PROFESSOR: Hi. Today we're going to conclude our study of vectors as applied to motion in the plane. Now recall that for the last two units, we were discussing polar coordinates. So today what we would like to do is investigate what velocity and acceleration vectors would have looked like, had we elected to pick representative vectors in terms of polar coordinates.

Now what do we mean by representative vectors? We mean, of course, analogs of i and j, just like T and N in tangential-normal components were parallels to i and j. Let's take a look and see what that means here. We call today's lecture "Vectors in Polar Coordinates."

The idea is that we're given a curve c, which we have for some reason or other elected to express in terms of polar coordinates. The polar equation of the curve c is r equals f(theta). A typical point on the curve c would be called (r, theta), say, where theta was the angle made by the horizontal and the radius vector, and r was the distance from the origin to the point.

Now what we're saying is, that in terms of polar coordinates, a very natural vector to pick-- especially if we think later in terms of the simple force fields that we've talked about earlier, the idea that there may be a force that has its line of action from the origin to the point on the curve-- a very natural vector to choose is the vector that we elect to call u sub r. And what that vector is, apparently, is the vector 1 unit long having the direction of the radius vector r. So here is u sub r over here.

If we think of u sub r as playing the role of i, then the vector which plays the role of j should be a positive 90-degree rotation of u sub r. And we elect to call that vector u sub theta, using the theta here more to indicate the fact that we're using polar coordinates than to indicate anything about the angle theta itself.

In other words, notice that u sub theta, by definition, is just a positive 90-degree rotation of u sub r, where u sub r is a unit vector in the direction of the radius vector. Now if we want to see this in terms of i and j components, what we're saying is that u sub r is a unit vector whose i component is what? Since this angle here is also theta, it's i component is cos(theta), and it's j component is sin(theta).

So u sub r is cos(theta) i plus sin(theta) j. u sub theta-- and I'm going to do a little twist here that I didn't do in the T and N components, just to show you another approach. Rather than to start with derivatives or anything like this, notice that what we know about u sub theta is that it's obtained from u sub r by a positive 90-degree rotation of theta. So that means if I replace theta in the expression for u sub r, by theta plus 90 degrees, that should give me u sub theta.

If I now remember my trigonometric identities, that tells me that u sub theta is minus sin(theta) i plus cos(theta) j. By the way, if we now look at this expression and compare it with the expression for u sub r, we see at once that u sub theta is the derivative of u sub r with respect to theta. And notice, as we said before, that part of this should have been known to us by now. Namely, since u sub r varies with theta but it has a constant magnitude, we know that the derivative of u sub r with respect to theta has to be perpendicular to u sub r.

In other words, we knew that u sub theta had to be either plus or minus the derivative of u sub r with respect to theta, but now we have a direct way of showing this. And by the way, going one step further, if we now differentiate u sub theta with respect to theta, we get what? Minus cos(theta) i minus sin(theta) j, which is just minus u sub r. In other words, if you differentiate u sub r with respect to theta once, you get u sub theta, just as we should. If you differentiate a second time, you get minus u sub r. And therefore, it appears that the operation of differentiating with respect to theta rotates u sub r by 90 degrees.

By the way, I should mention that I could have made all of these remarks when we were studying tangential and normal components. In other words-- I just wrote this out here, but this was true with T and N. But the point was, we never had to differentiate N with respect to T to find the acceleration vector a. In other words, recall that the key step in using tangential and normal vectors-- and I'll mention this in a little bit more detail later-- was that the velocity vector was simply ds/dt times the unit tangent vector T. The coefficient of N was 0.

In polar coordinates, notice that v in general will have both a u sub r and a u sub theta component. Therefore, to compute a, I have to differentiate v with respect to t. That means, among other things, I'm going to have to take the derivative of u sub theta with respect to t. By the chain rule, that's going to be the same as taking the derivative of u sub theta with respect to theta times d(theta)/dt. But the important point is, is that someplace along the line in studying kinematics and polar coordinates, I am going to have to differentiate u sub theta with respect to theta.

And just to show you again very, very quickly what I mean by this, all I'm saying is we already know in kinematics that the velocity vector is always tangential to the curve. Notice in this particular diagram, for example, that if you look at u sub r and u sub theta, if you think of a vector whose direction is tangent to the curve at this particular point, that vector will have, in general, both a u sub theta and a u sub r component. In fact, in this particular diagram, I shouldn't say "in general," it will have u sub r and a u sub theta component.

OK. So far so good, but now I want to make one little caution, a caution which is not at all self-evident, at least to me, and which gave me great difficulty myself when I was a student. And that is, my feeling was that u sub r was simply the unit vector in the direction of r. In fact, I said that earlier, that u sub r was the unit vector in the direction of the radius vector r. And this is one of the reasons why even though Professor Thomas in the textbook doesn't make such an issue over this, why I am such a bug on using the phrase "sense" as well as direction. And that is, my claim is that the unit vector u sub r need not be the radius vector R divided by the magnitude of R. It'll have the same direction, but watch what happens with sense.

Instead of talking about this thing abstractly, let me give you a concrete example.

Let's take the curve which in polar coordinates has the equation r equals cos(theta). OK? As you recall, this would be this particular circle here. Now let's take theta to be 120 degrees. If I use our definition for u sub r, which is cos(theta) i plus sin(theta) j, and replace theta by 120 degrees, what I get is, is that u sub r is cos(120 degrees) i plus sin(120 degrees) j. Remember that the cosine of 120 is minus 1/2 and the sine of 120 is plus 1/2 the square root of 3. u sub r turns out to be minus 1/2 i plus 1/2 the square root of 3 j.

On the other hand, my claim is that when theta is 120 degrees, what point are we at the curve? You see, if I take theta to be 120 degrees-- notice that when theta is 120 degrees, r is negative 1/2. And that therefore I'm at the point P0 here. Recall that by definition, r, the radius vector, is measured from the origin to the point. In other words, according to our previous definition, it's this vector, which would be called r. But our definition says that it's this vector, which is u sub r.

And in fact, if you just check the figures that we've obtained over here, notice that u sub r has its i component equal to minus 1/2, its j component being plus 1/2 the square root of 3. Therefore x is negative. y is positive. But if x is negative and y is positive, you're in the second quadrant, not the fourth quadrant. You see? In other words, u sub r is almost the radius vector. In fact, it would have been, if in polar coordinates, little r happened to be positive.

In fact, let me summarize that in a different way. Let's assume that we have a curve, c, whose polar equation is r equals f(theta). Then the idea is this. If f(theta) happens to be at least as big as 0, then u sub r is equal to the radius vector r divided by its magnitude. In other words, u sub r will be in the same direction as the radius vector. And by the way, recall this is just another way of saying r. What we're saying is again, if we had never let little r in polar coordinates be negative, no problem would have occurred.

But we do let little r be negative. So we have to be a mite careful. The careful part comes in where? If r happens to be negative. In which case, u sub r has the opposite sense of the radius vector the capital R, which we saw in the previous example. In other words, in this case, u sub r is minus-- the negative of the radius vector divided by its magnitude. In what cases is that? If r happens to be negative.

The important point to notice, however, that in either case, the radius vector r is equal to the polar coordinate r times u sub r. In other words, if r happens to be positive, these two vectors have the same sense. If r happens to be negative, these two vectors have the opposite sense. In other words, in either case, this expression is always correct. But the important thing to notice is that the sense of u sub r is determined by theta, not by r, not by f(theta). OK?

At any rate, once we have this particular recipe established, we can now go ahead and study motion in the plane. Namely, notice that our radius vector R is now given by the polar coordinate r times u sub r. Or I guess I should say here that I'm assuming that the equation of motion is given by r is some function of theta. That's the r that I'm using in here. At any rate, what is the velocity vector? By definition, it's just the derivative of the radius vector with respect to time. That's just d/dt of r times u sub r.

Now keep in mind, that r and u sub r are both functions of time-- namely, the distance of the particle from the origin as well as the direction of the line of action that joins the particle to the origin will, in general, depend on time. Consequently, I must use the product rule here. I already know that I can use the product rule for vector and or scalar functions and any combination thereof. So I just differentiate this thing with respect to time. I get what? This is dt/dt-- in other words, the derivative, first times the second, which is u sub r, OK?-- plus the first times the derivative of the second, which is the derivative of u sub r with respect to time.

Now keep in mind, again, there is nothing wrong with this recipe. But what I would like is to have the velocity expressed in u sub r and u sub theta components. So far, I have it expressed in terms of u sub r component and a d(u sub r)/dt component. But now, here again is why the chain rule is so important. Keep in mind that I already know that if this expression here had been the derivative of u sub r with respect to theta instead of with respect to t, what would this expression have been?

It would have been u sub theta. We solved that earlier in the lecture.

Well, here's what we do. We say, OK, these aren't the same. But let's cross this out. Let's use the chain rule. And by the chain rule, the derivative of u sub r with respect to t is the same as the derivative of u sub r with respect to theta-- times d(theta)/dt. And rewriting this so that it becomes legible, we have that the velocity vector is dr dt times u sub r plus r d theta dt times u sub theta.

And again notice that as far as u sub r and u sub theta are concerned, even if I did not notice my subtlety-- and by the way, I'm going to leave this for the exercises-- but even if I didn't notice the subtlety that u sub r need not have the same sense as the radius vector r-- notice that if I did not try to draw this thing to scale, I can still get the-- I shouldn't have said the scale-- but if I didn't try to graph the answer here given r as a function of theta and theta is a function of t, notice that dr/dt and r d(theta)/dt are well-defined arithmetically with no possible chance of making a geometrical mistake. The place that you can make the biggest mistake is if you automatically think that u sub r must have the same sense as capital R. But as I say, we'll leave any additional discussion of that for the exercises.

I should also point out that when I first learned this recipe myself, it turned out that we were ahead of-- the physics class was ahead of the math class. And we learned this thing in the physics class almost intuitively. In other words, as a geometric aside, notice that if I'm given the curve and say s indicates the direction of increasing arc length here, what I could do is think of a little differential region here.

Namely, here's my radius vector r, and here's my velocity vector in the direction of u sub r. Then I take a little increment of angle d(theta), and I now think of v sub theta, which is at right angles to v sub r, as being tangent to the circle I would have obtained if I had imagined that this particular point-- the particle was being viewed with respect to the circle rather than to the curve itself.

To make a long story short, what I'm driving at is that physically, it's very easy to justify that the magnitude of the u sub r component of the velocity is the magnitude of dr/dt-- how fast the radius vector is changing instantaneously. On the other hand, notice that for the u sub theta component, this arc length is given in differential form by r d(theta). If I divide the arc length by the time, which is dt, I get r d(theta) divided by dt, which leads to r d(theta)/dt, which is the same expression that we got analytically. But the point that I want to bring out here is that our derivation required no geometrical physical insight.

And the reason that I want to bring this out is I followed this argument fine in my elementary physics course. The place I got hung up is that the instructor then went into a fantastic hand-waving type of demonstration and showed us how the acceleration looked in terms of u sub r and u sub theta components.

And actually, that was a blessing for me, because it was that day that I decided to become a math major rather than a physics major, which was a blessing both for me and society, I guess. But the thing that I want to show you is that the beauty of our mathematical approach is that we can now obtain a, the acceleration vector, from the velocity vector without having to know any great physical insight. In fact, we have to know no physical insight to do this.

Namely, by definition, a is the derivative of the velocity vector with respect to time. We also have seen that the velocity vector is the expression that I have here in brackets. So I have to differentiate that with respect to time. Notice that this is the sum of two terms, one of which is a product of two factors, and the other of which is a product of three factors. And by the way, among other things to review here, this is the first time in this course that we have actually had to use the product rule for a function consisting of three variable factors.

Even though we discussed this in part one, here is a case where in a real-life situation, what we need is the derivative rule for a product of three functions. At any rate, this is done in great detail in the text. I do it more in the notes. So I'm just going to hit the highlights here.

The point is I now differentiate this sum term by term. Namely, to differentiate this, I take the derivative of the first term times the second plus the first term times the derivative of the second. Now to differentiate this term, I have to differentiate a product of three factors. And recall-- and by the way, as I told you in part one-- whenever I say "recall," that means if you don't recall, it's my polite way of saying, look it up. But to differentiate a product of three functions, we write the product down three times, and each time differentiate a different factor.

For example, the first time we'll differentiate r with respect to t, which is dr/dt. The second time, we'll differentiate d(theta)/dt with respect to t, which is the second derivative of theta with respect to t. And the third time, we'll differentiate u sub theta with respect to t, which is the derivative of u sub theta with respect to t. And summarizing that, what do I have here? I have dr/dt d(theta)/dt times u sub theta plus r d2 (theta)/dt^2 times u sub theta plus r d(theta)/dt times the derivative of u sub theta with respect to t.

Now the point is that if I look at these five terms, some of them are in nice form. Namely, here's a u sub r term. Here's a u sub theta term. Here's a u sub theta term. But these terms are sort of mongrelized. Namely, what I have to do here is utilize the chain rule. And remember that the derivative of u sub r with respect to theta would have been u sub theta. The derivative of u sub theta with respect to theta would have been minus u sub r. So by the chain rule, you see what I'm going to do is, I'll replace each of these terms by their chain rule expression. Then I'll collect terms. And the reason I'm going over this fairly rapidly is that it is a problem of sheer mechanics.

But the punch line is that if I now collect my terms, the acceleration vector has as its u sub r component d2(r)/dt^2 minus r [d(theta)/dt]^2. And the u sub theta component is r d2(theta)/dt^2 plus 2 dr/dt d(theta)/dt. And the beautiful part, from my point of view about all of this, is if I don't understand any physics at all, this particular result is valid. It's mathematically self-contained. Now certainly there is no harm in a man who understands physics well enough to say, look at it. This is the acceleration in the radius direction alone. And this is some kind of a correction factor proportional to the square of the angular velocity, see, d(theta)/dt being angular velocity and what have you. And go through this particular thing.

I'm saying fine, if you can do that. But notice the beauty. This complicated expression gives us the acceleration vector in terms of u sub r and u sub theta with no hand waving. It's mathematically self-contained. And by the way, keep in mind that one of the reasons that we study polar coordinate motion is the fact that, in many cases, we are going to be dealing with a central force field. And the interesting thing is that in a central-- I'll just abbreviate this-- in a central force situation, this expression is 0.

See central force means what? That the force is in the radial direction. That means all of the acceleration-- if you're using Newtonian physics, F equals ma-- all the acceleration is in the direction of u sub r. Therefore, the component in the direction of u sub theta must be 0.

So this fairly complicated expression-- r d2(theta)/dt^2 plus 2 dr/dt d(theta)/dt equals 0 becomes the fundamental equation for central force field motion. But again, we'll talk about that more in the exercises. What I wanted to do now was to make what I think is a very important summary. And that is that when we're studying the position vector R, and the velocity vector v, and the acceleration vector a, that none of these depend on the coordinate system. It's only their components that do. In other words, at the expense of having a fairly jumbled figure which I rationalize here-- it is small, but I think it is clear from context.

What I'm saying is, let's suppose I have a curve c, and some point P0 on this curve c. I can draw in the pair of orthogonal vectors i and j in the plane. I can draw in the pair of orthogonal vectors u sub r-- "orthogonal" means perpendicular, if we haven't said that before-- u sub r and u sub theta. I can draw those in. And I can draw in T and N. Now all I know is that if I have the velocity vector v, it must be tangential to the curve. Hopefully by this time, we realize that the acceleration vector has no such restriction. Let's just draw in a v and an a, call these the velocity vectors and the acceleration vectors.

The point is that v and a are determined by the motion-- not by the coordinate system. In other words, when we're talking about the velocity of this particle at the point P0, its velocity is the same, no matter what coordinate system we're using. It just happens that if we're dealing with Cartesian coordinates, the velocity vector is dx/dt i plus dy/dt j. In other words, it's this particular combination of i and j.

If we're using T and N components, the particular combination of T and N is what? ds/dt times the unit tangent vector plus 0 N. And if we happen to be using polar coordinates, the expression is dr/dt * u sub r plus r * d(theta)/dt * u sub theta. But let me circle these, because it's the same velocity in each case. We use this when horizontal and vertical motion are important. We use this when we're interested in motion along the curve. And we use this primarily in central force fields. But it makes no difference. It's the same velocity vector.

And in a similar way, it's also the same acceleration vector, whichever system you happen to use. Namely, if we use Cartesian coordinates, the acceleration vector is the second derivative of x with respect to t times i plus the second derivative of y with respect to t times j. That same vector, if we express it in T and N components, is d2(s)/dt^2 times T, plus kappa, the curvature number, times (ds/dt)^2 times N. And if we express it in terms of polar coordinates, as we just saw earlier in our lecture, this is the expression that we get.

In other words then, this summarizes our study of motion in the plane using either Cartesian or polar or tangential and normal components. You see, the point is that we pick whichever coordinate system happens to be of the greatest interest to us, the greatest value to us. We make the coordinate system our slave, rather than the other way around, and tackle the problem from that particular point of view.

At any rate, that ends this phase of our particular course. And in the next phase of our course, we get to probably what is the most fundamental building block of the entire course. We get to that particular topic which by and large most courses in functions of several variables begin with. But we'll talk about that more the next time we meet. And until that time, goodbye.

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