PROFESSOR: Hi. Our lesson today involves a rather subtle difference between a curve and a coordinate system. In other words, given a curve, that curve has a certain shape, has a certain position, independently of where our coordinate axes are if we're using Cartesian coordinates, or whether we're using other coordinate systems, or what have you, but the equation of the curve may very well depend on what coordinate system we're using.

The thing that we would like to do in this particular lecture is to hit a very important highlight that comes up-- well, you can motivate it from a purely mathematical point of view, but physically there's an even more natural interpretation. And basically what the thing hinges on is this: If you're given a curve in space-- we'll start with a curve in the plane, but it applies to curves in space as well. Given that curve in the plane, does that curve have certain properties regardless of whether you know what your coordinate system is or not?

Or in still other words, can you measure the shape of the curve, can you measure the speed along the curve as a particle traverses it, if you had never heard of the x- and y-coordinate system? And what this leads to is a new system of vectors by which we study motion in space called tangential and normal vectors when we're dealing in the plane. And there's a third number called the binormal vector, which we'll talk about later when we deal in three-dimensional space.

Any rate, just for brevity, I call this lecture "Tangential and Normal Vectors". And the idea is something like this. We're given a curve C. Now, given this particular curve C, it happens that we have a Cartesian coordinate system here. And it also happens that we prefer, at least as we've done things in the past, we write everything in terms of i and j components.

Notice that if a person were restricted to his universe being the curve C, i and j have no basic meaning to him. What does have a basic meaning to him, as he's moving along this curve, I would imagine, would be what?-- what is his motion tangential to the curve? In other words, if you want to look at this from a calculus point of view, if this is a smooth curve, in a sufficiently small neighborhood of this point, you cannot distinguish between the curve and the tangent line. And consequently, one could interpret that at a given instance the motion was always along the straight line tangential to the curve.

What this leads to is the notion of inventing what we call a unit tangent vector, which I'll call T. And what is that tangent vector? It's not a constant, mind you. It shifts with position as you move along the curve. What is constant is its magnitude. It has constant magnitude 1.

I guess what I'm trying to say, in sort of a surrealistic or metamathematical way, is that T plays to a person who's living on the curve C the same role that i plays to a person living in our ordinary space, but somehow or other he sees T as a constant vector as he moves along the curve If he visualizes the curve as being a straight line. He always sees it tangential to his motion.

Now, in the same way that j was a 90-degree positive rotation of i, one would like to mimic the i and j Cartesian coordinate system by inventing another unit vector, which is again what?-- a positive 90-degree rotation of T. And we'll call that vector N.

So that now we have a new system of coordinates, T and N, new system of variables, or vectors, whereby we can now study motion along a curve in a very natural way. In other words, we talk about the unit tangent direction and the unit normal direction. And we have this thing now established.

What we would like to do is to see what happens in our study of kinematics, motion in the plane, motion in space, if we work in terms of tangential and normal components now rather than in terms of i and j components. The first thing that we'd probably like to do is figure out how in the world do you compute T?

Well, for example, let's take a particular application. Let's take the example that we were dealing with last time where we had the radius vector R-- in other the scalar function t, where t denoted time-- and we were dealing what?-- motion in space where the radius vector R was a function of t. And remember what we showed last time?

We showed that the dR/dt, the velocity vector, has its what? Its direction is always tangent to the curve. We proved that last time. Well, as long as the dR/dt is always tangent to the curve, what prevents it from being a unit tangent vector?

Well, nothing prevents it from being a tangent vector, because it's already tangent to the curve. All that could go wrong is that the magnitude of the dR/dt is not 1. Well, look at that again-- a very simple point to fix up-- namely, if the magnitude of the dR/dt is not 1, suppose we divide that vector by its magnitude? We have already seen that given any non-zero vector, if you divide that vector by its magnitude, you get the unit vector in the same direction as the vector that you started with.

In other words, if I take the vector dR/dt, which is already tangential to the curve at the given point, and I divide that by the magnitude of the dR/dt, then I automatically get the unit tangent vector. Is that clear? Well, since nobody says no, I assume it is clear.

Look it. We also showed last time that the magnitude of dR/dt is speed along the curve. Speed along the curve happens to be called ds/dt. So, again, another name for the unit tangent vector is dR/dt divided by ds/dt.

By the chain rule, we can cancel dt. And by the way, notice the chain rule applies for vector functions like this, the same as it did in part one of our course, by virtue of what we showed in the last unit-- namely, that every formula for derivatives that was true for scalar functions also happens to be true for what?-- vector functions of a scalar variable. At any rate, notice then by the chain rule, another way of saying T is that it's the derivative of a position vector R with respect to the arc length s.

I'd like to make one comment on this. It's important enough so that I will also make this comment in the notes as well when we're doing the exercises. The point is there are many textbooks that will define T by saying it's dR/ds. Now, 999 times out of 1,000-- in fact, 999 times out of 998 even-- you will never be given R as a function of arc length in the real world. In the real world, R is a function of some parameter, usually time.

And the trouble that happens is if you try to use this definition, you find yourself trying to convert things into s, and this makes sort of a mess for you. The thing I would like to show you-- and, by the way, this does not depend on t standing for time. If t is any variable-- and we show this in the notes in the exercises again-- if t is any scalar, if you differentiate R with respect to that scalar, and divide that result by the magnitude of this vector, you wind up with the unit tangent vector.

In other words, in a real life problem, do not worry about converting R into a function of s. Simply differentiate R as it stands with respect to the given variable, divide by the magnitude of the derivative, and, presto, you have the unit tangent vector.

Of course, you may ask, if it's so simple to do what I just said, why is it that every book defines it this way? The answer is rather interesting, and that is, we have just mentioned that we would like to believe that the shape of a curve depends only on the curve itself, not on how we parameterize it. The beauty of this particular definition simply says the natural parameter is arc length-- namely, arc length doesn't depend on any coordinate system.

Given the curve, start at any point you want, and you can measure the arc length. So s is a very natural parameter that does not depend on the coordinate system. In other words, by defining the unit tangent vector to be dR/ds, you have a beautiful philosophically pure mathematical definition, because you have a definition which does not depend on any coordinate system or any unnatural parameter. But in practice, this is the way we compute the unit tangent vector.

The question that comes up is how do you find the vector N? And I'm going to show you the traditional way of doing this before I jazz it up with a more modern approach. Let's look at T over here. Let's call phi the angle that the unit tangent vector makes with the curve here. Notice that in terms of this diagram, since the unit tangent vector has magnitude 1, the i component of it will be cosine phi and the j component will be sine phi.

In other words, T is equal to cos(phi) i plus sin(phi) j. Let's just differentiate. See, T is a function of phi here. Let's just take the derivative of T with respect to phi, and we get right away-- Remember, now, we're getting the mileage out of this basic definition of derivative. That hasn't changed since last time. We just differentiate term by term here.

We get minus sin(phi) i plus cos(phi) j. Right away we observe that dT/d(phi) is still a unit vector. You see it's components are minus sin(phi) and cos(phi), so its magnitude is still 1. And its slope is cos(phi) over minus sin(phi) , which is the negative reciprocal of the slope here.

In other words, what this shows us through the traditional approach is that whatever vector the dT/d(phi) is, it's a unit vector perpendicular to T. By the way, what that tells us right away is that dT d phi must be either plus N or minus N before we go any further. Why? Because we already saw that positive N, the vector that we called N, was a positive 90-degree rotation of T. If we only knew that dT d phi was a positive 90-degree rotation rather than a negative 90-degree rotation, we'd be home free.

And, again, the beauty of trigonometry, in the non-surveyor's sense of the word, analytically is this-- that sort of having a premonition of what we'd like to be true, we simply verify the trigonometric identities that the cosine of phi plus 90 degrees is minus sine phi, and the sine of phi plus 90 degrees is cosine phi, so that dT d phi is what? It's [cos(phi) + 90 degrees] i, + [sin(phi) + 90 degrees] j.

And if we now compare this with this, we notice that we have exactly the same expression, except that the angle has been increased by a positive 90 degrees. In other words, dT d phi is a positive 90-degree rotation of T. Consequently, dT d phi is the vector that was called N. OK? That's dT d phi.

Now, let's go back to our kinematics. We have T and N now. Let's talk about our velocity vector v, where R is still some function of time. By definition, v is dR/dt. That isn't going to change. V was dR/dt last time. It's going to be dR/dt this time. It's going to be dR/dt whenever we want to use it.

The only difference is that instead of expressing this in terms of i and j, we now want to express it in terms of T and N. And notice that since v has as its direction the direction of the tangent line, and as its magnitude ds/dt-- we saw that last time-- notice that in terms of T, v is just a scalar multiple of the unit tangent vector T. And what scalar multiple is it? It's ds/dt. All right. All that says is what? That v is the vector in the direction of T whose magnitude is ds/dt, which is speed along the curve.

I now want to find a. a is acceleration. It's the same acceleration that I was talking about before. It's dv/dt. The only thing that's going to change now is I am not going to change the acceleration vector. I am going to change how it looks, because now I'm going to try to find it in terms of T and N components.

So what do I do here? Look at this expression for v. ds/dt is speed along the curve. That changes from time to time, in general. The unit tangent vector T is also a variable function of T, unless T happens to be a straight line through the origin-- namely, notice that the unit tangent vector, even though it always has unit length, changes its direction as we move along the curve.

So in other words, both of these factors are functions of T. Consequently, to differentiate this with respect to T, we must use the product rule. And the fact that all of our differentiation formulas are true for vector and scalar combinations as well as the scalars, I now use the regular product rule-- namely, it's the derivative of the first factor times the second, plus the first factor times the derivative of the second.

And I now have a expressed in terms of two vectors, T and the derivative of the unit vector T with respect to time t. And somehow or other, all that's wrong here is I would like to get this thing expressed in terms of N. You see, when I'm working with

T and N components, I want my answer to depend on T and N.

Now, here's where I become very shrewd. And, by the way, this is an insight that, if you're going to pick it up at all, you're either born with it or you pick it up with experience. But you just have to work with these things. There are tricks, if you want. I guess the novice calls them "tricks." The expert calls it "keen analytical insight,"

The point is I want to get an N out of this thing. I already know how to express N in terms of dT d phi. In fact, N is dT d phi. So what I do is I take dT/dt and say, let me write it so I can get a dT d phi factor out of this. I also want everything to be in terms of arc length so I can ultimately have an answer which doesn't depend on a coordinate system.

So what I really do is I use the chain rule to express this factor in terms of what?-- these three factors. You see, according to the chain rule, the d(phi) here cancels the d(phi) here, the ds here cancels the ds here, and all I'm saying is that dT/dt can be written as dT d phi times d(phi)/ds times ds/dt.

Now we're in very good shape, you see. dT/d(phi) we already know is N. And ds/dt we already know can go with this. And the only new thing that we have to worry about is what is d(phi)/ds.

See, again what so often happens, you apply logic, you get to a certain inescapable conclusion, and then if you have brand new terms, you have a choice between doing what?-- saying I don't like the new terms, I'm going to throw them away, or saying I like the result, I had better interpret what this new term means.

All I want to show you is, is that the d(phi)/ds has a very natural interpretation-- namely, what is d(phi)/ds? Let me tell you what it's called first. It's usually denoted by the Greek letter kappa, and it's called curvature. Its reciprocal, 1 over kappa, is usually denoted by rho, and it's called the radius of curvature. And I give you plenty of drill on the stuff. I just want to mention what these words are now.

In fact, part of the drill is that d(phi)/ds is not a very convenient thing to compute.

Usually you're given y as some function of x. And many of the drill problems that we have in calculus ask questions like, how do you express d(phi)/ds in terms of y, dy, dx, et cetera? Those are problems that we can get into in more detail as we do the exercises. But all I wanted to do in this lecture is to show you why d(phi)/ds is such a natural thing.

Look at the curve s. As you move along this curve, notice that the change in phi with respect to s in a way tells you how the shape of the curve is changing. In other words, d(phi)/ds measures how-- what could be a more infactual word than curvature? See, as phi changes as you move along the curve, that's measuring how your curvature is changing.

As an extreme case, notice if the curve where a straight line, d(phi)/ds would be 0 because phi would be a constant. d(phi)/ds would be 0, and the curvature of a straight line should be 0. At any rate, one defines d(phi)/ds to be the curvature.

And, in fact, to play it safely, since s can have two different senses-- in other words, why couldn't somebody else say why don't you go this way along the curve, I don't know what the sense is? Usually what one does to play it safe is we put the absolute value signs around d(phi)/ds and just call the magnitude the curvature.

And the punch line is that once I call d(phi)/ds the curvature, what I wind up with is what? Just substituting in here now, the acceleration vector is d^2s/dt^2 * T + kappa * (ds/dt)^2 * N.

By the way, this entire recipe is derived in the text. I have you do it again as a learning exercise because I want you to practice with this. And I make additional comments on this in the notes. The textbook makes additional comments on it in the text, which is where you'd expect it to be.

And all I want you to see is that this is the same acceleration vector that we were talking about in the last lecture, only now we're talking about how it looks in terms of tangential and normal components instead of i and j components. OK? And what's so good about tangential and normal? What's so good about tangential and normal is that you're now moving along the curve rather than with respect to some isolated x- and y-coordinate system.

By the way, in the last unit we showed a rather interesting result, that if T was any vector function of the scalar x, and the magnitude of T was a constant, then dT/dx was perpendicular to T. That was an exercise in the last unit.

Now, the interesting point is that the modern approach to calculus says this-- why should we single out the xy-plane? After all, you can be given a particle moving through space, or you can be using a different coordinate system. The natural parameter is arc length. Consequently, the modern approach never talks about the angle phi or anything like this. The modern approach simply says this-- define the unit tangent vector as before.

Because the magnitude of T is a constant, since dT/ds is already perpendicular to T, let's define a second vector N to be dT/ds divided by its magnitude. Again, the same old trick. What have we done here? We have simply taken dT/ds, which we know is perpendicular to T-- any scalar multiple of the dT/ds will still be perpendicular to T-- but now this is what? It's a unit vector because we've divided this vector by its magnitude. Therefore, N is a unit vector.

And where is it? It's perpendicular to T. If we now cross-multiply, notice that dT/ds is equal to the magnitude of dT/ds * N. See, just cross-multiply. I now claim that the magnitude of dT/ds is just d(phi)/ds. Now, why is that?

I guess I should have planned this better, but let me come back to the previous board over here. Notice that since T is a constant vector, since T is a constant vector, how does it change? It can't change in magnitude because it has constant magnitude. Therefore, its only change must be due to direction alone.

But the direction of T is measured by phi. In other words, if dT/ds is changing at all-- in other words, if this is a variable, it must be changing only in direction, because the magnitude of T is always 1. In other words, T cannot change in magnitude. It must therefore change only in direction. In other words, the magnitude of dT/ds is the same as the magnitude of d(phi)/ds.

Recall that we just defined the magnitude of d(phi)/ds to be kappa, and therefore dT/ds is kappa N, the same way as in the traditional approach. The beauty of this approach is that we're no longer restricted to the xy-plane. We're not restricted to any plane. We're not restricted to any coordinate system. We can now, in fact, generalize this to go out into three dimensions.

And, in fact, some of you will probably have enough difficulty with what we've done so far that you won't want to go into three dimensions. What I've done is I have made up an optional unit that follows this one, a unit which has no lecture. It simply has a batch of exercises for those who have mastered the material in this unit and would like to see what happens in three-dimensional space. And, after all, when you deal with real life orbit-type problems and things like this, notice that you do need the geometry of three-dimensional space for this.

If you so desire, you can then do the optional unit. That's why it's called optional. You can skip it if you want. There's no loss of continuity if you should skip it, but in that optional unit I devote computational drill to what happens when our curve happens to be a three-dimensional space curve-- in other words, a curve that winds through space.

Notice, by the way, that in the same way as before, I can write $R(t) = x(t)\, i + y(t)\, j + z(t)\, k$, where that is the vector form of the curve in Cartesian coordinates given in scalar form by the three equations, x is some function of t, y is some function of t, z is some function of t. Again, my claim is that if I just take the dR/ds, I still have the unit tangent vector.

And to see that, just notice what we're saying here. This is a space curve now. I've taken a small segment of it. Here's R, here's R plus delta R, so this difference is delta R. Look what happens as you take delta R and divide it by delta s.

First of all, the direction of delta R does become the direction of the tangent line as delta s approaches 0. So certainly we can believe that the direction of dR/ds is going to be the tangential direction. Also, if we invoke the result of geometry that we talked about in part one of our course, when we talk about sin(theta) over theta as theta approaches 0, the length of the arc is approximately the length of the chord for small segments. So, therefore, delta R over delta s in magnitude approaches 1.

In other words, dR/ds is still the unit tangent vector, the same as before. Again from a computational point of view, to find dR/ds you do not rewrite this in terms of s. You simply do what? You take dR/dt, the same as before, divide by its magnitude, and you automatically have dR/ds.

Similarly, once T is given, to find N you simply differentiate T with respect to s and divide it by its magnitude. And again notice, even though I've written it again, if you look back to the first third of our blackboard, this is the same definition for N as before, because our original definition did not specify that the curve had to be in a particular plane. See? The same general definition.

So I now have T and N. Now what do T and N do? T and N determine a plane. It's a plane which we call the osculating plane to the curve. That's the plane which sort of touches the curve at that particular moment. Remember, this curve is winding through space.

And, again, this is done in more detail in the notes. Not quite as elegantly as going like this, but the idea is you have this plane that's shifting along with the curve. The only thing that's missing that causes new complications when you deal in three-dimensional space is that in the same way that T and N take the place of i and j in two-space, you need something that takes the place of k in three-dimensional space.

What we do is-- again look at the structure-- we mimic how k is related to i and j and invent a new vector called the binormal, hence abbreviated B, which is simply defined to be T cross N, the vector that you get by rotating the unit vector T into the unit vector N through the smaller-- namely, the positive 90-degree-- angle.

Now what is B? B is perpendicular to both T and N. In other words, B is a vector which is perpendicular to the osculating plane. Since B always has a constant magnitude, because T and N are always perpendicular-- see, B always has magnitude 1-- the point is that dB/ds, the magnitude of dB/ds, measures the twist of the curve.

In other words, here's this tangent plane following a point, a particle, along the curve. And what you're saying is how fast the direction of that tangent plane is changing is measured by dB/ds. That is called the "twist." I call it the "twist." I put it in quotation marks because nobody else calls it the "twist." The formal name is the "torsion." See? This is called the torsion. I talk about that more in the notes.

The point being, by the way, that notice that if dB/ds happens to be 0-- in other words, if B happens to be constant-- then the curve lies in the plane. We certainly recognize that. For example, if the curve happens to be in the xy-plane, notice that if T and N were i and j, i cross j would just be k, B would then be a constant. The derivative of a constant with respect to any variable is 0, and, therefore, when the curve does lie in the plane, the torsion, the twist, is 0.

In other words, the torsion does for three-dimensional space what the curvature in a sense does for two-dimensional space. At any rate, our main aim is to get you familiar with some vector calculus, and if in doing this we can also help you learn how to use this stuff in some physical applications, that happens to be frosting on the cake.

Next time, we are going to talk about the fact that we still have to invent additional coordinate systems, that i and j isn't enough, T and N isn't enough. Next time we're going to show why we need polar coordinates, but we'll worry about that next time. Until next time then, goodbye.

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