PROFESSOR: Hi. Our lesson today is called the 'Inverse Logarithm'. And what it will do, among other things, is give us a very nice chance to revisit inverse functions in general, only now applied specifically to the natural logarithm function that we talked about last time. So I call today's lesson, as I say, 'Inverse Logarithms'.

And to see what's coming up here, simply recall that last time, we invented, so to speak, a new function called the 'natural log of x'. This had nothing to do with exponents. It was what? This was the function whose derivative with respect to 'x' was '1/x' and passed through the point (1 , 0). That uniquely defined this particular function.

The point being what? That this particular function, since the domain is for positive 'x', '1/x' is positive. The curve is always rising, which means that the function itself must be one to one. And because the function is one to one, it means that the inverse function exists. This is no different from any other example of forming f inverse, given a one to one function called 'f'.

So what we do is now, and remember how you invert this function. If you want to do this thing in slow motion, you first rotate this through 90 degrees, then flip it over. If you want to do it faster, the inverse graph of this is the reflection of this curve with respect to the line 'y' equals 'x'.

In any event, if we do this, we find that the graph of the function 'y' equals 'inverse log x' is this particular curve. And notice the correspondence. If this is the point (1 , 0) on the log curve, the inverse point is (0 , 1). In other words, the inverse log of 0 is 1.

OK. Now, the idea is something like this. Just to have a review, again, of how-- see, the punchline I want to make for today's lesson, right at the beginning, is that the nice thing about studying inverse functions is that once you know the function of which you're taking the inverse, you automatically get all the mileage you want out of the inverse function. Well, by way of illustration, if the function natural log had the usual logarithmic property, we would suspect that its inverse should have the usual exponential property.

Now what is the usual exponential property? The exponential function is characterized by what? That if you take 'f of 'x1 plus x2'', that's 'f of x1' times 'f of x2'. So you think of it in terms of the usual exponential notation. If you have, say, 10 raised to the 'x1 plus x2' power, that's the same as 10 to the 'x1' power times 10 to the 'x2' power. In other words, if our ln function is genuinely a logarithmic function, we would expect the inverse of that function to be a genuine exponential function.

And just to work with inverses again, let's see if this is indeed true. Let's see if it's really true that the 'inverse log of 'x1 plus x2'' is the 'inverse log of x1' times the 'inverse log of x2'. You see, the whole idea is what? Let's give these things names. Let 'y1' be the 'inverse log of x1' and 'y2' be the 'inverse log of x2'. Now, since we've already studied the natural log function, the most natural thing to do here is to invert these, namely 'y1' equals the inverse 'natural log of x1' is the same as saying 'x1' is the 'natural log of y1'. And similarly, this is the same as saying that 'x2' is the 'natural log of y2'. And now we add equals to equals, and we get that 'x1 plus x2' is 'natural log y1' plus 'natural log y2'.

But allegedly, we understand the properties of the natural log, because if we didn't understand those, it would be kind of futile to be studying the inverse function. What do we know about the natural log? We know that the 'natural log of 'y1 times y2', by definition of a logarithmic function, is 'log y1' plus 'log y2'. In other words, 'natural log y1' plus 'natural log y2', by the property of being logarithmic, is just the 'natural log of 'y1 times y2''.

And now, taking this relationship and inverting it to say this is the same as saying what? The 'inverse natural log of 'x1 plus x2'' is equal to 'y1' times 'y2'. And if we now observe that 'y1' was the 'inverse natural log of x1', and 'y2' is the 'inverse natural log of x2', we have the result that we claimed we would have at the beginning here.

Now again, here's one example where it's not the result that I'm not interested in. But I'm interested in showing, again, what we meant when we said that once we learn a new concept, we can bring to it all of our old knowledge. You see, all of these things come from our basic definition. Well, let's carry on one step further. You see, this was an arithmetic property. Let's see if we can get some calculus properties about the inverse natural log based on our knowledge of the natural log itself.

Well, for example, in a course of this type, the most natural thing to do, I guess, is given any function, we would always like to be able to talk about its derivative, provided, of course, the function is differentiable. So for example, a typical question that one might ask is find 'dy dx', if 'y' is 'inverse natural log x'. Again, this works the same way as it did for the inverse trig functions, for the inverse of everything that we've done so far in this course.

We start with this particular relationship and right away translate it into that which we're more familiar with, namely we translate 'y' equals the 'inverse natural log of x' into 'x' equals 'natural log y'. Now you see again, since I know how to differentiate 'log y' with respect to 'y', the derivative of 'natural log y' with respect to 'y' was by definition '1/y'. From this relationship here, I can find that 'dx dy' is '1/y'. And knowing that 'dx dy' is '1/y', using my inverse function general theory that tells me that to find the derivative of 'y' with respect to 'x', all I have to do is invert the derivative of 'x' with respect to 'y', I wind up with the fact that 'dy dx' is equal to 'y' itself.

In other words, the 'inverse log x' function is characterized by the fact that it's its own derivative. In other words, it's a non-destructible type of function with respect to differentiation. This is a rather powerful property. You see what this thing says? It says that the derivative of 'y' with respect to 'x' is 'y' itself.

By the way, just as an aside, we can do the converse of this particular problem. You see, we started with 'y' equals 'inverse log x' and showed that this particular equation was satisfied. Notice that if we start with this particular equation, starting with 'dy dx' equals 'y', separate the variables, and lo and behold, we wind up with 'dy over y' equals 'dx'. And without carrying out the details, I leave this to you, because it is straightforward.

Notice that as we look at the left hand side here, we hopefully will think immediately of the natural logarithm. Namely, what do we want here? The function whose derivative with respect to 'y' is '1/y'.

Again, you see what I'm saying is notice that if we had started with this particular differential equation, we could have shown that a logarithm, and hence solving for 'y' explicitly, would have led to the inverse logarithm also. But at any rate, notice then that we can find the derivative of the inverse log just by knowing how to find the derivative of the natural log itself, which is as we expect things should be. Again, and this is a notational thing, we do not use the language in general "inverse log x."

In other words, it's quite conventional to talk about 'inverse sine x'. You may notice that sometimes in the text, sometimes in my lectures I may have used the word 'arcsin x'. But the general notation is sine to the minus 1. However, when it comes to the natural logarithm function, the inverse is usually not written as 'inverse natural log x'. It's usually abbreviated, and let me say this, by the symbol 'e to the x'.

Now the reason I say by the symbol is this. If I wanted to, I can say, look, If I've never heard of exponents before, let this symbol be an abbreviation for 'inverse log x'. Does 'inverse log x' exists? Yes, I even drew the graph of it at the beginning of this particular lecture.

However, again, if you are tempted to use exponents, the notation 'e to the x' is in keeping in line with the idea of our previous lecture. I say what? This matches the identification of 'natural log x' with the 'log of x' to the base 'e'. Remember, in our previous lecture, we mentioned that this particular function exists without having to talk about a base. But if you wanted to identify it with a traditional logarithm, the base you would have to pick is the base 'e', where we showed that 'e' was some number between 2 and 4.

In other words, again, it's just like our 'dy dx' versus 'dy' divided by 'dx' and other symbols that we invented this way. Namely, if I look at 'e to the x' as being the inverse of the natural log, or whether I look at it as being 'e' raised to the 'x' power, where 'e' is that number that's someplace between 2 and 4, I don't get into any trouble either way because of the natural identification of choosing e to be the base of my system. But I just mention that in passing.

Now you see, the rest of today's lecture will go fairly quickly. And the reason that it will go fairly quickly is that once we've established what the function is that we're talking about, every other property of that function is going to follow from the principles that we've already learned. For example, in terms of our new notation, since the derivative of 'e to the x' with respect to 'x' is 'e to the x' itself, remember, 'e to the x' now can be viewed as what? 'e to the x' power, or, to be on safer grounds, if you want to be consistent, it's just an abbreviation for 'inverse natural log x'.

But at any rate, if 'u' is now any differentiable function of 'x', notice that to find the derivative of 'e to the u' with respect to 'x', by the chain rule, I can say what? It's the derivative of 'e to the u' with respect to 'u' times 'du dx'. We've just shown that the derivative of 'e to the u' with respect to 'u' is 'e' to the 'u' again. And therefore, the derivative of 'e to the u' with respect to 'x' is 'e to the u' times 'du dx'. By the way, this again leads to another interesting thing that makes integrals tough to handle, in a way. We've mentioned this before, but here's another nice, natural environment to bring this problem up in again.

When we discussed the second fundamental theorem of integral calculus to show how one actually would have to understand areas to be able to find a function whose derivative was 'e to the 'minus x squared', we mentioned, we didn't prove it, we mentioned that there was no familiar function whose derivative with respect to 'x' was 'e to the 'minus x squared'. So this would be a very difficult problem to handle. Now, let's look at this one instead. Let's look at integral '2x 'e to the minus x squared' dx'.

To the untrained eye, it would appear that this is even messier than this. In other words, this is just 'e to the 'minus x squared''. This one has 'e to the 'minus x squared'' with a '2x' dangling in front of it, and that looks even tougher. But notice that if we look at our exponent, 'minus x squared', the derivative, or the differential of 'minus x squared' is what? It's 'minus '2x dx''. In other words, the '2x dx' is precisely what you need to reduce this to the form integral ''e to the u' du'.

In other words, working this thing out in more specific detail, if I let 'u' equal 'minus x squared', 'du' becomes 'minus '2x dx'', and therefore integral '2x 'e to the 'minus x squared'' dx' just becomes what? Well, the 'e to the 'minus x squared'' just becomes 'e to the minus u'. And '2x dx' is just 'minus du'. OK?

I'm sorry, 'u' is 'minus x squared'. 'u' is 'minus x squared'. So 'minus x squared' is 'u'. So this becomes 'minus e to the 'u du''. And that, of course, is just minus. See, the minus will come outside. The integral of 'e to the u' with respect to 'u' is just 'e to the u'. And since 'u' is equal to 'minus x squared', all we're saying is that if you differentiate minus 'e' to the 'minus x squared', you wind up with what? '2x 'e to the 'minus x squared'', the reason being that you must multiply this by the derivative of the exponent with respect to 'x'.

The root of the exponent is 'minus 2x'. 'Minus 2x' times minus 1 is '2x'. If you want to see this in more concise differential notation, you see, what we're saying is that the integral of 'e' to some power with respect to that same power is 'e' to that power plus a constant. In other words, the function that one would have to integrate 'e to the 'minus x squared'' with respect to to wind up with 'e to the 'minus x squared'' would be 'minus x squared' itself.

And you see, now, in the next step, since this is a true statement, the differential of 'minus x squared' is minus '2x dx'. And now you see, factoring out the minus sign and multiplying through by minus 1, we arrive at the same result that we wound up with before. But again, this is the kind of material that we can drill on extensively in the exercises in this particular unit. The technique is what? That the basic building block of the so-called exponential function, the inverse natural log, is that when you differentiate it, you essentially do not destroy it. In other words, the derivative of 'e to the u' with respect to 'x' is 'e to the u' times 'du dx'.

And I thought that in closing today's lesson, I might as well show you a very powerful application of this particular result in a non-obvious situation. And it's something that we call second-order differential equations. I've picked out here a differential equation.

Let me just show you what I have in mind over here. Suppose I tell you that 'y' is a twice-differentiable function of 'x' that satisfies the following identity, that the second derivative of 'y' with respect to 'x' minus 5 times the first derivative of 'y' with respect to 'x' plus 6 times 'y' is identically 0. And the question is, if this equation is to be obeyed, what must 'y' be? And a very interesting technique for solving this kind of a problem, you see, this is called a second-order differential equation because the highest derivative that appears is the second derivative.

OK, a rather powerful technique using the exponential is available to us for problems of this sort. And the idea hinges on this. As a trial solution, which I'll call 'y sub t', let's try 'e to the rx', where 'r' happens to be a constant. You see, the whole idea is if I differentiate 'e to the rx' with respect to 'x', I get 'e to the rx' back again, only with a factor of 'r', namely the derivative of my exponent in this case.

See, 'r' is a constant. The derivative of 'rx' with respect to 'x' is just 'r'. So notice that the first derivative of 'y sub t' with respect to 'x' is 'r 'e to the rx''. The second derivative of 'y sub t' with respect to 'x' is what? 'r' is a constant. I differentiate 'e to the rx'. That brings down another factor of 'r' and leaves me with 'e to the rx'. You see, the whole idea being, notice that 'e to the rx' is a common factor of 'y sub t', 'y sub 't prime'', 'y sub 't double prime''.

If I now substitute these results back into my original equation, look what I wind up with. 'y double prime' becomes 'r squared 'e to the rx''. Minus '5y prime', that's minus '5r 'e to the rx'', plus '6y', '6e to the rx'. And that must equal zero.

And here's the key point. 'e to the rx' is now a common factor. I factor that out. If the product of two numbers is 0, one of the factors must be 0. But notice from our graph of the exponential, the inverse logarithm, 'e to the x', 'e to the rx' can never be negative and can never be 0, in fact. 'e to the rx' is always positive. Therefore, if 'e to the rx' is always positive, it must be the other factor which is 0.

But 'r squared' minus '5r' plus 6 is not a second-order differential equation. It's a seconddegree polynomial equation. It's a familiar quadratic equation, which I can solve quite easily.

In other words, I find what? That 'r' must be either 2, or 'r' must equal 3. In other words, my claim is that either 'e to the 2x' or 'e to the 3x' must be a solution of this particular equation.

Of course, we can do more with that, which we will in a later part of calculus, not in this package's work. But we're not going to study differential equations in great detail here. But for our present purposes, I think this illustrates how one can use the fact that it's a rather powerful structural property when the derivative of a function with respect to 'x' is the function itself.

By the way, as a quick check, notice that if 'y' equals 'e to the 2x', 'y prime' is twice 'e to the 2x'. 'y double prime' is '4 e to the 2x'. And therefore 'y double prime' minus '5y prime' plus '6y' is what? It's '4e to the 2x' minus '10 e to the 2x', '6 e to the 2x'. And that, in fact, is genuinely, identically zero.

In a similar way, checking out 'y' equals 'e to the 3x', we get 'y prime' is '3e to the 3x', 'y double prime' is '9e to the 3x'. Therefore, '9e to the 3x' minus '15 e to the 3x' plus '6e to the 3x' is again, identically 0. And again, you see what this powerful technique is.

In general, if 'a' and 'b' are constants, the substitution 'y sub t' equals 'e to the rx'. And there'll be a problem in the exercises on this to give you additional drill. But that substitution transforms the second or the differential equation, 'y double prime' plus 'ay prime' plus 'by' equals 0, into an equivalent quadratic equation, 'r squared' plus 'ar' plus 'b' equals 0. And you see from this equation, using the quadratic formula, we can find the values of 'r' that satisfy this, and that gives us a couple of solutions to the equation.

Well, at any rate, I deliberately want this lecture to stay short to make the most important impact. And that is again what? That once we knew what the natural log function was, and we defined the inverse natural log function, everything that we wanted to know about the inverse natural log followed from the properties of the log itself. And this is basically the lesson for today. We will continue the discussion of exponential functions from a different point of view in our next lecture. But until that next lecture, goodbye.

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