dots, y_m)</math></p> <p><math>\vdots</math></p> <p><math>x_n = g_n(y_1, \dots, y_m)</math></p> <p>etc</p>	<p><u>Then</u></p> $w = h(y_1, \dots, y_m)$ and $\frac{\partial w}{\partial y_1} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial y_1} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial y_1} + \dots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial y_1}$ <p>etc.</p>	<p><u>Higher Order Derivatives</u></p> <p>Suppose <math>w = f(x, y)</math></p> $x = r \cos \theta, y = r \sin \theta$ <p>Then</p> $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}$ $= \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta$ <p><math>\frac{\partial w}{\partial x}</math> and <math>\frac{\partial w}{\partial y}</math> are also functions of <math>x</math> and <math>y</math> - so may use chain rule again</p>
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b.

$\frac{\partial^2 w}{\partial r^2} = \frac{\partial}{\partial r} \left[ \cos \theta \frac{\partial w}{\partial x} + \sin \theta \frac{\partial w}{\partial y} \right]$ $= \cos \theta \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial x} \right) + \sin \theta \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial y} \right)$ <p>Let <math>h(x, y) = \frac{\partial w}{\partial x}</math></p> $\therefore \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial h}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial h}{\partial y} \frac{\partial y}{\partial r}$ $= \frac{\partial^2 w}{\partial x^2} \cos \theta + \frac{\partial^2 w}{\partial x \partial y} \sin \theta$	$\frac{\partial}{\partial r} \left( \frac{\partial w}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial y} \right) \frac{\partial y}{\partial r}$ $= \frac{\partial^2 w}{\partial y \partial x} \cos \theta + \frac{\partial^2 w}{\partial y^2} \sin \theta$ $\frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 w}{\partial x^2} \cos^2 \theta + \frac{\partial^2 w}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 w}{\partial y^2} \sin^2 \theta + \frac{\partial^2 w}{\partial y \partial x} \sin \theta \cos \theta$	<p><u>Example</u></p> <p>Let <math>w = x^2 + y^2</math></p> $\therefore w = r^2 \quad \therefore \frac{\partial^2 w}{\partial r^2} = 2$ $\frac{\partial w}{\partial x} = 2x, \frac{\partial^2 w}{\partial x^2} = 2$ $\frac{\partial w}{\partial y} = 2y, \frac{\partial^2 w}{\partial y^2} = 2$ $\frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) = 0$ $\therefore \frac{\partial^2 w}{\partial r^2} = 2 \cos^2 \theta + 0 + 2 \sin^2 \theta + 0 = 2$
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c.

2. Read Thomas, Section 15.7.

3. Exercises:

3.5.1(L)

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Let  $f: E^3 \rightarrow E$  be defined by  $f(x, y, z) = x^2 + y^2 + z^2$ , and let  $P$  be the plane whose equation is given by

$$x = r + s$$

$$y = 2r + 3s$$

$$z = 4r + 7s$$

- Letting the domain of  $f$  be restricted to  $P$ , express  $w$  as a function of  $r$  and  $s$  where  $w = f(x, y, z)$ .
- Use the result of (a) to compute  $w_r$ .
- Compute  $w_r$  directly from the chain rule [i.e., without using (a) or (b)].

3.5.2

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Let  $f: E^4 \rightarrow E$  be defined by

$$f(\underline{x}) = f(x_1, x_2, x_3, x_4) = x_1 x_4 + x_1 x_2 x_3 + x_2 x_3 x_4,$$

and let  $R$  denote the subset of  $E^4$  defined by  $x_1 = h_1(t)$ ,  $x_2 = h_2(t)$ ,  $x_3 = h_3(t)$ , and  $x_4 = h_4(t)$ , where  $h_1$ ,  $h_2$ ,  $h_3$ , and  $h_4$  are differentiable functions of  $t$ .

- If we let  $w = g(x_1, x_2, x_3, x_4)$ , where  $g$  denotes the restriction of  $f$  to  $R$ , use the chain rule to compute  $\frac{dw}{dt}$ .
- Compute  $\frac{dw}{dt}$  both by the chain rule and by direct substitution if  $h_k(t) = kt$ , where  $k = 1, 2, 3$ , or  $4$ .

3.5.3(L)

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Suppose  $w = f(u + v, u - v)$ . Let  $x = u + v$  and  $y = u - v$ .

- Use the chain rule to show that

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3.5.3(L) continued

$$w_u w_v = f_x^2 - f_y^2.$$

- b. Use the result of part (a) to find a solution to the partial differential equation,  $f_x^2 - f_y^2 = 0$ .

(Note: In general, it is a difficult task to solve partial differential equations. In this exercise, part (a) gives us a result that we can use to solve part (b). It would be much more difficult to invent part (a) had we been asked to solve part (b) without any "hints." Yet, from the point of view of this section, the important thing is the use of the chain rule, and any application to differential equations is more in the vein of an interesting aside.)

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3.5.4(L)

Suppose that  $w = f(ax + by)$  where  $a$  and  $b$  are non-zero constants and  $f$  is a differentiable function of a single variable. (Note that  $ax + by$  is a single variable despite the fact that  $x$  and  $y$  are two variables.)

- a. Compute  $w_x$  and  $w_y$ .
- b. Show that with  $w$  as above,  $bw_x - aw_y = 0$ .
- c. Use the result of part (b) to compute  $4w_x - 3w_y + 2$  if  $w = e^{3x+4y} \tan^{-1}(3x + 4y)^2 \cosh(3x + 4y) \sinh^4(3x + 4y)$ .
- d. Use (b) to solve the equation  $w_x + w_y = 0$ .
- e. Solve the equation  $w_x + w_y = 0$  subject to the boundary value condition that  $w = e^x$  along the  $x$ -axis.

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3.5.5(L)

(Note: This exercise defines a particular type of function called a homogeneous function. No knowledge of calculus is necessary in this exercise. What this problem does is set the stage for the next two exercises, which are calculus problems.)

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3.5.5(L) continued

A function  $f(x_1, \dots, x_n)$  is defined to be homogeneous of degree  $k$  if and only if

$$f(tx_1, \dots, tx_n) = t^k f(x_1, \dots, x_n)$$

where  $t$  is an arbitrary non-zero variable.

- Show that if  $f(x, y, z) = x^3 yz$ , then  $f$  is homogeneous of degree 5.
- If  $f(x, y) = \sin\left(\frac{y}{x}\right)$ , show that  $f$  is homogeneous of degree 0.
- If  $f(x_1, x_2, x_3, x_4) = x_1^3 x_4 + x_2^2 x_3^2$ , show that  $f$  is homogeneous of degree 4.

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3.5.6(L)

- Let  $f(x_1, \dots, x_n)$  be homogeneous of degree  $k$  (unless stated to the contrary, all functions will be assumed to be continuously differentiable). Let  $w = f(tx_1, \dots, tx_n)$  where  $t$  is any variable which is independent of  $x_1, \dots, x_n$ . Compute  $\frac{\partial w}{\partial t}$  in two different ways (one using the chain rule and the other using the definition of homogeneous) to conclude that

$$x_1 f_{x_1} + \dots + x_n f_{x_n} = kf.$$

- Use the result of (a) to find a family of solutions for the equation  $xw_x + yw_y + zw_z = w$ , and give at least one explicit solution of this equation.

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3.5.7

Use the result of part (a) of the previous exercise to solve the equation  $xw_x + yw_y = 3w$ .

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**Resource: Calculus Revisited: Multivariable Calculus**  
Prof. Herbert Gross

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