PROFESSOR: Hi. Having already studied area and volume and its relationship to calculus, today, we turn our attention to the study of length. And this may seem a bit strange because, intuitively, I think it's fair to assume that you would imagine that length would be simpler than area which, in turn, would be simpler than volume and, hence, that perhaps we should have started with length in the first place. The interesting thing is in terms of our structure, which we so far have called two-dimensional area, three-dimensional area, and which, today, we shall call one-dimensional area, a rather peculiar thing that causes a great deal of difficulty, intellectually speaking, occurs in the study of arc length that does not occur in either the study of area or volume. And I think that we'll start our investigation today leading up to what this really means.

So, as I say, I call today's lesson 'One-dimensional Area', which is arc length. And let's show that there is a parallel, at least in part, between the structure of arc length and the structure of area and volume. You may recall that for area, our initial axiom was that the building block of area was a rectangle. And for volumes, the building block we saw was a cylinder. For arc length-- I think it's fairly obvious to guess what we're going to say-- the basic building block is a straight line segment.

And so without further ado, that becomes our first rule, our first axiom, axiom number one. We assume that we can measure the length of any straight line segment. That's our building block.

The second axiom that we assume is that the length of the whole equals the sum of the lengths of the parts. In other words, if an arc is broken down into constituent bases, the total arc length is equal to the sum of the arc lengths of the constituent parts. And at this stage, we can say, so far, so good. This still looks like it's going to be the same as area or volume.

But now remember what one of the axioms for both area and volume were, namely, what? That if region 'R' was contained in region 'S', the area or the volume of 'R' was no greater than that of the area or volume of 'S'. However, for arc length, this is not true. It need not be true. I shouldn't say it's not true.

It need not be true that if region 'R' is contained in 'S' that the perimeter of region 'R' is less than or equal to the perimeter of 'S'. In fact, this little diagram that I've drawn over here, I hope will show you what I'm driving at. Notice that it's rather clear that the region 'R' here, which is shaded, is contained inside the region 'S', which is my rectangle. And yet, if you look at the perimeter here, all these finger-shaped things in here, I think it's easy to see that the perimeter of 'R' exceeds the perimeter of 'S'.

And if it's not that easy to see, heck, just make a few more loops inside here and keep wiggling this thing around until you're convinced that you have created this particular situation. All I want you to see here is that it's plausible to you that we cannot talk about lengths by squeezing them, as we did areas and volumes, between regions that we already knew contained the given region and were contained in the given region.

Now let me just pause here for one moment to make sure that we keep one thing straight. We're talking now about an analytical approach to length. In other words, an approach that will allow us to bring to bear all of the power of calculus to the study. I don't want you to forget for a moment that intuitively, we certainly do know what arc length is, just as we intuitively had a feeling for what area and volume were.

Just to freshen our memories on this, remember the intuitive approach. That if you have an arc from 'A' to 'B', the typical way of measuring the arc length is to take, for example, a piece of string, lay it off along the curve from 'A' to 'B'. After you've done this, pick the string up. And then straighten the string out, whatever that means, and measure its length with a ruler.

And we won't worry about how you know whether you're stretching the string too taut or what have you. We'll leave out these philosophic questions. All we'll say is we would like a more objective method that will allow us to use mathematical analysis. And so what we're going to try to do next is to find an analytic way that will allow us to use calculus, but at the same time will give us a definition which agrees with our intuition.

And the first question is how shall we begin. And as so often is the case in mathematics, we begin our new quest by going back to an old way that worked for a previous case. And hopefully, we'll find a way of extending the old situation to cover the new.

Now what does this mean in this particular instance? Well, let me just call it this. I'll call it analytical approach, trial number one. What I'm going to do is try to imitate exactly what we did in the area case. For example, if I take the region 'R', which I'll draw this way here, if this is the region 'R', namely bounded above by the curve 'y' equals 'f of x', below by the x-axis, on the left, by the line 'x' equals 'a', and on the right, by the line 'x' equals 'b', how did we find the area of the region 'R'? Well, what we did is we inscribed and we circumscribed rectangles. And we took the limit of the circumscribed rectangles, et cetera, and put the squeeze on as 'n' went to infinity.

Now the idea is we might get the idea that maybe we should do the same thing for arc length. In other words, let me call one of these little pieces of arc length 'delta w'. In other words, I'm just isolating part of the diagram here. Here's 'delta w'. Here's 'delta x'. Here's 'delta y'.

The idea is in the same way that I approximated a piece of area by an inscribed and a circumscribed rectangle, why can't I say something like, well, let me let 'delta w' be approximately equal to 'delta x'? And just to make sure that our memories are refreshed over here, notice that 'delta x' is just the length of each piece if the segment from 'a' to 'b', namely of length 'b - a', is divided into 'n' equal parts. See, the idea is why can't we mimic the same approach.

And let me point out what is so crucial here in terms of what I mentioned above, namely, notice that the reason that we can say that the area of the region 'R' is just the limit of 'U sub n' as 'n' approaches infinity, where 'U sub n' is the area of the circumscribed rectangles. The only reason we can say that is because we squeezed 'A sub r' between 'L sub n', the inscribed rectangles, and 'U sub n', the circumscribed rectangles. And the limits of these lower bounds and upper bounds were equal. 'A sub 'r was squeezed between these two. Hence, it had to equal the common limit.

That was the structure that we used. On the other hand, we can't use that when we're dealing with arc length. And I'll mention that in a few moments again. But let me just point out what I'm driving at this way.

Suppose we mimic this as we did before. And we say, OK, let the element of arc length, 'delta w', be approximately equal to 'delta x'. And now what I will do is define script 'L' from 'a' to 'b'. I don't want to call it arc length because it may not be.

But as a first approximation, let me define this symbol to be the limit of the sum of all these 'delta x's when we divide this region into 'n' parts as 'n' goes to infinity. Now look, I have the right to make up this particular definition. Now if I compute this limit, what happens? Recall that we mentioned that 'delta x' was 'b - a' over 'n'.

Consequently, if I have 'n' of these pieces, the total sum would be what? 'n' times b' - a' over 'n'. And 'n' times 'b - a' over 'n' is just 'b - a'. In other words, script 'L' from 'a' to 'b' is defined. And it's 'b - a', not 'w'.

In other words, coming back to our diagram, notice what happened. What we wanted was a recipe that would give us this length here. What we found was a recipe that gave us the length from 'a' to 'b'. Now intuitively, we know that the length from 'a' to 'b' is not the arc length that we're looking for.

In other words, what we defined to be script 'L' existed as a limit, but it gave us an answer which did not coincide with our intuition. Since we intuitively know what the right answer is, we must discard this approach in the sense that it doesn't give us an answer that we have any faith in.

And by the way, notice where we went wrong over here if you want to look at it from that point of view. Notice that when we approximated 'delta w' by 'delta x', it's clear from this diagram that 'delta x' was certainly less than 'delta w'. But notice that we didn't have an upper bound here.

Or we can make speculations like, maybe 'delta x + delta y' would be more than 'delta w', and things like this. We'll talk about that more later. But for now, all I want us to see is the degree of sophistication that enters into the arc length problem that didn't bother us in either the area or the volume problems, namely, we are missing now the all-important squeeze element. Well, no sense crying over spilt milk. We go on, and we try the next type of approach.

In other words, what we sense now is why don't we do this. Instead of approximating 'delta w' by 'delta x', why don't we approximate 'delta w' by the cord that joins the two endpoints of the arc. In other words, I think that we began to suspect intuitively that, somehow or other, for a small change in 'delta x', 'delta s' should be a better approximation to 'delta w' than 'delta x' was.

Of course, the wide open question is granted that it's better, is it good enough. Well, we'll worry about that in a little more detail later. All we're saying is let 'delta w' be approximately equal to 'delta s'.

In other words, we'll approximate 'delta w' by 'delta s'. And we'll now define 'L' from 'a' to 'b', 'L' from 'a' to 'b' to be the limit not of the sum of 'delta x's now, but the sum of the 'delta s's, as 'k' goes from 1 to 'n', taken in the limit as 'n' goes to infinity.

And for those of us who are more familiar with 'delta x's and 'delta y's, and the symbol delta s bothers us, simply observe that by the Pythagorean theorem, 'delta s' is related to 'delta x' and 'delta y' by ''delta s' squared' equals ''delta x' squared' plus ''delta y' squared'. So we can rewrite this in this particular form.

In other words, I will define capital 'L' hopefully to stand for length later on. But we'll worry about that later too. But 'L' from 'a' to 'b' to be this particular limit.

And now I claim that there are three natural questions with which we must come to grips. The first question is does this limit even exist. Does this limit exist? And the answer is that, except for far-fetched curves, it does. You really have to get a curve that wiggles uncontrollably to break the possibility of this limit existing. Unfortunately, there are pathological cases, one of which is described in the text assignment for this lesson, of a curve that doesn't have a finite limit when you try to compute the arc length this way. Just a little idiosyncrasy.

However, for any curve that comes up in real life, that doesn't oscillate too violently with infinite variations, et cetera, et cetera, which we won't, again, talk about right now, the idea is that this limit does exist. As far as this course is concerned, we shall assume the answer to question one is yes. In fact, the way we'll do it without being dictatorial is we'll say, look, if this limit doesn't exist, we just won't study that curve. In fact, we will call a curve rectifiable if this limit exists. And so we'll assume that we deal only with rectifiable curves, in other words, that this limit does exist.

Question number two. OK, the limit exists. So how do we compute it? And that, in general, is not a very easy thing to answer. What's even worse though is that after you've answered this, you have to come to grips with a question that we were able to dodge when we studied both area and volume, namely, the question is once this limit does exist and you compute it, how do you know that it agrees with our intuitive definition of arc length.

In other words, if you recall what we did just a few minutes ago, we defined script 'L' from 'a' to 'b' to be a certain limit. We showed that that limit existed. The problem was is that limit, even though it existed, did not give us an answer that agreed intuitively with what we believed arc length was supposed to mean. In other words, you see, we've assumed the answer to the first question is yes.

Now we have two questions to answer. How do you compute this limit, which is a hard question in it's own right? Secondly, once you do compute this limit, how do you know that it's going to agree with the intuitive answer that you get for arc length? And this shall be what we have to answer in the remainder of our lesson today.

Let's take these in order. And let's try to answer question number two first. The idea is we've defined capital 'L' from 'a' to 'b' to be this particular limit, and we'd like to know if this limit exists. Not only that, but we have a great command of calculus at our disposal now. All of the previous lessons can be brought to bear here to help us put this into the perspective of what calculus is all about.

For example, when I see an expression like this, I like to think in terms of a derivative. A derivative reminds me of 'delta y' divided by 'delta x', et cetera. So what I do here is I factor out a ''delta x' squared'. In other words, I divide through by ''delta x' squared' inside the radical sign, which is really the same equivalently as dividing by 'delta x'. And I multiply by 'delta x' outside.

In other words, factoring out with ''delta x' squared', the square root of ''delta x' squared' plus ''delta y' squared' can be written as the square root of '1 + ''delta y' over 'delta x'' squared' times 'delta x'. Now the idea is that 'delta y' over 'delta x' is the slope of that cord that joins the two endpoints of 'delta w'. It's not a derivative as we know it. It's the slope of a straight line cord, not the slope of a curve.

Now the whole idea is this. We know from the mean value theorem that if our curve is smooth, there is a point in the interval at which the derivative at that point is equal to the slope of the cord. In other words, if 'f' is differentiable on [a, b], we may invoke the Mean Value Theorem-- here abbreviated as MVT, the Mean Value Theorem-- to conclude that there is some point 'c sub k' in our 'delta x' interval for which ''delta y' over 'delta x'' is 'f prime of 'c sub k''.

And in order to help you facilitate what we're talking about in your minds, look at the following diagram. This is all we're saying. What we're saying is here's our 'delta x', here's our 'delta y'. We'll call this point 'x sub 'k - 1'', this point 'x sub k'. This is our k-th partition. 'Delta y' divided by 'delta x' is just the slope of this line. See, that's just the slope of this line.

And what the Mean Value Theorem says is if this curve is smooth, some place on this arc, there is a point where the line tangent to the curve is parallel to this cord. And that's what I'm calling the point 'c sub k'. 'c sub k' is the point at which the slope of the curve is equal to the slope of the cord.

In other words, if 'f' is continuous, I can conclude that 'L' from 'a' to 'b' is the limit as 'n' approaches infinity, summation 'k' goes from 1 to 'n', square root of '1 + ''f prime 'c sub k'' squared' times 'delta x'. And notice that this now starts to look like my definite integral according to the definition that we were talking about in our earlier lectures in this block. In fact, how can we invoke the first fundamental theorem of integral calculus? Remember, if this expression here-- it's not an integral yet-- happens to be a continuous function, then we're in pretty good shape.

In other words, if I can assume that 'f prime' is continuous-- let's go over here and continue on here. See, what I'm saying is if I can assume that 'f prime' is continuous, well, look, the square of a continuous function is continuous. The sum of two continuous functions is continuous. And the square root of a continuous function is continuous.

In other words, and this is a key point, if the derivative is continuous, I can conclude that the 'L' from 'a' to 'b' can be replaced by the definite integral from 'a' to 'b' square root of '1 + ''dy/dx' squared'' times 'dx', which I quickly point out may be hard to evaluate. In other words, one thing I could try to do over here is to find the function g whose derivative with respect to 'x' is the square root of '1 + ''dy/dx' squared'' and evaluate that between 'a' and 'b'. I can put approximations on here, whatever I want.

In fact, let's summarize it down here. If 'f' is differentiable on the closed interval from 'a' to 'b' and if 'f prime' is the derivative-- you see, 'f prime'-- is also continuous on the closed interval from 'a' to 'b', then not only does capital 'L' from 'a' to 'b' exist, but it's given computationally by this particular integral. And that answers question number two, that the limit exists, and this is what it's equal to.

The problem that we're faced with-- and I've written this out. I think it looks harder than what it says. But I've taken the trouble to write this whole thing out, so that if you have trouble following what I'm saying, that you can see this thing blocked out for you. The idea is this.

What we have done is we have approximated 'delta w' by 'delta s'. Then what we said is 'w' is the sum of all these 'delta w's. And since each 'delta w' is approximately 'delta s', then what we can be sure of is that 'w' is approximated by this sum over here.

Now here's what we did. We didn't work with 'w' at all after this. We turned our attention to this. This is what we did in our case here. And we showed that this limit existed. We showed that the limit, as 'k' went from 1 to 'n' and then went to infinity of these pieces here, was 'L' of 'ab'. And that existed. What we did not show is that this limit was w itself.

Intuitively, you might say, if I put the squeeze on this, doesn't this get rid of all the error for me? We haven't shown that we've gotten rid of all the error. In essence, how do we know if all the error has been squeezed out? This is precisely what question three is all about.

Again, going back to what we did earlier, remember, when we approximated 'delta w' by 'delta x', then we said, OK, add up all these 'delta x's, and take the limit as 'n' goes to infinity. We found that that limit was 'b - a', which was not the length of the curve.

In other words, somehow or other, even though the limit existed, we did not squeeze out all the error. And this is why the study of arc length is so difficult. Because we don't have a sandwiching effect. It is very difficult for us to figure out when we've squeezed out all the error.

So at any rate, let me generalize question number three. Remember what question number three is? How do we know that if the limit exists, it's equal to 'w'? All I'm saying is don't even worry about arc length. Just suppose that 'w' is any function defined on a closed interval from 'a' to 'b' and that we've approximated 'delta w' by something of the form 'g of 'c sub k'' times 'delta x', where 'g' is what I call some intuitive function defined on [a, b].

For example, in our earlier example, we started with 'delta w' being arc length. And we approximated 'delta w' by 'delta x' in which case 'g' would've been the function which is identically 1. In the area situation, remember we approximated 'delta A' by something times 'delta x'. Well, what times 'delta x'? Well, it was the height of a rectangle.

In other words, we look at the thing we're trying to find, we use our intuition-- and this is difficult because intuition varies from person to person-- and we say, what would make a good approximation here. What would be an approximation? We say, OK, let's approximate 'delta w' by 'g of 'c sub k'' times 'delta x', where 'c' is some point in the interval, et cetera.

Then we add up all of these 'delta w's as 'k' goes from 1 to 'n'. We say, OK, that's approximately this thing over here. Now what we have shown is that if 'g' is continuous on [a, b] then as 'n' goes to infinity, this particular limit exists and is denoted by the integral from 'a' to 'b', ''g of x' dx'. This is what we've shown so far.

What the big question is is, granted that this limit exists, does it equal 'w'? In other words, is 'w' equal to the integral from 'a' to b', ''g of x' dx'? That's what the remainder of today's lesson is about as far as arc length is concerned. And I'm going to solve this problem in general first and then make some applications about this to arc length itself.

And by the way, what we're going to see next is you may remember that very, very early in our course, we came to grips with something called infinitesimals. We came to grips with this delta y tan infinitesimals of higher order. And now we're going to see how just as this came up in differential calculus, these same problems of approximation come up in integral calculus. The only difference, as we've mentioned before, is instead of having to come to grips with the indeterminate form 0/0, we're going to have to come to grips with the indeterminate form infinity times 0. Let me show you what I mean by that.

The idea is this. Let's suppose that our case 'delta w'-- we've broken up 'w' now into increments-- and let's suppose that we're approximating 'delta w', as we said before, by 'g of 'c sub k'' times 'delta x'. Well, what do we mean by we're approximating this? What we mean is there's some error in here.

Let's call the error 'alpha sub k' times 'delta x'. In other words, this is just a correction factor. This is what we have to add on to this to make this equality whole. Once I add on the error, I'm no longer working with an inequality. I'm working with an equality. And that allows me to use some theorems.

What I can say now is by definition, w is the sum of all these 'delta w's. But 'delta w' being a sum, we can use theorems about the sigma notation. In other words, what is the sum of all these 'delta w's? It's the sum of all of these pieces plus the sum of all of these pieces, which I've written over here.

And now you see, if I transpose, I get that 'w' minus this sum is equal to the 'sum k' goes from 1 to 'n', 'alpha k' times 'delta x'. Now the next thing I do is take the limit as 'n' goes to infinity. By definition, since 'g' is a continuous function, this limit here is just the definite integral from 'a' to 'b', ''g of x' dx'. On the other hand, this limit here is what we have to investigate.

In other words, we would like to know whether 'w' is equal to the definite integral or not. If we look at this particular equation, what we have now shown is whatever the relationship is between these two terms, it's typified by the fact that this difference is this particular limit. In other words, if this limit happens to be 0, then the integral will equal what we're setting out to show it's equal to, namely, this function itself. On the other hand, what we're saying is we do not know that this limit is 0.

By the way, notice what's happening over here. As 'n' goes to infinity, 'delta x' is going to 0. In other words, each individual term in the sum is going to 0, but the number of pieces is becoming infinite. There's your infinity times 0 form here.

And let me show you a case where the pieces are growing too fast in number to be offset by the fact that their size is going to 0. For the sake of argument, let me suppose that 'alpha sub k' happens to be some non-0 constant for all 'k'. If I come back to this expression here, if 'alpha sub k' is equal to a constant, I'll replace 'alpha sub k' by that constant, which is 'c'. I now have what?

That the limit that I'm looking for is the 'sum k' goes from 1 to 'n', 'c' times 'delta x', taking the limit as 'n' goes to infinity. 'c' is a constant, so I can take it outside the integral sign. Since 'c' is a constant and it's outside the integral sign, let's look at what 'delta x' is. 'Delta x' is 'b - a' divided by 'n', same as we were talking about earlier in the lecture. I have 'n' of these pieces. The 'n' in the denominator cancels the 'n' in the numerator when I add these up. And notice that this particular sum here, no matter what 'n' is, is just 'b - a'.

In other words, in the case that 'alpha sub k' is a constant, notice that this limit is 'c' times 'b - a'. 'c' is not 0. 'b' is not equal to 'a'. We have an interval here. Therefore, this will not be 0. And notice that if this is not 0, these two things here are not equal.

And by the way, the aside that I would like to make here is that even though this error is not negligible, notice the fact that if 'alpha sub k' is a constant that as 'delta x' goes to 0, this whole term will go to 0. But it doesn't go to 0 fast enough. In other words, eventually, we're taking this sum as 'n' goes to infinity. And here's a case where, what? The pieces went to 0, but not fast enough to become negligible.

Well, let me give you something in contrast to this. Situation number two is suppose instead 'alpha k' is a constant times 'delta x'. 'B' times 'delta x', where 'B' is a constant. In that case, notice that summation 'k' goes from 1 to 'n', 'alpha k' times 'delta x' is just summation 'k' goes from 1 to 'n', 'B' times ''delta x' squared'.

Now keep in mind again that 'delta x' is still 'b - a' over 'n'. So ''delta x' squared', of course, is 'b - a' over 'n squared'. Notice that what's inside the summation sign here does not depend on 'k'. It's a constant. I can take it outside the summation sign.

How many terms of this size do I have? Well, 'k' goes from 1 to 'n', so I have 'n' of those pieces. Therefore, this sum is given by this. This is an 'n squared' term. One of the 'n's in the denominator cancels with my 'n' in the numerator. And in this particular case, I find that the sum, as 'k' goes from 1 to 'n', 'alpha sub k' times 'delta x', is 'B', which is a constant, times ''b - a' squared', which is also a constant, divided by 'n'.

Now look, if I now allow 'n' to go to infinity, my numerator is a constant. My denominator is 'n'. As 'n' goes to infinity, my denominator increases without bound. My numerator remains constant. So the limit is 0.

In other words, in the case where 'alpha sub k' is a constant times 'delta x', this limit is 0, the error is squeezed out, and, in this particular case, 'w' is given by the integral from 'a' to 'b', ''g of x' dx' exactly in this particular situation.

Well, the question is how many situations shall we go through before we generalize. And the answer is since this lecture is already becoming quite long, let's generalize now without any more details. And the generalization is this. In general, if you break down 'w' into increments, which we'll call 'delta 'w sub k'', and 'delta 'w sub k'' is equal to-- well, I've made a little slip here. That should be a 'g' in here. I'm using 'g's rather than 'f's.

If 'delta 'w sub k'' is 'g of 'c sub k'' times 'delta x' plus the correction factor 'alpha k' times 'delta x', and, for each 'k', the limit of 'alpha k' as 'delta x' approaches 0 is 0. In other words, what we're saying is that 'alpha sub k' times 'delta x' must be a higher order infinitesimal. If this is a higher order infinitesimal, if 'alpha k' goes to 0 as 'delta x' goes to 0, that says, what? That 'alpha k' times 'delta x' is going to 0 much faster than 'delta x' itself.

So you compare this with our discussion on infinitesimals earlier in our course. I think that was in block two, but that's irrelevant here. But all I'm saying is if that is the case, in that particular case, the limit, that integral is exactly what we're looking for. The error has been squeezed out.

In other words, now, in conclusion, what we must do in our present problem to answer question number three, remember, we have approximated delta wk by this intricate little thing, '1 + ''f prime 'c sub k' squared' times 'delta x'. In other words, in our particular illustration in this lecture, the role of 'g' is played by the square root of '1 + 'f prime squared''. What we must show is that this difference is a higher order differential. And this really requires much more advanced work than we really want to go into.

The only trouble is, as a student, I always used to be upset when the instructor said, the proof is beyond our ability or knowledge. Whenever he used to say, the proof is beyond our knowledge at this stage of the game, I used to say to myself, ah, he doesn't know how to prove it. I think there's something upsetting about this. So what I'm going to try to do for a finale here is to at least give you a plausibility argument that we really do squeeze the error out in our approximation of 'delta w' in this case.

In other words, let me draw this little diagram to bring in the infinitesimal idea here. Here's my 'delta w'. Here's my 'delta s'. And what I'm doing now is I am going to take the tangent line to the curve at 'A', use that rather than 'delta x'.

In other words, what I'm going to say is we're going to assume that our curve doesn't have infinite oscillations. So I can assume the special case of a monotonically increasing function, use the intuitive approach that in this diagram, 'delta w' is caught between 'delta s' and 'AB' plus 'BC', observing that 'BC' is just what's called 'delta y' minus 'delta y-tan'. And that, by the Pythagorean theorem, 'AB' is the square root of ''delta x' squared' plus ''delta 'y sub tan'' squared', which, of course, can be written this particular way, namely, notice that the slope here is the slope of this curve when 'x' is equal to 'x sub 'k - 1''.

And again, this is written out, so I think you can fill in the details as part of your review of the lecture and your homework assignment. All I want to do here is present a plausibility argument using 'AB', 'AC', and 'delta s' as they occur in this diagram. All we're saying is, look, if we're willing to make the assumption that this curve has the right shape, 'delta w' is squeezed between 'delta s' and 'AB' plus 'BC'.

As we showed on our little inset here, 'AB' is the square root of '1 + ''f prime' evaluated 'x sub 'k - 1'' squared' times 'delta x'. What is 'BC'? Remember, 'BC' was 'delta y' minus 'delta y-tan' . That's just your epsilon 'delta x' of your infinitesimal idea, where the limit of epsilon as 'delta x' approaches 0 is 0.

In fact, let me just come over here and make sure we write that part again. Remember what we saw was that 'delta y-tan' was 'dy/dx' evaluated at the point in question plus what? An error term which was called epsilon 'delta x', where epsilon went to 0 as 'delta x' went to 0. And that's all I'm saying over here.

In other words, where is delta s squeezed between right now? Well, let me put it this way, delta s itself, by definition, is the square root of ''delta x' squared' plus ''delta y' squared'. That we saw was this. That was our beginning definition in fact. Now if you look at our diagram once more, notice that since our curve is always holding water and rising, that the slope of the line 'delta s' is greater than the slope of the line 'AB'.

Putting all of this together, we now have 'delta w' squeezed. And it was not at all trivial in putting the squeeze on 'delta w'. There was no self-evident way of saying just because one region was contained in another, it must have a smaller arc length. We really had to be ingenious in how we put the squeeze in to catch this thing. But in the long run, what we now have shown is what? That 'delta w' is equal to this. With an error of no greater than epsilon 'delta x'.

In other words, the exact delta w is what? It's the square root of ''1 + 'f prime 'x sub 'k - 1'' squared' 'delta x' plus 'alpha delta x', where alpha can be no bigger than epsilon. In other words, this is the maximum error that we have here because it's caught between this. Well, look, as 'delta x' approaches 0, so does epsilon. And since alpha is no bigger than epsilon, it must be that as 'delta x' approaches 0, so does alpha approach 0.

In other words, if we now write 'delta w' in this form, observe that, in line with what we're saying, this is a higher order infinitesimal. And as a result, the intuitive approach can be used as the correct answer. The idea is we could have said earlier, look, why don't we approximate the arc length by the straight line segment that joins the two endpoints of the arc. And the answer is you can do that. But you are really on shaky grounds if you say it's self-evident that all the error is squeezed out in the limit.

This is a very, very touchy thing. In other words, in the same way that 0/0 is a very, very sensitive thing in the study of differential calculus, infinity times 0 is equally as sensitive in integral calculus. The whole upshot of today's lecture, however, is now that we've gone through this whole, hard approach, it turns out that we can justify our intuitive approach of approximating the arc length by straight line segments. At any rate, this concludes our lesson for today. And until next time, good-bye.

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