Calculus Revisited
Part 1
A Self-Study Course

Study Guide

Block III
The Circular Functions

Block IV
The Definite Integral

Center for Advanced
Engineering Study

Herbert I. Gross

Massachusetts
Institute of Technology

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STUDY GUIDE: Calculus of a Single Variable

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BLOCK III: THE CIRCULAR FUNCTIONS

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PRETEST

1. Evaluate $\lim_{x \to 0} \frac{\sin 5x}{\sin 3x}$ and $\lim_{x \to 0} \frac{\cos 5x}{\cos 3x}$

2. Evaluate $\lim_{h \to 0} \frac{\sin(a + h) - \sin a}{h}$

3. Find dy/dx if $y^3 = \sin^3 x + \cos^3 x$

4. Find the maximum height of the curve $y = 6\cos x + 8\sin x$

5. Find the following indefinite integrals:

   a. $\int x\sin(x^2)dx$ b. $\int \sin^2 x dx$ c. $\int \sin^2 x \cos x \, dx$

6. Find cosA, tanA, cotA, secA, and cscA if $A = \sin^{-1}(1/2)$.

7. Simplify $\sin(2\sin^{-1}0.8)$

8. Find dy/dx if $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$

9. Find y if it is known that $dy/dx = 1/(x^2+1)$ and that $y = 3$ when $x = 0$.

UNIT 1: Trigonometry Revisited

1. View: Lecture 3.010
2. Read: Supplementary Notes, Chapter VII, Sections A and B.
3. Read: Thomas 5.4
4. Exercises:

3.1.1(L) Evaluate each of the following limits:

(a) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{3x}$ (b) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 4x}$ (c) $\lim\limits_{x \to 2} \dfrac{\sin(x^2 - 4)}{x - 2}$

(d) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x}$ (e) $\lim\limits_{x \to 0} \dfrac{\cos x}{x}$ (f) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{x - \frac{\pi}{2}}$

(g) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos \frac{\pi}{2}}{x - \frac{\pi}{2}}$ (h) $\lim\limits_{x \to \infty} x \sin \dfrac{1}{x}$

3.1.2 Evaluate each of the following limits:

(a) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$ (b) $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x}$ (c) $\lim\limits_{x \to 0} \dfrac{\cos 5x}{\cos 3x}$

(d) $\lim\limits_{x \to 0} \left[\dfrac{\sin 3x}{3x^4 + 2}\right]$ (e) $\lim\limits_{x \to \infty} x(1 - \cos \frac{1}{x})$ (f) $\lim\limits_{x \to 3} \left[\dfrac{\sin(x-3)}{x^2 - 9}\right]$

3.1.3(L) Evaluate $\lim\limits_{\Delta x \to 0} \left[\dfrac{\cos(x + \Delta x) - \cos x}{\Delta x}\right]$

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STUDY GUIDE: Calculus of a Single Variable - Block III: The Circular Functions

UNIT 2: Calculus of Circular Functions

1. Read: Supplementary Notes, Chapter VII, Section C

2. Read: Thomas 5.5 and 7.1 as well as the "trig" part of 2.3

3. Exercises:

3.2.1 (L) Find $\frac{dy}{dx}$ in each of the following cases:

a. $y = \sin(x^2)$ b. $y = \sin^2 x$ c. $y = \sin^2 x + \cos^2 x$

d. $y = \frac{1}{\cos x}$ e. $y = 3 \sin 2x - 4 \cos 2x$

3.2.2 Find $\frac{dy}{dx}$ in each of the following cases:

a. $y = \tan x$ b. $y = \tan(3x^2 + 4)$

c. (1) $y = 2 \sin x \cos x$
   (2) $y = \sin 2x$
   (3) Explain why the answers to (1) and (2) are equal.

d. $y^3 = \sin^3 x + \cos^3 x$

3.2.3 Let $f(x) = \sin^2 x$ and $g(x) = -\cos^2 x$. Show that $f'(x)$ and $g'(x)$ are identical. What can we therefore conclude about the equality of $\sin^2 x$ and $\cos^2 x$? $-\cos^2 x$? *

3.2.4 Let the curve C be defined by $y = 6 \cos x + 8 \sin x$ where $0 \leq x \leq \frac{\pi}{2}$.

a. Find the maximum value of y (that is, the highest point of C) by computing $\frac{dy}{dx}$ directly.

b. Find the maximum value of y by transforming the equation into the form $y = A \sin(x + \alpha)$, showing why in this case the maximum value of y is $|A|$.

3.2.5(L) Determine each of the following integrals:

a. $\int \sin^2 t\cos t\,dt$ b. $\int \cos^3 t\sin t\,dt$

c. $\int (\sin^2 t\cos t + \cos^3 t\sin t)\,dt$ d. $\int \frac{\sin 2x\,dx}{\sqrt{2 + \cos 2x}}$

3.2.6 Determine each of the following integrals:

a. $\int x\sin(x^2)\,dx$ b. $\int \sin^4 x\cos x\,dx$ c. $\int \sin^2 x\,dx$

d. $\int \frac{\sec^2 x\,dx}{\tan^3 x}$

3.2.7(L) Use the Mean Value Theorem to show that

$$|\sin b - \sin a| \leqslant |b - a|$$

3.2.8 A particle moves on the curve

$$x = a\cos\omega t$$
$$y = a\sin\omega t$$

where a, b, and ω are constants. Show that the components of the particle's acceleration are given by:

$$a_x = -\omega^2 x \text{ and } a_y = -\omega^2 y$$

3.2.9 Two ships A and B are sailing away from the point O along routes such that the angle AOB is 120°. How fast is the distance between them changing if at a certain instant, OA is eight miles, OB is six miles, ship A is sailing at the rate of 20 mi/hr, and ship B is sailing at the rate of 30 mi/hr?

UNIT 3: The Inverse Circular Functions

1. View: Lecture 3.020

2. Read: Supplementary Notes, Chapter VII, Sections D and E

3. Read: Thomas 7.2 and 7.3

4. Exercises

3.3.1 (L)

a. Given that $\sin^{-1}(-\frac{3}{5}) = A$ determine cos A, tan A, csc A, sec A, and cot A.

b. Simplify $\sin(2 \sin^{-1} 0.8)$.

3.3.2

a. Find sin A and tan A if $A = \text{ccs}^{-1}(-\frac{5}{13})$.

b. Simplify $\cos(2 \sin^{-1}(-\frac{5}{13}))$.

c. Express sin 2x as a function of u if $x = \tan^{-1}u$.

3.3.3 (L)

a. Prove that if $y = \tan^{-1}x$ then $\frac{dy}{dx} = \frac{1}{1 + x^2}$.

b. Determine f(x) if y = f(x) and we are given that $\frac{dy}{dx} = \frac{1}{1 + x^2}$ and that when x = 0, y = 3 .

3.3.4

a. By appropriate use of a reference triangle show that $\int \frac{dx}{\sqrt{1 - x^2}}$ $(|x| < 1)$ is equal to either $\sin^{-1}x + C$ or $- \cos^{-1}x + C$.

b. Explain why the two answers in a. are equivalent.

c. Find $\frac{dy}{dx}$ if $y = \sin^{-1}(\frac{x - 1}{x + 1})$.

3.3.5 A particle moves along the x-axis with a velocity v in feet/sec after t seconds given by $v = \frac{1}{1+t^2}$. We know that the particle starts at x = 0 when t = 0. Express x as a function of t. In particular, where is the particle located when t = 1 ?

3.3.6 (L) A particle moves along the x-axis in such a way that its acceleration is always given by a = -9x. In addition, we know that when v = 0, x = 4 and t = 0.

a. Express the velocity of the particle in terms of its displacement x. From this information, describe the motion of the particle.

b. Express the displacement of the particle, x, as a function of time (t), and sketch the graph of the function.

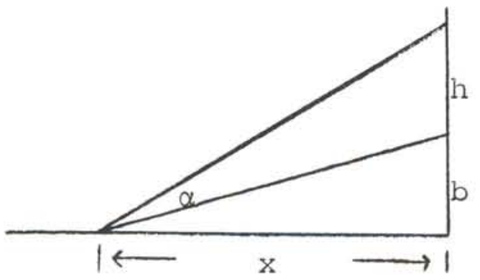
3.3.7 (L) C(x) and S(x) are known to be differentiable functions of x. Moreover it is known that C'(x) = - S(x) and S'(x) = C(x). Define $h(x) = C^2(x) + S^2(x)$. Show that h(x) is a constant.

3.3.8 (L) A picture of height h is hung with its base b feet above the eye-level of an observer who is x feet from the wall on which the picture is hung. Let α denote the angle of vision with which the observer sees the picture.

a. Determine α as a function of x .

b. Where should the observer stand if α is to be maximum ?

(Diagram:



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QUIZ

1. Compute the following limits:

   (a) $\lim_{x\to 0} \frac{\sin 4x}{3x}$ (b) $\lim_{x\to 0} \frac{\cos 4x}{3x}$ (c) $\lim_{x\to 0} \frac{\tan 4x}{3x}$

   (d) $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$ (e) $\lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$

2. Find $\frac{dy}{dx}$ if:

   (a) $y = \sin^4 x - \cos^4 x$ (b) $y = \tan^3 2x$

3. (a) For what values of x will $\frac{dy}{dx} = 0$ if $y = \sin^4 x + \cos^4 x$ $(0 \leqslant x \leqslant 2\pi)$?

   (b) Sketch the curve $y = \sin^4 x + \cos^4 x$.

4. Determine each of the following numbers:

   (a) $\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x \, dx$ (b) $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$.

5. A particle moves along the x-axis in such a way that at any time t its velocity is given by

$$v = \cos 2t \quad 0 \leqslant t \leqslant \frac{\pi}{3}$$

   (t in seconds, v in ft. per sec.).

   (a) What is the displacement of the particle?

   (b) How far does the particle travel?

6. Determine

(a) $\cos\left[\sin^{-1}\left(-\frac{5}{13}\right)\right]$ (b) $\sin\left[2\sin^{-1}\left(-\frac{5}{13}\right)\right]$

7. (a) Use the fact that $y = \cos^{-1}u$ means $u = \cos y$ to show that

$$\frac{d(\cos^{-1}u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

(b) Find $\frac{dy}{dx}$ if $y = \cos^{-1}3x^2$

(c) Find $\frac{dy}{dx}$ if $y = \cos^{-1}(3x^2 + 5)$

(d) Evaluate $\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$

STUDY GUIDE: Calculus of a Single Variable

BLOCK IV: THE DEFINITE INTEGRAL

Content

PRETEST

1. Using the definition of the definite integral,

   (a) Express $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f(\frac{k}{n})$ as a definite integral.

   (b) Evaluate $\{\lim\limits_{n\to\infty} \frac{1}{n}[\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \ldots + \sin \frac{n\pi}{n}]\}$.

2. Find the area of the region R if R is bounded above by $y = \frac{1}{1 + x^2}$, below by the x-axis, on the left by the y-axis and on the right by the line $x = 1$.

3. Let R be the "triangular" region which is bounded above by $y = \cos x$, below by $y = \sin x$, on the left by the y-axis, and on the right by the line $x = \frac{\pi}{4}$. Find the area of R.

4. Let G be defined by $G(x) = \int_0^x \frac{\sin \pi t}{1 + t^2} dt$, $0 \leq x \leq 1$. Compute $G'(\frac{1}{6})$.

5. Let R be the region bounded above by $y = x^2$ and below by $y = x^3$. Find the volume generated if R is revolved about (a) the x-axis (b) the y-axis .

6. Find the length of the segment of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$.

STUDY GUIDE: Calculus of a Single Variable - Block IV: The Definite Integral

UNIT 1: Area

1. View: Lecture 4.010
2. Read: Supplementary Notes, Chapter VIII, Section A.
3. Read: Thomas 5.6, 2.5, 5.7
4. Exercises:

4.1.1 Find a formula for the area of a trapezoid by decomposing the trapezoid into two triangles.

4.1.2 (L) Let C denote the circle whose radius is 1 unit. By circumscribing a regular hexagon about C and by inscribing a regular hexagon in C, obtain upper and lower bounds for π .

4.1.3 (L) Deduce each of the following results

a. $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

b. $\sum_{k=1}^{n} c = nc$ where c is any constant

c. $$\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k$$

4.1.4 Use the results of Exercise 4.1.3 (L) to show that

$\sum_{k=1}^{n} (2k - 1) = n^2$. (You may also use the fact that

$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$.)

4.1.5 (L)

a. Show that $\sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1$.

b. In a. let $a_k = k^2$. Then use the properties developed in Exercise 4.1.3 (L) to obtain a new proof that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} .$$

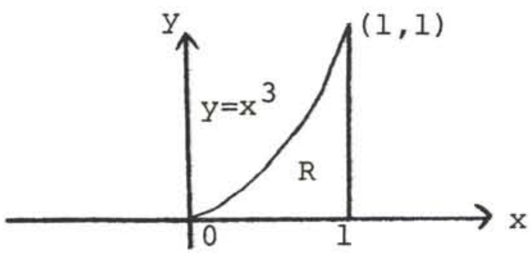
4.1.6 Let $a_k = k^3$. Mimic the procedure in Exercise 4.1.5 (L) to deduce that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

4.1.7 (L)

a. Extend the technique of Exercise 4.1.6 to show that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} .$$

b. With U_n as in the text and our notes, compute $\lim_{n\to\infty} U_n$ to find the area of the region R, where



c. The region R is bounded above by the line y = 1, below by the curve $y = x^3$, on the left by the y-axis, and on the right by the line x = 1. Determine the area of R.

4.1.8 (L) Let R be the region bounded above by $y = \sin x$, below by the x-axis, on the left by the y-axis and on the right by the line $x = \frac{\pi}{2}$.

a. Write an expression for U_n.

b. Determine A_R by computing $\lim_{n\to\infty} U_n$. (You may use the identity $\sum_{k=1}^{n} \sin k\theta = (\cos \frac{\theta}{2} - \cos \frac{2n+1}{2}\theta)/2\sin\frac{\theta}{2}$.)

UNIT 2: Area as a Differential Equation

1. View: Lecture 4.020

2. Read: Supplementary Notes, Chapter VIII, Section B

3. Read: Thomas 5.8

4. Exercises

4.2.1 (L) Compute $\int_0^1 \sqrt{1 - x^2}\,dx$ in two different ways. Namely,

a. Describe the region whose area is the infinite sum $\int_0^1 \sqrt{1 - x^2}\,dx$

b. Find a function $G(x)$ such that $G'(x) = \sqrt{1 - x^2}$

4.2.2

a. Find $G(x)$ such that $G'(x) = \sqrt{16 - x^2}$ and use this to to evaluate $\int_2^4 \sqrt{16 - x^2}\,dx$

b. Describe the region whose area is denoted by $\int_2^4 \sqrt{16 - x^2}\,dx$ and use this information to compute $\int_2^4 \sqrt{16 - x^2}\,dx$

4.2.3 Let R be the region which is bounded above by the curve $y = \sin x$, below by the x-axis, on the left by the y-axis and on the right by $x = \frac{\pi}{2}$. Find A_R using the notion of inverse differentiation and compare your answer with that obtained in Exercise 4.1.8 (L).

4.2.4 Let R be the "triangular" region which is bounded above by $y = \cos x$, below by $y = \sin x$, on the left by the y-axis, and on the right by $x = \frac{\pi}{4}$. Find the area of R.

4.2.5 (L)

a. Show that if $f(x)$ is continuous and non-increasing on $[a,b]$ then:

4.2.5 (L)

$$U_n - L_n = \frac{[f(a) - f(b)](b - a)}{n}$$

b. Let R be the region which is bounded above by $y = \frac{1}{1 + x^2}$ below by the x-axis, on the left by the y-axis and on the right by x = 1. Sketch R making use of y' and y".

c. With R as above, compute L_4 .

d. Use a. and c. to compute U_4 .

e. Based on c. and d. is $\frac{\pi}{4}$ a "reasonable" estimate for A_R ?

f. Find the area of R <u>exactly</u> by finding a function G such that:

$$G'(x) = \frac{1}{1 + x^2} \quad .$$

4.2.6 (L)

a. Let f(x) be a continuous function. Express

$$\lim_{n\to\infty} \frac{1}{n}[f(\tfrac{1}{n}) + \ldots + f(\tfrac{n}{n})] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\tfrac{k}{n})$$

as a definite integral.

b. Use a. to compute $\lim_{n\to\infty} \frac{1}{n}[\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \ldots + \sin \frac{n\pi}{n}]$.

4.2.7

Find $\lim_{n\to\infty} \left[\frac{1^6 + 2^6 + \ldots + n^6}{n^7}\right]$.

(Observe that if we "let n = ∞" we get on $\frac{\infty}{\infty}$ form which is indeterminate in the same sense that $\frac{0}{0}$ is indeterminate.)

UNIT 3: The Second Fundamental Theorem of Integral Calculus

1. Read: Supplementary Notes; Chapter VIII; Section C
2. Read: Thomas 5.9, 5.11, 5.12
3. Exercises:

4.3.1(L) The function G(x) has the following properties. It is defined for all $x \geqslant 1$, $G(1) = 0$ and $G'(x) = \frac{1}{x}$.

a. Interpret G(x) as an area.

b. By computing L_4 and U_4 on [1,2] show that $0.6 < G(2) < 0.8$.

4.3.2(L) Let x and y be defined by $x = \int_0^y \frac{du}{\sqrt{1 - u^2}}$ $|y| < 1$.

Show that $\frac{d^2y}{dx^2} = -y$.

4.3.3 Let G be defined by $G(x) = \int_0^x \frac{\sin \pi t}{1 + t^2}\,dt$ $0 \leqslant x \leqslant 1$. Compute $G'(\frac{1}{6})$.

4.3.4(L) Determine f(x) from the following set of conditions:

(1) The curve $y = f(x)$ passes through the origin.

(2) It exists only in the first quadrant and is continuous.

(3) The area of the region which is bounded above by $y = f(x)$, below by the x-axis, on the left by the y-axis and on the right by the line $x = t$ is always equal to $f^3(t)$ $\left[=[f(t)]^3\right]$.

4.3.5 Let $H(x) = \int_0^{x^2} \frac{\sin \pi t}{1 + t^2} dt$. Determine $H'(x)$.

4.3.6 Compute $G'(t)$ if $G(t) = \int_{t^2}^{t^4} \frac{\sin x \, dx}{1 + x^4}$.

4.3.7(L) Let f be continuous on [a,b] and let $G'=f$. Then, by the mean value theorem, there exists $c\varepsilon(a,b)$ such that $G(b) - G(a) = (b - a)G'(c)$.

a. Apply this result to show that if f is continuous on [a,b] there exists $c\varepsilon[a,b]$ such that $\int_a^b f(x)dx = (b - a)f(c)$.

b. Referring to (a) find c if $f(x) = x^2$.

c. What is the average value of x^2 on the interval [1,7]?

4.3.8(L) Let f be continuous on [a,b] and define G on [a,b] by $G(x) = \int_a^x f(t)dt$. Use the definition of G' (i.e. the delta method) together with the fact that $\int_{x_1}^x f(t)dt = f(c)[x - x_1]$ for some c between x_1 and x to prove that $G' = f$.

4.3.9

a. Evaluate $\lim_{x\to x_1} \left[\frac{x^2}{x - x_1} \int_{x_1}^x f(t)dt\right]$ where f is continuous in a neighborhood of $x = x_1$.

b. Evaluate $\lim_{x\to 2} \left[\frac{4}{x-2} \int_2^x \frac{\cos^2 t \, dt}{1 + t}\right]$.

STUDY GUIDE: Calculus of a Single Variable - Block IV: Differentiation

UNIT 4: Some Simple Applications of the Definite Integral

1. Read Thomas 6.1, 6.2, 6.3

2. Exercises:

4.4.1(L)

a. Sketch the curve $y = 12x\ (x-1)^2$

b. Find the area of R if R is the region bounded above by $y = 12x\ (x-1)^2$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = 1$.

c. Find the area of the region which is enclosed between the two curves $y = 12x\ (x-1)^2$ and $y = \frac{4}{3}x$.

4.4.2(L) Find the area of the region which is bounded above by the curve $x = 12y\ (y-1)^2$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = \frac{16}{9}$.

4.4.3 Find the area of the region R which is enclosed between the curves $y = x^3$ and $y = 7x - 6$

4.4.4 A particle travels along the x-axis according to the rule $v = t^3 - 7t + 6 \quad 0 \leqslant t \leqslant 2$. Find the total distance travelled by the particle. What is the displacement of the particle during this time interval?

4.4.5 The area of the region R which is enclosed between $y = x^2$ and $y = 4$ is divided into two regions of equal area by the line $y = c$. Determine the value of c.

4.4.6

a. Find the area of the region R which is enclosed between the y-axis and the curve $x = y^2\ (y-3)$

4.4.7 Find the area of R if R is the region bounded above by the single-valued curve $x = y^2 (y-3)$ where $0 \leq y \leq 2$, below by the x-axis, on the left by $x = -4$ and on the right by the y-axis.

UNIT 5: Volume

1. View: Lecture 4.030

2. Read: Thomas 6.4

3. Exercises:

4.5.1 The base of a solid is the circular disc $x^2 + y^2 = 9$. Every cross-section of this solid made by a plane perpendicular to the x-axis is a square having a chord of the circle as one of its sides. Find the volume of this solid.

4.5.2 Let R be the region bounded above by $y = x^2$ and below by $y = x^3$.

a. Find the volume generated when R is revolved about the x-axis.

b. Find the volume generated when R is revolved about the y-axis.

4.5.3 Let R be the region contained between $y = 4x - x^2 - 3$ and the x-axis. Find the volume of the solid generated when

a. R is revolved about the x-axis

b. R is revolved about the y-axis

4.5.4(L) The circle centered at (b,0) with radius equal to r (b > r) is revolved about the y-axis. What is the volume of the resulting solid?

4.5.5(L) The curve $y = f(x)$ $0 \leqslant x \leqslant a$ has the property that if it is revolved about the x-axis the volume of the resulting solid is a^5 for each $a > 0$. Determine $f(x)$.

UNIT 6: Arc Length and Approximations

1. View: Lecture 4.040

2. Read: Thomas 6.6, 6.7, and skim 5.10, 6.5

3. Exercises:

4.6.1(L) Find the length of the segment of the curve

$$x = \frac{y^3}{3} + \frac{1}{4y} \qquad \text{from } y = 1 \text{ to } y = 3.$$

4.6.2(L) Find the length of the segment of the curve $y^2 = x^3$ from $y = -1$ to $y = 8$.

4.6.3(L) Let L denote the length of the segment of the curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

a. Express L exactly as a definite integral.

b. Use trapezoidal approximations with $n = 3$ to estimate L from the answer to (a).

c. Divide the segment $y = \sin x$, $0 \leqslant x \leqslant \frac{\pi}{2}$ at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ to estimate L directly.

4.6.4(L) Given the straight line $y = mx + b$, express Δs in the form $\Delta s = \Delta x + \alpha\Delta x$. Then show that α is not an infinitesimal except in one special case, and describe that special case.

4.6.5 A particle moves in the xy-plane in such a way that its position at any time t is given by:

$$\begin{cases} x = \frac{1}{2}t^2 \\ y = \frac{1}{3}(2t + 1)^{\frac{3}{2}} \end{cases}$$

Find the distance travelled by the particle during the time from t = 2 to t = 6.

4.6.6(L) The curve described by the parametric equations:

$$\begin{cases} x = t + 1 \\ y = \frac{t^2}{2} + t \end{cases}$$

from t = 0 to t = 4 is rotated about the y-axis. Find the surface area that is generated.

4.6.7 The segment of the curve described in Exercise 4.6.1(L) is rotated about the y-axis. Find the surface area which is generated.

QUIZ

1. For $0 \leq x \leq 1$, define g by $g(x) = \int_0^x \frac{du}{u^6 + 1}$.

   a. Evaluate $g'(\frac{1}{2})$

   b. Compute $\lim_{h \to 0} \left\{ \frac{1}{h} \int_x^{x+h} \frac{du}{u^6 + 1} \right\}$

2. Let R denote the region which is bounded above by $y = 4x - x^2$ and below by the x-axis.

   a. Find the area of R.

   b. Find the volume generated when R is rotated about the x-axis.

   c. Find the volume generated when R is rotated about the y-axis.

3. The base of a certain solid is the circle $x^2 + y^2 = 4$. Each plane section of the solid cut out by a plane perpendicular to the x-axis is a square with one edge of the square in the base of the solid. Find the volume of the solid.

4. C is the curve $y = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^{\frac{1}{2}}$ where $0 \leq x \leq 4$.

   a. Find the length of C.

   b. Find the surface area generated when C is rotated about the y-axis.

5. Let f be defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } 0 < x \leqslant \dfrac{\pi}{2} \\ 1, & \text{if } x = 0 \end{cases}$$

a. Show that f is continuous on [0,1]

b. Let R be the region bounded above by $y = f(x)$ [where f is as in (a)], below by the x-axis, on the left by the y-axis and on the right by $x = \frac{\pi}{2}$. Use trapezoidal approximations with $n = 3$ to estimate the area of R.

Calculus of a Single Variable

SOLUTIONS

SOLUTIONS: Calculus of a Single Variable - Block III: The Circular Functions

PRETEST

1. (a) $\frac{5}{3}$ (b) 1

2. cos a

3. $\frac{\sin x \cos x(\sin x - \cos x)}{y^2}$

4. 10

5. (a) $-\frac{1}{2}\cos x^2 + c$ (b) $\frac{x}{2} - \frac{1}{4}\sin 2x + c$ (c) $\frac{1}{3}\sin^3 x + c$

6. $\frac{1}{2}\sqrt{3}$, $\frac{1}{\sqrt{3}}$, $\sqrt{3}$, $\frac{2}{\sqrt{3}}$, 2

7. $\frac{24}{25}$

8. $\frac{1}{(x + 1)\sqrt{x}}$

9. y = arc tan x + 3

SOLUTIONS: Calculus of a Single Variable - Block III: The Circular Functions

UNIT 1: Trigonometry Revisited

3.1.1(L)

The major purpose of this exercise, aside from supplying a drill on trigonometry, is to show that the limit theorems we learned earlier remain intact. What is new is that we can now handle a wider variety of choices for f(x). In other words, $\lim_{x\to a} f(x) = L$ means the same as it did before, but now, for example, we can examine f(x) = sin x. In any event, let us proceed.

(a) We notice that our major theorem is of the form $\lim_{(\ )\to 0} \frac{\sin (\ )}{(\ )} = 1$, where our parentheses indicate that the name of the variable is not important but that it be the same in each case. For example, we could say that:

$$\lim_{5x\to 0} \frac{\sin 5x}{5x} = 1$$

So, our first task is to try to bring $\frac{\sin 5x}{5x}$ into the expression $\lim_{x\to 0} \frac{\sin 5x}{3x}$. To this end we multiply by 1 in the form $\frac{5x}{5x}$ where $\lim_{x\to 0}$ insures that $x \neq 0$. That is, if $x \neq 0$, then $\frac{\sin 5x}{3x} = (\frac{\sin 5x}{5x})\ (\frac{5x}{3x})$. Therefore,

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{x\to 0} \left[(\frac{\sin 5x}{5x})\ (\frac{5x}{3x})\right]$$

and since the limit of a product is still the product of the limits, we have:

[3.1.1(L) cont'd]

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \left[\lim_{x\to 0} \frac{\sin 5x}{5x}\right] \left[\lim_{x\to 0} \frac{5x}{3x}\right] \qquad (1)$$

In (1), it is clear from our previous work with limits that $\lim_{x\to 0} \frac{5x}{3x} = \frac{5}{3}$.

We should also "suspect" that $\lim_{x\to 0} \frac{\sin 5x}{5x} = 1$. We know that $\lim_{5x\to 0} \frac{\sin 5x}{5x} = 1$, and we sense that 5x approaches zero if and only if x approaches zero.*

At any rate, putting these results into (1) yields:

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = (1)\ \left(\frac{5}{3}\right) = \frac{5}{3} \qquad (2)$$

---

*We do not have to rely on our suspicions. For example, once we suspect that $\lim_{x\to 0} \frac{\sin 5x}{5x} = 1$, we can appeal directly to the epsilon-delta method to substantiate our claim. In this case, we must show that given $\varepsilon > 0$, we can find $\delta > 0$ such that $0 < |x| < \delta \rightarrow \left|\frac{\sin 5x}{5x} - 1\right| < \varepsilon$. Since $\lim_{5x\to 0} \frac{\sin 5x}{5x} = 1$, we know that for the given choice of ε, we can find δ_1 such that $0 < |5x| < \delta_1 \rightarrow \left|\frac{\sin 5x}{5x} - 1\right| < \varepsilon$. Then since $|5x| = 5|x|$ we can divide by 5, and we have that $0 < |x| < \frac{\delta_1}{5} \rightarrow \left|\frac{\sin 5x}{5x} - 1\right| < \varepsilon$. If we now choose $\delta = \frac{\delta_1}{5}$, our proof is complete. Notice, again, how intuition and rigor are combined in a mathematical demonstration. We use our intuition to form the conjecture and rigor to substantiate the truth of the conjecture. (Without the conjecture, we wouldn't even know what we wanted to prove!)

[3.1.1(L) cont'd]

What (2) says from a more intuitive point of view (and once the correct result is proved it is safe to talk about an intuitive interpretation) is that for small values of x, sin 5x "behaves like" 5x and, hence, $\frac{\sin 5x}{3x}$ "behaves like" $\frac{5x}{3x}$. Obviously, we can generalize the result of part (a) by observing that:

$$\lim_{x\to 0} \frac{\sin mx}{nx} = \frac{m}{n} \quad (n \neq 0) \qquad (3)$$

To obtain (3), we need only imitate our earlier procedure:

$$\frac{\sin mx}{nx} = \frac{\sin mx}{mx}\frac{mx}{nx} \quad (x \neq 0)$$

$$\therefore \lim_{x\to 0} \frac{\sin mx}{nx} = \lim_{x\to 0} \left[\left(\frac{\sin mx}{mx}\right)\left(\frac{mx}{nx}\right)\right]$$

$$= \left[\lim_{x\to 0} \frac{\sin mx}{mx}\right]\left[\lim_{x\to 0} \frac{mx}{nx}\right]$$

$$= (1)\left(\frac{m}{n}\right)$$

$$= \frac{m}{n}$$

(Notice that x is a number, and if we view it as an angle we are using radians. Had x been in degrees, the result would be $\lim_{x\to 0} \frac{\sin x°}{x} = \frac{\pi}{180}$ not 1.)

[3.1.1(L) cont'd]

(b) Using the same hints as in part (a), we elect to rewrite $\frac{\sin 5x}{\sin 4x}$ as $\left(\frac{\sin 5x}{5x}\right)\left(\frac{5x}{4x}\right)\left(\frac{4x}{\sin 4x}\right)$ $(x \neq 0)$. That is, we use algebraic manipulation to introduce the form $\frac{\sin (\ )}{(\ )}$ wherever we can. Then:

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x\to 0} \left[\left(\frac{\sin 5x}{5x}\right)\left(\frac{5x}{4x}\right)\left(\frac{4x}{\sin 4x}\right)\right]$$

$$= \left[\lim_{x\to 0} \frac{\sin 5x}{5x}\right] \left[\lim_{x\to 0} \frac{5x}{4x}\right] \left[\lim_{x\to 0} \frac{4x}{\sin 4x}\right]$$

$$= (1)^{*} \left(\frac{5}{4}\right) (1)^{**}$$

$$= \frac{5}{4}$$

---

*In our previous footnote we proved specifically that $\lim_{x\to 0} \frac{\sin 5x}{5x} = 1$. The proof is easily modified to show that if $m \neq 0$, $\lim_{x\to 0} \frac{\sin mx}{mx} = 1$. All we have to do is replace 5 by m in the earlier proof.

**This illustrates our contention that all limit theorems remain intact. For example, we proved earlier that if $\lim_{x\to a} f(x) = L$ and $L \neq 0$ then $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{L}$. If we take this result with $f(x) = \frac{\sin 4x}{4x}$, $a = 0$ and $L = 1$ we see that $\lim_{x\to 0} \frac{\sin 4x}{4x} = 1$ implies that $\lim_{x\to 0} \frac{4x}{\sin 4x} = 1$ since in this case $\frac{1}{f(x)} = \frac{4x}{\sin 4x}$ and $\frac{1}{L} = 1$.

[3.1.1(L) contd']

(<u>Note</u>: If we are not careful, it looks as if we are "cancelling" sin from numerator and denominator to obtain:

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x\to 0} \frac{5x}{4x} = \frac{5}{4}$$

It is <u>not</u> true that $\frac{\sin 5x}{\sin 4x} = \frac{5x}{4x}$. It is only true that $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x\to 0} \frac{5x}{4x}$. In other words, $\frac{\sin 5x}{\sin 4x}$ can be approximated rather well by $\frac{5x}{4x}$ in sufficiently small neighborhoods of x = 0.)

(c) Here the limit is a somewhat different form from what we are used to. In particular, we are more familiar with the form $\lim_{(\ )\to 0}$ than with $\lim_{(\ )\to 2}$ in dealing with $\frac{\sin (\ )}{(\ )}$. So we first invoke the result that x approaches c if and only if (x - c) approaches 0. With this in mind, we may write:

$$\lim_{x\to 2} \frac{\sin(x^2 - 4)}{x - 2} = \lim_{x - 2\to 0} \frac{\sin(x^2 - 4)}{(x - 2)}$$

This, in turn, suggests the change of variable (substitution) u = x - 2. In this event, x = u + 2, $x^2 = u^2 + 4u + 4$, $x^2 - 4 = u^2 + 4u$. Thus:

$$\lim_{x\to 2} \frac{\sin(x^2 - 4)}{x - 2} = \lim_{u\to 0} \frac{\sin(u^2 + 4u)}{u} \qquad (4)$$

From (4) we may either proceed rigorously and write:

[3.1.1(L) cont'd]

$$\frac{\sin(u^2 + 4u)}{u} = \left[\frac{\sin(u^2 + 4u)}{u^2 + 4u}\right]\left[\frac{u^2 + 4u}{u}\right] \text{ etc.}$$

[where we shall explain "etc." in more detail in Exercise 3.1.3(L).]

Or we may now fairly safely take the more intuitive path and notice that for small values of u, $\sin(u^2 + 4u)$ "behaves like" $u^2 + 4u$. Thus, $\frac{\sin(u^2 + 4u)}{u}$ "behaves like" $\frac{u^2 + 4u}{u}$ and this in turn "behaves like" u + 4 which, in turn, approaches 4 as u approaches 0.

In either case, we arrive at

$$\lim_{x \to 2} \frac{\sin(x^2 - 4)}{x - 2} = 4$$

(In fact, we could have appealed to the intuitive approach sooner and said that "near" x = 2, $\sin(x^2 - 4)$ "behaved like" $x^2 - 4$, hence

$$\lim_{x \to 2} \frac{\sin(x^2 - 4)}{x - 2} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2},$$

but we already know that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.)

(d) Here we see a good example of how we may need a knowledge of some trigonometric identities to reduce an unfamiliar form to a more familiar form. In this case we would like, if possible, to transform $\lim_{x \to 0} \frac{(1 - \cos x)}{x}$ into a form which allows us to utilize our knowledge about $\frac{\sin(\ )}{(\ )}$.

[3.1.1(L) cont'd]

Well, suppose we recall that $1 - \cos^2 x = \sin^2 x$. This would motivate us to write:

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \left[\left(\frac{1 - \cos x}{x}\right)\left(\frac{1 + \cos x}{1 + \cos x}\right)\right]$$

We must be sure that $1 + \cos x \neq 0$. But near 0, $\cos x \neq -1$. So we are on safe ground here.

$$= \lim_{x\to 0} \left[\frac{1 - \cos^2 x}{x(1 + \cos x)}\right]$$

$$= \lim_{x\to 0} \left[\frac{\sin^2 x}{x(1 + \cos x)}\right]$$

$$= \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1 + \cos x}\right)\right]$$

$$= \left[\lim_{x\to 0} \frac{\sin x}{x}\right] \left[\lim_{x\to 0} \frac{\sin x}{1 + \cos x}\right]$$

$$= (1)\ (0)^*$$

$$= 0$$

---

*Again notice our reliance on the "old" limit theorems. That is:

$$\lim_{x\to 0} \left[\frac{\sin x}{1 + \cos x}\right] = \frac{\lim_{x\to 0}(\sin x)}{\lim_{x\to 0}(1 + \cos x)} = \frac{\lim_{x\to 0}(\sin x)}{\lim_{x\to 0} 1 + \lim_{x\to 0} \cos x}$$

$$= \frac{0}{1 + 1}$$

$$= 0$$

[3.1.1(L) cont'd]

(e) Here we observe that for values of x near 0, cos x "behaves like" 1 while $\frac{1}{x}$ grows without bound. That is, for small values of x, $\frac{\cos x}{x}$ grows without bound. That is, $\lim_{x \to 0} \frac{\cos x}{x} = \infty$. (Another way of saying this, is to observe $\lim_{x \to 0} \frac{x}{\cos x} = 0$.)

One aim of this exercise is to emphasize the basic difference between $\frac{\sin x}{x}$ and $\frac{\cos x}{x}$ near 0. Unless the limit takes on the form 0/0, we need no great ingenuity to find the limit.

(f) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$

In this case we get a 0/0 form. One approach is to let $u = x - \frac{\pi}{2}$ whereupon $x = \frac{\pi}{2} + u$. Then

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} = \lim_{\frac{\pi}{2} + u \to \frac{\pi}{2}} \frac{\cos(\frac{\pi}{2} + u)}{u} = \lim_{u \to 0} \frac{\cos(\frac{\pi}{2} + u)}{u}$$

But since $\cos(\frac{\pi}{2} + u) = \cos \frac{\pi}{2} \cos u - \sin \frac{\pi}{2} \sin u = -\sin u$, we see

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} = \lim_{u \to 0} \left[\frac{-\sin u}{u}\right] = -1$$

[3.1.1(L) cont'd]

(g) At first glance, this seems to be just like part (f), but the basic difference here is that now our numerator no longer "approaches" 0, it is exactly 0. Thus:

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos \frac{\pi}{2}}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2}} \frac{0}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2}} 0 = 0$$

The major point here is to observe that 0/0 is indeterminate in the sense that the quotient of two small <u>non-zero</u> numbers is indeterminate. However, 0 divided by a small <u>non-zero</u> number is always 0.

(h) Here, we invoke the idea that $\lim_{x \to \infty} f(x)$ is equivalent to $\lim_{x \to 0} f(\frac{1}{x})$. If we apply this to (h) with $f(x) = x \sin \frac{1}{x}$, we obtain

$$f(\tfrac{1}{x}) = \frac{1}{x} \sin \left(\frac{1}{\frac{1}{x}}\right) = \frac{1}{x} \sin x = \frac{\sin x}{x}$$

$$\therefore \lim_{x \to 0} f(\tfrac{1}{x}) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\therefore \lim_{x \to \infty} f(x) = \lim_{x \to 0} f(\tfrac{1}{x}) \to \lim_{x \to \infty} x \sin \frac{1}{x} = 1$$

(Graphically, this means that the curve $y = x \sin \frac{1}{x}$ has the line $y = 1$ as an asymptote for large values of x.)

3.1.2

(a) $$\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)\left(\frac{2x}{x}\right)$$

$$= \left[\lim_{x\to 0} \frac{\sin 2x}{2x}\right] \left[\lim_{x\to 0} \frac{2x}{x}\right]$$

$$= (1)(2)$$

$$= 2$$

(b) $$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right) \sin x\right]$$

$$= \left[\lim_{x\to 0} \frac{\sin x}{x}\right] \left[\lim_{x\to 0} \sin x\right]$$

$$= [1][0]$$

$$= 0$$

(c) $$\lim_{x\to 0} \frac{\cos 5x}{\cos 3x} = \frac{\lim_{x\to 0} \cos 5x}{\lim_{x\to 0} \cos 3x} = \frac{\cos 0}{\cos 0} = \frac{1}{1} = 1$$

[3.1.2 cont'd]

(d) $$\lim_{x\to 0}\left[\frac{\sin 3x}{3x^4 + 2}\right] = \lim_{x\to 0}\left[\frac{\sin 3x}{3x(x^3 + \frac{2}{3x})}\right]$$

$$= \lim_{x\to 0}\left[\left(\frac{\sin 3x}{3x}\right)\left(\frac{1}{x^3 + \frac{2}{3x}}\right)\right]$$

$$= \lim_{x\to 0}\left[\left(\frac{\sin 3x}{3x}\right)\left(\frac{3x}{3x^4 + 2}\right)\right]$$

$$= \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\lim_{x\to 0}\left(\frac{3x}{3x^4 + 2}\right)$$

$$= (1)(0)$$

$$= 0$$

(e) $$\lim_{x\to\infty} x\left(1 - \cos\frac{1}{x}\right) = \lim_{x\to 0}\frac{1}{x}(1 - \cos x)$$

$$= \lim_{x\to 0}\frac{1 - \cos x}{x}$$

$$= 0 \quad \text{(why?)}$$

[3.1.2 cont'd]

(f) $$\lim_{x\to 3} \frac{\sin(x-3)}{x^2-9} = \lim_{x-3\to 0} \frac{\sin(x-3)}{(x-3)(x+3)}$$

$$= \lim_{u\to 0} \frac{\sin u}{u(u+6)} \quad \text{(where } u = x-3\text{)}$$

$$= \lim_{u\to 0} \left[\left(\frac{\sin u}{u}\right)\left(\frac{1}{u+6}\right)\right]$$

$$= \left[\lim_{u\to 0} \frac{\sin u}{u}\right]\left[\lim_{u\to 0} \frac{1}{u+6}\right]$$

$$= [1]\left[\frac{1}{6}\right]$$

$$= \frac{1}{6}$$

3.1.3(L)

The main aim of this exercise is to motivate the use of limits in calculus. Recalling that f'(x) is defined by $\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. We observe that $\lim_{\Delta x\to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$ is f'(x) where f(x) = cos x.

Thus in this exercise we are being asked to find f'(x) if f(x) = cos x. This seems to anticipate the next unit. Indeed, it does, and we shall do more of this then; however, it is

[3.1.3(L) cont'd]

imperative to understand how in a subject as logical as mathematics we can on our own derive large amounts of material once a few "new" things are known.

In this exercise, if we were asked to find f'(x) for f(x) = cos x, we could by definition write:

$$f'(x) = \lim_{x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (1)$$

Equation (1) is always true. All we do now is let f(x) = cos x to obtain

$$f'(x) = \lim_{\Delta x\to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} \qquad (2)$$

and (2) brings us to the present exercise.

Using the indicated identities, we obtain:

$$\lim_{\Delta x\to 0}\left[\frac{\cos(x+\Delta x) - \cos x}{\Delta x}\right] = \lim_{\Delta x\to 0}\left[\frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\frac{\cos x(\cos \Delta x - 1) - \sin x \sin \Delta x}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[-\cos x\left\{\frac{1-\cos \Delta x}{\Delta x}\right\} - \sin x\left\{\frac{\sin \Delta x}{\Delta x}\right\}\right]$$

$$= -\cos x\left[\lim_{\Delta x\to 0}\left(\frac{1-\cos \Delta x}{\Delta x}\right)\right] - \sin x \left[\lim_{\Delta x\to 0}\frac{\sin \Delta x}{\Delta x}\right] \qquad (3)$$

[3.1.3(L) cont'd]

Equation (3) now supplies us with a motivation for solving problems such as those in Exercises 3.1.1 and 3.1.2. In fact, from 3.1.1(d) we see that $\lim_{\Delta x \to 0} \left(\frac{1 - \cos \Delta x}{\Delta x}\right) = 0$, while our basic limit tells us that $\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$.

Putting these results into (3), we obtain:

$$\lim_{\Delta x \to 0} \left[\frac{\cos(x+\Delta x) - \cos x}{\Delta x}\right] = (-\cos x)(0) - \sin x(1)$$

$$= -\sin x$$

In other words, if f(x) = cos x, then f'(x) = - sin x.

Notice, above all, that we did not, in truth, have to study the next unit to obtain the results of that unit. Often in this course we will be able to derive new results by applying "old" ideas to "new" concepts.

UNIT 2: Calculus of Circular Functions

3.2.1 (L)

(a) $$y = \sin(x^2)$$

We know that $\frac{d(\sin u)}{du} = \cos u$. Hence, we may view this exercise as

$$\begin{cases} y = \sin u \\ u = x^2 \end{cases}$$

and invoke the chain rule.

That is, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$= \cos u(2x)$$

$$= 2x\cos(x^2)$$

The major learning value of this example is to reinforce certain principles which have already been emphasized.

(1) The notion that the derivative of sin is cos is a loose paraphrase of $\frac{d\sin(\;)}{d(\;)} = \cos(\;)$. It certainly does not mean that $\frac{d\sin(x^2)}{dx} = \cos(x^2)$. Rather: $\frac{d\sin(x^2)}{d(x^2)} = \cos(x^2)$.

This, in turn, suggests the chain rule, and

(2) we wish to emphasize that the concept of the chain rule depends on the general notion of functions. All that happened in the block is that we learned more about those specific functions known as the trigonometric functions. In any event, once we know that $\frac{d\sin(\;)}{d(\;)} = \cos(\;)$, the "same, old" chain rule allows us to conclude

$$\frac{d\sin u}{dx} = \frac{d\sin u}{du}\cdot\frac{du}{dx} = \cos u\frac{du}{dx}$$

where u is any differentiable function of x.

[3.2.1 (L) cont'd]

(b) $y = \sin^2 x$

This means the same as $y = (\sin x)^2$. Again we may invoke the chain rule to write

$$\begin{cases} y = u^2 \\ u = \sin x \end{cases}$$

We then obtain:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u \cos x$$

and recalling that $u = \sin x$, we obtain

$$\frac{dy}{dx} = 2 \sin x \cos x$$

(which can, of course, also be written as $\frac{dy}{dx} = \sin 2x$ if we remember the identity $\sin 2x = 2 \sin x \cos x$).

It is worthwhile comparing parts (a) and (b) and observing the difference between $\sin^2 x$ and $\sin x^2$.

(c) $y = \sin^2 x + \cos^2 x$

The fact that we know that the derivative of a sum equals the sum of the derivatives allows us to write

$$\frac{dy}{dx} = \frac{d(\sin^2 x + \cos^2 x)}{dx} = \frac{d(\sin^2 x)}{dx} + \frac{d(\cos^2 x)}{dx} \quad . \qquad (1)$$

(Again, these theorems were proven for differentiable functions in general and they do not have to be proven again for the trigonometric functions.)

[3.2.1 (L) cont'd]

From (b), we know that $\frac{d(\sin^2 x)}{dx} = 2 \sin x \cos x$. Hence, (1) becomes

$$\frac{dy}{dx} = 2 \sin x \cos x + \frac{d(\cos^2 x)}{dx} \qquad (2)$$

We may compute $\frac{d(\cos^2 x)}{dx}$ in the same way as we handled $\frac{d(\sin^2 x)}{dx}$.
Namely, we let $\begin{cases} y_1 = u^2 \\ u = \cos x \end{cases}$ (or $y_1 = \cos^2 x$)

Then $\frac{d(\cos^2 x)}{dx} = \frac{dy_1}{dx} = \frac{dy_1}{du} \cdot \frac{du}{dx} = 2u(-\sin x)$ (where we assume that we already know $\frac{d \cos x}{dx} = -\sin x$)

$$= 2 \cos x(-\sin x)$$

$$= -2 \sin x \cos x$$

Putting this result into (2), we see that

$$\frac{dy}{dx} = 2 \sin x \cos x - 2 \sin x \cos x$$

or

$$\frac{dy}{dx} \equiv 0$$

The answer seems to indicate that we went through a lot of work for "nothing." In fact, since $\frac{dy}{dx} \equiv 0$ it means that y is a constant. With this as a hint we might now reexamine this problem and observe that $\sin^2 x + \cos^2 x \equiv 1$ (that is, $\sin^2 x + \cos^2 x$ is nothing more than a "hard way" of writing 1). Therefore, we could have written

$$y = \sin^2 x + \cos^2 x \longrightarrow$$

$$y = 1 \longrightarrow$$

$$\frac{dy}{dx} = 0$$

[3.2.1 (L) cont'd]

and we see once again, the difference between an equation and an identity.

(d) $y = \frac{1}{\cos x}$

Here, since we know how to differentiate cos x, we invoke the quotient rule to obtain:

$$\frac{dy}{dx} = \frac{\cos x \frac{d(1)}{dx} - (1)\frac{d}{dx}(\cos x)}{(\cos x)^2} = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

We could also have obtained this result by writing

$$y = \frac{1}{\cos x} = (\cos x)^{-1}$$

$$\therefore \frac{dy}{dx} = -1(\cos x)^{-2}\frac{d}{dx}(\cos x) = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}.$$

Quite in general, there will often be several ways to solving the same problem.

The major learning experience of this exercise is to emphasize the fact that, since the remaining trig functions are obtained from our knowledge of the sine and cosine functions, we can derive many results this way without recourse to memory. For example, this exercise is precisely how we would compute $\frac{d(\sec x)}{dx}$. Namely:

$$y = \sec x \longrightarrow$$

$$y = \frac{1}{\cos x} \longrightarrow \frac{d(\sec x)}{dx} = \frac{\sin x}{\cos^2 x}$$

* NOTE

[3.2.1 (L) cont'd]

If we wish to invoke some basic trigonometric identities, we can write:

$$\frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x \quad .$$

Hence: $\frac{d(\sec x)}{dx} = \sec x \tan x$, where the advantage, if any, of this form is that the entire expression can be written "on one line" without fractions or exponents.

(e) $y = 3 \sin 2x - 4 \cos 2x$

We have:

$$\frac{d}{dx} \sin 2x = \frac{d(\sin 2x)}{d(2x)} \frac{d(2x)}{dx} = 2 \cos 2x$$

$$\frac{d}{dx} \cos 2x = \frac{d(\cos 2x)}{d(2x)} \frac{d(2x)}{dx} = -2 \sin 2x$$

$$\therefore \frac{dy}{dx} = 3(2 \cos 2x) - 4(-2 \sin 2x)$$

$$= 6 \cos 2x + 8 \sin 2x$$

Aside from getting the answer, we would like to use this exercise to motivate a rather clever use of trigonometric identities.

Suppose we have $f(x) = A \sin mx + B \cos mx$ {where not both A and B are 0.

* The "trick" is that we multiply and divide by $\sqrt{A^2 + B^2}$.

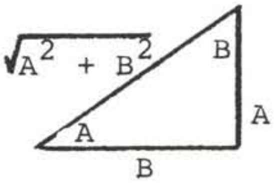
Then $f(x) = \sqrt{A^2 + B^2}\left[\frac{A}{\sqrt{A^2 + B^2}} \sin mx + \frac{B}{\sqrt{A^2 + B^2}} \cos mx\right]$.

The point is that $\frac{A}{\sqrt{A^2 + B^2}}$ and $\frac{B}{\sqrt{A^2 + B^2}}$ suggest the right triangle:

*

[3.2.1 (L) cont'd]

* this is a different B: angle associated with the side B



$$\cos \overset{*}{B} = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\sin \overset{*}{B} = \frac{B}{\sqrt{A^2 + B^2}}$$

Hence $f(x) = \sqrt{A^2 + B^2}\ [\cos B \sin mx + \sin B \cos mx]$

$$= \sqrt{A^2 + B^2} \sin(mx + B)$$

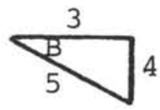
(since $\sin(mx + B) = \sin mx \cos B + \cos mx \sin B$) .

In our particular exercise, A = 3, B = -4, and m = 2. Thus,

$$y = 3 \sin 2x - 4 \cos 2x$$

$$= 5[\tfrac{3}{5} \sin 2x - \tfrac{4}{5} \cos 2x]$$

$$= 5 \sin(2x + B) \text{ where } \sin B = -\tfrac{4}{5},\ \cos B = +\tfrac{3}{5}$$



$$\therefore \frac{dy}{dx} = 5 \cos(2x + B)[\tfrac{d}{dx}(2x + B)]$$

$$= 10 \cos(2x + B)$$

$$= 10[\cos 2x \cos B - \sin 2x \sin B]$$

$$= 10[\cos 2x(\tfrac{3}{5}) - \sin 2x(-\tfrac{4}{5})]$$

$= 6 \cos 2x + 8 \sin 2x$ which agrees with our previously-obtained answer).

Our main point is that $\sin(2x + B)$ <u>might</u> look more suggestive than $3 \sin 2x - 4 \cos 2x$.

*

3.2.2

(a) $y = \tan x = \frac{\sin x}{\cos x}$

$$\therefore \frac{dy}{dx} = \frac{d\left(\frac{\sin x}{\cos x}\right)}{dx} = \frac{\cos x \frac{d(\sin x)}{dx} - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{(\cos x)(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

(b) $y = \tan(3x^2 + 4)$

$$\frac{dy}{dx} = \frac{d\tan(3x^2+4)}{d(3x^2+4)} \frac{d(3x^2+4)}{dx}$$

$$= [\sec^2(3x^2+4)]6x$$

$$= 6x\sec^2(3x^2+4) \quad \text{or} \quad \frac{6x}{\cos^2(3x^2+4)}$$

(c) (1) $y = 2\sin x \cos x$

$$\frac{dy}{dx} = 2\left[\sin x \frac{d\cos x}{dx} + \frac{d\sin x}{dx} \cdot \cos x\right]$$

$$= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)]$$

[3.2.2 cont'd]

$$= 2[\cos^2 x - \sin^2 x]$$

(2) $y = \sin 2x$

$$\frac{dy}{dx} = \frac{d(\sin 2x)}{d(2x)} \frac{d(x)}{x}$$

$$= 2 \cos 2x$$

(3) Since $\cos 2x = \cos^2 x - \sin^2 x$ we see that

$$\frac{d(\sin 2x)}{dx} \equiv \frac{d(2 \sin x \cos x)}{dx}$$

This is as it should be since $\sin 2x \equiv 2 \sin x \cos x$

(d) $y^3 = \sin^3 x + \cos^3 x$

By implicit differentiation,

$$\frac{d(y^3)}{dx} = \frac{d}{dx}(\sin x)^3 + \frac{d}{dx}(\cos x)^3$$

$$3y^2 \frac{dy}{dx} = 3 \sin^2 x \frac{d(\sin x)}{\cos x} + 3 \cos^2 x \frac{d(\cos x)}{dx}$$

$$y^2 \frac{dy}{dx} = \sin^2 x \cos x - \cos^2 x \sin x$$

$$\frac{dy}{dx} = \frac{\sin x \cos x (\sin x - \cos x)}{y^2}$$

3.2.3

$$\frac{d}{dx} \sin^2 x = \frac{d(u^2)}{dx}, \quad u = \sin x$$

$$= 2u \frac{du}{dx} = 2 \sin x \cos x$$

[3.2.3 cont'd]

$$\frac{d}{dx}\cos^2 x = 2\cos x\left(\frac{d\cos x}{dx}\right) = -2\sin x\cos x$$

$$\therefore \frac{d}{dx}(-\cos^2 x) = 2\sin x\cos x$$

By the corollary to the mean value theorem all we can conclude is that

$$\sin^2 x = -\cos^2 x + c \qquad (1)$$

We can<u>not</u> conclude that $\sin^2 x = -\cos^2 x$. All we can conclude is that the two functions differ by at most a constant.

If we now recall that $\sin^2 x + \cos^2 x \equiv 1$, then $\sin^2 x \equiv -\cos^2 x + 1$ and this is compatible with (1) merely by letting $C = 1$.

In terms of integrals, what this says is that

$$\left.\begin{array}{l}\int 2\sin x\cos x\,dx = \sin^2 x + C_1 \\ \text{and} \\ \int 2\sin x\cos x\,dx = -\cos^2 x + C_2\end{array}\right\}$$

We must <u>not</u> conclude that $\sin^2 x = -\cos^2 x$ since C is just a generic name for an arbitrary constant.

It would have been better to write:

$$\int 2\sin x\cos x\,dx = \sin^2 x + C_1 \qquad (2)$$

$$\int 2\sin x\cos x\,dx = -\cos^2 x + C_2 \qquad (3)$$

We could then write:

[3.2.3 cont'd]

$$\int 2 \sin x \cos x \, dx = \sin^2 x + c_1$$

$$= 1 - \cos^2 x + c_1$$

$$= -\cos^2 x + (1 + c_1)$$

$$= -\cos^2 x + c_2$$

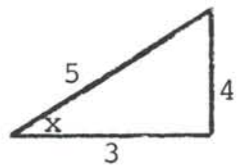
thus showing that (2) and (3) are compatible if $c_2 = 1 + c_1$.

3.2.4

(a) $y = 6 \cos x + 8 \sin x \qquad (0 \leqslant x \leqslant \frac{\pi}{2}) \qquad (1)$

$$\frac{dy}{dx} = -6 \sin x + 8 \cos x$$

$$\frac{dy}{dx} = 0 \longleftrightarrow \frac{\sin x}{\cos x} = \frac{8}{6} \longleftrightarrow \tan x = \frac{4}{3}$$



(Figure 1)

Now the above reference triangle shows that if $\tan x = \frac{4}{3}$ $(0 \leqslant x \leqslant \frac{\pi}{2})$ then $\sin x = \frac{4}{5}$ and $\cos x = \frac{3}{5}$. Thus (1) becomes:

$$y = 6(\tfrac{3}{5}) + 8(\tfrac{4}{5}) = \frac{18}{5} + \frac{32}{5} = 10 \quad .$$

Hence $y_{max} = 10$.

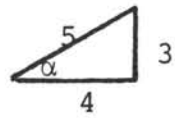
*reference III 2.6

[3.2.4 cont'd]

(b) $y = 10[\frac{6}{10} \cos x + \frac{8}{10} \sin x]$

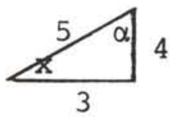
$= 10[\frac{3}{5} \cos x + \frac{4}{5} \sin x]$

* $= 10[\sin \alpha \cos x + \cos \alpha \sin x]$ where



$= 10 \sin(\alpha + x)$

but $-1 \leqslant \sin(\alpha + x) \leqslant 1$ implies that $-10 \leqslant y \leqslant 10$ and $y_{max} = 10$ and occurs when $\alpha + x = \frac{\pi}{2}$ or $x = \frac{\pi}{2} - \alpha$. The fact that $x = \frac{\pi}{2} - \alpha$ is seen at once from the fact that Figure 1 and Figure 2 can be combined as



3.2.5 (L)

(a) The "trick" here hinges on the fact that $d(\sin t) = \cos t\, dt$. Thus:

$$\int \sin^2 t \cos t\, dt = \int \sin^2 t\, d(\sin t) \quad ,$$

and our last integral is of the form $\int (\ )^2 d(\ )$ which is $\frac{1}{3}(\ )^3 + C$. Hence,

$$\int \sin^2 t \cos t\, dt = \int \sin^2 t\, d(\sin t) = \frac{1}{3} \sin^3 t + C \quad .$$

Quite in general, for $n \neq -1$,

$$\int \sin^n t \cos t\, dt = \int \sin^n t\, d(\sin t) = \frac{1}{n+1} \sin^{n+1} t + C \quad .$$

[3.2.5 (L) cont'd]

Notice that while $\int \sin^n t\, dt$ may seem simpler than $\int \sin^n t \cos t\, dt$, the factor cos t is what allows us to transform the integral into the form $\int u^n\, du$ by letting u = sin t.

(Notice also that if one is tempted to say $\int \sin^2 t\, dt = \frac{1}{3}\sin^3 t + C$ a quick check shows this to be incorrect. Namely:

$$\frac{d}{dt}[\tfrac{1}{3}\sin^3 t] = \sin^2 t \frac{d(\sin t)}{dt} = \sin^2 t \cos t$$

not $\sin^2 t$.)

(b) Here we let u = cos t, whereupon du = -sin t dt. Hence

$$\int \cos^3 t \sin t\, dt = \int u^3(-du)$$

$$= -\int u^3\, du$$

$$= -\frac{1}{4}u^4 + C$$

$$= -\frac{1}{4}\cos^4 t + C$$

Again, the substitution y = cos t allows us to transform

$$\int \cos^n t \sin t\, dt \qquad (n \neq -1)$$

into:

$$\int -u^n\, du \quad .$$

Whence:

$$\int \cos^n t \sin t\, dt = -\frac{\cos^{n+1} t}{n+1} + C \qquad (n \neq -1) \quad .$$

[3.2.5 (L) cont'd]

(c) Here we invoke the fact that the integral of a sum is the sum of the integrals. Hence,

$$\int (\sin^2 t \cos t + \cos^3 t \sin t)dt = \int \sin^2 t \cos t\, dt + \int \cos^3 t \sin t\, dt$$

and this, in turn, from (a) and (b) yields

$$\int (\sin^2 t \cos t + \cos^3 t \sin t)dt = \frac{1}{3} \sin^3 t - \frac{1}{4} \cos^4 t + C \quad .$$

(Actually: $= (\frac{1}{3} \sin^3 t + C_1) + (-\frac{1}{4} \cos^4 t + C_2)$, but, since C_1 and C_2 are arbitrary constants, so also is $C_1 + C_2 = C$.)

(d) Here we again observe that $d(2 + \cos 2x) = -2 \sin 2x\, dx$ and this should tempt us to try the substitution

$$\begin{cases} u = 2 + \cos 2x \\ du = -2 \sin 2x\, dx \quad \text{or} \quad \sin 2x\, dx = -\frac{1}{2}\, du \quad . \end{cases}$$

Hence:

$$\int \frac{\sin 2x\, dx}{\sqrt{2 + \cos 2x}} = \int \frac{-\frac{1}{2}\, du}{u^{1/2}} = -\frac{1}{2} \int u^{-1/2}\, du$$

$$= -\frac{1}{2}[2u^{1/2} + C_1]$$

$$= -u^{1/2} + C$$

$$= -\sqrt{2 + \cos 2x} + C \quad .$$

(Notice here that we did <u>not</u> let $u = \sqrt{2 + \cos 2x}$, rather we let $u = 2 + \cos 2x$.)

3.2.6

(a) We know that $\int \sin(\ )d(\ ) = -\cos(\ ) + C$ since $\frac{d[-\cos(\cdot)]}{d(\ )}$ $= \sin(\ )$. Hence, $\int \sin(x^2)d(x^2) = -\cos(x^2) + C$

$$\therefore \int \sin(x^2)[2x\ dx] = -\cos(x^2) + C$$

$$\therefore 2\int x \sin(x^2)dx = -\cos(x^2) + C$$

$$\therefore \int x \sin(x^2)dx = -\frac{1}{2}\cos(x^2) + C$$

(b) Here, we let $u = \sin x$ then $du = \cos x\ dx$. Hence:

$$\int \sin^4 x \cos x\ dx = \int u^4\ du = \frac{1}{5}u^5 + C$$

$$= \frac{1}{5}\sin^5 x + C$$

(Another method is:

$$\int \sin^4 x\ d(\sin x) = \frac{1}{5}\sin^5 x + C$$

$$\therefore \int \sin^4 x \cos x\ dx = \frac{1}{5}\sin^5 x + C$$

(c) The answer to (b) rules out the possibility that

$$\int \sin^2 x\ dx = \frac{1}{3}\sin^3 x + C \quad .$$

Here, we must invoke the identity $\sin^2 x = 1 - \frac{\cos 2x}{2}$. Then

$$\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2}\ dx$$

$$= \frac{1}{2}\int (1 - \cos 2x)\ dx$$

[3.2.6 cont'd]

$$= \frac{1}{2}[x - \frac{1}{2}\sin 2x] + C$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(Check: $\frac{d}{dx}[\frac{x}{2} - \frac{\sin 2x}{4}] = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}(\cos^2 x - \sin^2 x)$

$$= \frac{1}{2} - \frac{1}{2}(1 - 2\sin^2 x)$$

$$= \frac{1}{2} - \frac{1}{2} + \sin^2 x$$

$= \sin^2 x$.)

(d) $$\int \frac{\sec^2 x\, dx}{\tan^3 x} = \int \frac{d(\tan x)}{\tan^3 x}$$

$$= \int (\tan^{-3}) d(\tan x)$$

$$= -\frac{1}{2}\tan^{-2} x + C$$

$$= \frac{-1}{2\tan^2 x} + C$$

3.2.7

This problem shows us that the mean value theorem still holds as before.

Namely, if $f(x) = \sin x$ and $a < b$ then f is continuous on $[a,b]$ and differentiable in (a,b). Hence, there exists c such that $a < c < b$ and

$$\frac{f(b) - f(a)}{b - a} = f'(c) .$$

That is:

$$\frac{\sin b - \sin a}{b - a} = f'(c)$$

[3.2.7 cont'd]

$$\therefore \left|\frac{\sin b - \sin a}{b - a}\right| \quad \left(= \frac{|\sin b - \sin a|}{|b - a|}\right) \quad = |\cos c|$$

Now for <u>any</u> real number c, $-1 \leqslant \cos c \leqslant 1$ $\therefore |\cos c| \leqslant 1$

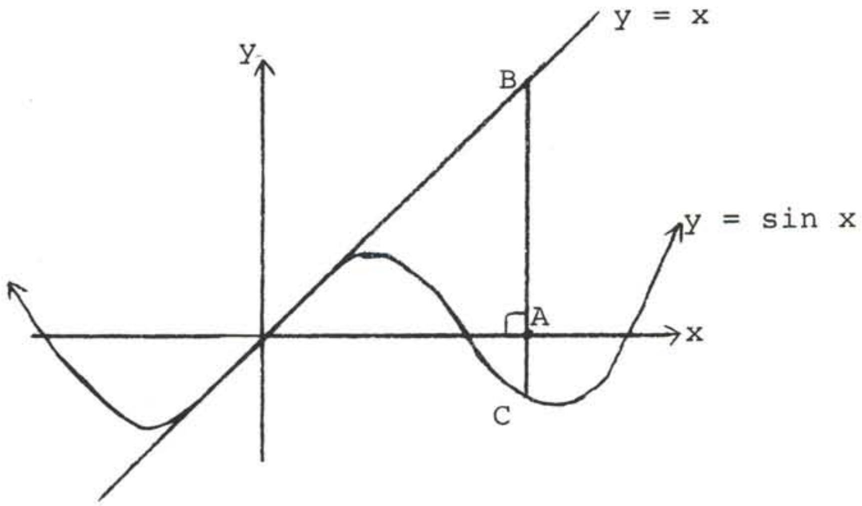
$$\therefore \frac{|\sin b - \sin a|}{|b - a|} \leqslant 1$$

$$\therefore |\sin b - \sin a| \leqslant |b - a| \qquad \text{q.e.d.}$$

As a corollary to this exercise, we may let a = 0. Then sin a = 0 and the result becomes:

$$|\sin b| \leqslant |b|$$

Pictorially,



For any point A on the x-axis the distance from A to C can never exceed the distance from A to B.

Another way of obtaining the same result is to let

[3.2.7 cont'd]

$$f(x) = x - \sin x \quad (1)$$

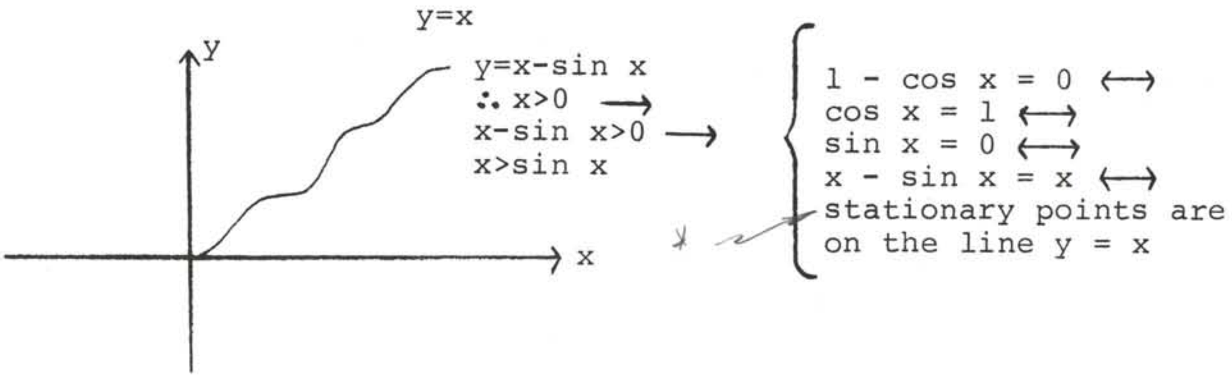
From (1) it follows:

$$f'(x) = 1 - \cos x \quad (2)$$

and

$$f''(x) = \sin x \quad (3)$$

Since $-1 \leqslant \cos x \leqslant 1$, it follows that $0 \leqslant 1 - \cos x \leqslant 2$. From (2) this tells us that the curve $y = x - \sin x$ is never falling. From (1), $f(0) = 0$. Hence the curve $y = x - \sin x$ looks like



Quite in general a common device for proving the $f(x) \geqslant g(x)$ for all x is to show that the curve $y = f(x) - g(x)$ can never be below the x-axis.

A final observation here is that we have already solved this problem more generally in Block II when we showed that $|f(b) - f(a)| \leqslant |b - a|$ if $|f'(x)| \leqslant 1$.

3.2.8

Observe that the basic definitions are the same as before. For example:

$$a_x = \frac{d^2x}{dt^2} \quad \text{and} \quad a_y = \frac{d^2y}{dt^2} \quad .$$

Again, all that's new, compared with previous sections, is that we can now differentiate sines and cosines.

Thus:

$$x = a \cos \omega t \longrightarrow \qquad (1)$$

$$\frac{dx}{dt} = -a\omega \sin \omega t \longrightarrow$$

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos \omega t \longrightarrow \qquad (2)$$

$$a_x = -a\omega^2 \cos \omega t = -\omega^2(a \cos \omega t) = -\omega^2 x \text{ [from (1)]} \quad .$$

Similarly

$$y = b \sin \omega t \longrightarrow \qquad (3)$$

$$\frac{dy}{dt} = b\omega \cos \omega t \longrightarrow$$

$$a_y = \frac{d^2y}{dt^2} = -b\omega^2 \sin \omega t = -\omega^2(b \sin \omega t) = \underline{\underline{-\omega^2 y}} \quad \text{[by (3)]} \quad .$$

Notice also that we can analyze the path of the particle just as in previous sections - only now we know a few trigonometric facts as well.

For instance:

[3.2.8 cont'd]

$$x = a\cos\omega t \longrightarrow \cos\omega t = \frac{x}{a} \longrightarrow \frac{x^2}{a^2} = \cos^2\omega t \qquad (4)$$

$$y = b\sin\omega t \longrightarrow \sin\omega t = \frac{y}{b} \longrightarrow \frac{y^2}{b^2} = \sin^2\omega t \qquad (5)$$

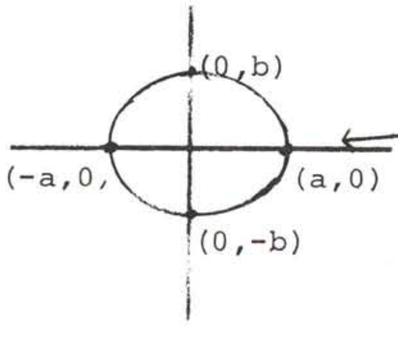
Combining (4) and (5), we obtain:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\omega t + \sin^2\omega t = 1$$

∴The equation of the path of the particle is:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

which happens to be an ELLIPSE



when $t=0$, $x=a\cos\omega t \rightarrow x=a\cos 0 \rightarrow x=a$

$y=b\sin\omega t \rightarrow y=b\sin 0 \rightarrow y=0$

so the particle starts at (a,0)

when $t=\frac{\pi}{2\omega}$, $x=a\cos\frac{\pi}{2} \rightarrow x=0$

$y=b\sin\frac{\pi}{2} \rightarrow y=b$

So at $t=\frac{\pi}{2\omega}$ particle is at (0,b)

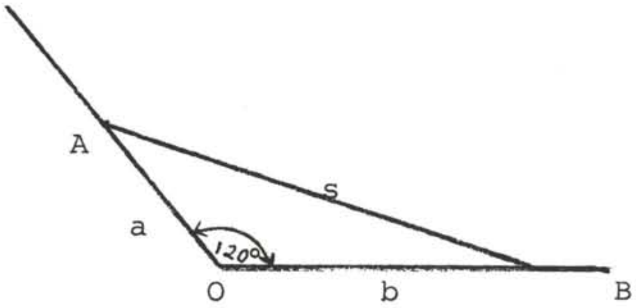
∴particle is moving counterclockwise (if ω is positive).

3.2.9

This exercise shows how the law of cosines plays an important analytical role in certain types of problems. In particular (see diagram below) in this exercise we wish to find ds/dt knowing

[3.2.8 cont'd]

a,b,da/dt,db/dt, and s at a given instant.



$$s^2 = a^2 + b^2 - 2ab\cos 120°$$

$$\cos 120° = -\cos 60° = -\frac{1}{2}$$

$$\therefore s^2 = a^2 + b^2 + ab \qquad (1)$$

If we now differentiate (1) implicitly with respect to t, we obtain:

$$2s\frac{ds}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt} \qquad (2)$$

(While we may not be used to seeing letters like a and b denote variables, the fact is that we are only assuming that a and b are both differentiable functions of t. Among other things, this is why we must use the product rule when we differentiate ab.)

Now, at the instant in question, we have that a = 8, da/dt = 20, b = 6, and db/dt = 30. Knowing a and b, we can find s from (1). Namely,

$$s^2 = 64 + 36 + 48 = 148 = 4(37)$$

therefore:

$$s = 2\sqrt{37}\ .$$

[3.2.8 cont'd]

With this additional piece of information, (2) becomes $4\sqrt{37}\frac{ds}{dt} = 2(8)(20) + 2(6)(30) + 8(30) + 6(20)$ whence $\frac{ds}{dt} = \frac{260}{\sqrt{37}} \approx 43$ mph.

Notice that, except for the law of cosines, we could have solved this exercise prior to Block III as a related rates problem. In fact notice that in Block II we solved a more special case of this exercise wherein $\measuredangle AOB = 90°$ whereupon we were able to use the Pythagorean Theorem.

UNIT 3: The Inverse Circular Functions

3.3.1 (L)

(a) $A = \sin^{-1}(-\frac{3}{5})$

This, by definition, says that $\sin A = -\frac{3}{5}$ where $-\frac{\pi}{2} \leqslant A \leqslant \frac{\pi}{2}$. Since $A \geqslant 0 \rightarrow \sin A \geqslant 0$, we have that $\sin A = -\frac{3}{5} \rightarrow -\frac{\pi}{2} \leqslant A < 0$. We could now proceed analytically and observe that since $\sin^2 A + \cos^2 A = 1$, $\cos^2 A = 1 - (-\frac{3}{5})^2 = \frac{16}{25}$.

Thus $\cos A = \pm\frac{4}{5}$. We can next discard the minus sign since we know that $-\frac{\pi}{2} \leqslant A < 0$ and that in this interval $\cos A \geqslant 0$.

Hence:

$$\cos A = \frac{4}{5} \quad . \tag{1}$$

Notice that in deriving this result it was <u>crucial</u> that we realize that $f(x) = \sin^{-1}x$ implies the restriction that $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. If this restriction is not imposed, then $\sin^{-1}x$ is multi-valued and, among other things, the fact that $\sin x = -\frac{3}{5}$ could imply that $\pi < x < \frac{3\pi}{2}$ in which case $\cos x = -\frac{4}{5}$ <u>not</u> $\frac{4}{5}$. In other words, to reinforce our remarks in the supplementary notes, $\sin^{-1}x$ is <u>not</u> the inverse of $\sin x$ but rather of $S_1(x)$ where S_1 is defined by $S_1(x) = \sin x$ for all x in the domain of $S_1 = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

At any rate, once (1) is obtained the rest falls out. Namely,

$$\tan A = \frac{\sin A}{\cos A} = (-\tfrac{3}{5})/(\tfrac{4}{5}) = -\frac{3}{4}$$

$$\csc A = \frac{1}{\sin A} = -\frac{5}{3}$$

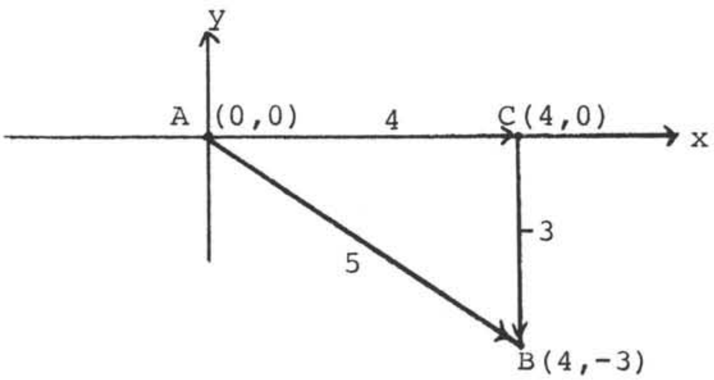
$$\sec A = \frac{1}{\cos A} = +\frac{5}{4}$$

$$\cot A = \frac{1}{\tan A} = -\frac{4}{3} \quad .$$

[3.3.1 (L) cont'd]

As a final remark, let us use this example as an illustration of the use of radian measure. In this exercise, A is a number (although we will admit that the symbol A probably reminds us more of an angle than a number). However, it we wish to view A as an angle then, as our theme has been all along, we must view it as measured in radians if we want sin A to be unambiguously defined regardless of whether we view A as a number or as an angle.

Thus we have $\sin A(\text{radians}) = -\frac{3}{5}$ where $-\frac{\pi}{2} < A < \frac{\pi}{2}$. Hence, A is a fourth quadrant angle and we have:



whereupon we can read all the relationships directly from the diagram. Notice that even with the diagram we are using the fact that $-\frac{\pi}{2} \leq A \leq \frac{\pi}{2}$, otherwise we could not conclude from the fact that if $\sin A = -\frac{3}{5}$, then A is in the fourth quadrant. In general, $\sin A < 0$ merely implies that A terminates in either the 2nd or 4th quadrant.

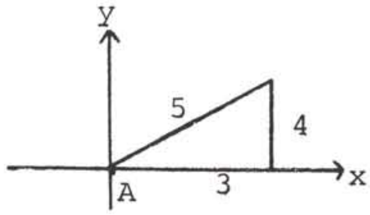
(b) We have $\sin(2\sin^{-1}0.8)$

To translate this into an easier picture to see, we may choose an angle A (measured in radians) such that $\sin A = 0.8 = \frac{4}{5}$ and $-\frac{\pi}{2} \leq A \leq \frac{\pi}{2}$. If we do this, then by definition $\sin^{-1}0.8 = A$. Our problem becomes:

[3.3.1 (L) cont'd]

$\sin(2A)$ where $-\frac{\pi}{2} \leq A \leq \frac{\pi}{2}$

and $\sin A = \frac{4}{5}$



A terminates in first quadrant since $\sin A > 0 \rightarrow A > 0 \rightarrow 0 < A \leq \frac{\pi}{2}$ since $-\frac{\pi}{2} \leq A \leq \frac{\pi}{2}$.

Now:

$$\sin 2A = 2 \sin A \cos A$$

$$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) \quad \text{(from our diagram, or from } \cos A = \sqrt{1 - \sin^2 A}\text{)}$$

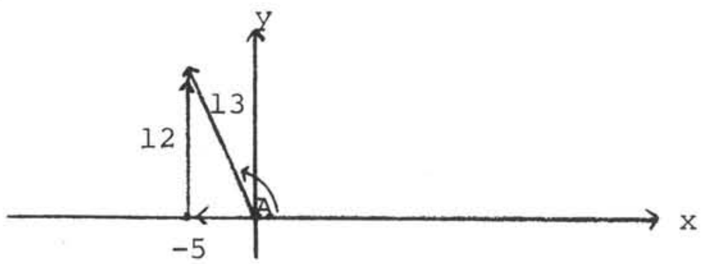
$$= \frac{24}{25} \quad .$$

Finally resubstituting $\sin^{-1} 0.8$ for A, we obtain

$$\sin(2 \sin^{-1} 0.8) = \frac{24}{25} \quad .$$

3.3.2

(a) $\cos^{-1}\left(-\frac{5}{13}\right) = A \rightarrow \cos A = -\frac{5}{13}$ and $0 \leq A \leq \pi$. Since $\cos A < 0$, it follows that $\frac{\pi}{2} < A \leq \pi$. Thus if A is measured in radians, we have:

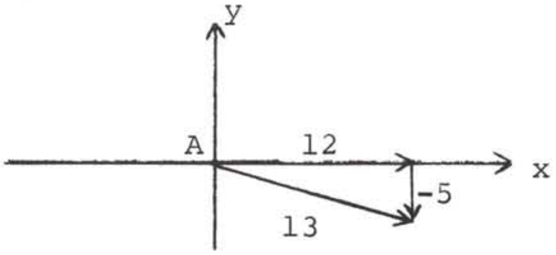


From the diagram

$$\tan A = \frac{12}{-5} \text{ , } \sin A = \frac{12}{13} \quad .$$

[3.3.2 cont'd]

(b) If we are given $\cos(2\sin^{-1}(-\frac{5}{13}))$ we may let $A = \sin^{-1}(-\frac{5}{13})$ which tells us:



$$\begin{cases} \sin A = -\frac{5}{13} \\ \cos A = \frac{12}{13} \end{cases}$$

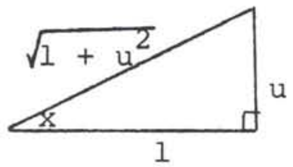
$$\cos(2\sin^{-1}(-\tfrac{5}{13})) = \cos 2A$$

$$= \cos^2 A - \sin^2 A$$

$$= (\tfrac{12}{13})^2 - (\tfrac{-5}{13})^2$$

$$= \frac{119}{169}$$

(c) We have $\tan^{-1}u = x$, hence $\tan x = u$ where $-\frac{\pi}{2} < x < +\frac{\pi}{2}$. Thus, pictorially, if x is measured in radians



$$\begin{cases} \tan x = u \longrightarrow \\ \sin x = \dfrac{u}{\sqrt{1 + u^2}} \\ \cos x = \dfrac{1}{\sqrt{1 + u^2}} \end{cases}.$$

(Note: In the diagram above, we have drawn the picture as if x is a first quadrant angle. Certainly u could be negative. This would place x in the 4th quadrant. In this event $\frac{u}{\sqrt{1 + u^2}} = \sin x$ would still be correct since both $\frac{u}{\sqrt{1 + u^2}}$ and $\sin x$ would be negative, as

[3.3.2 cont'd]

would $\cos x = \dfrac{1}{\sqrt{1+u^2}}$ since each number is then positive. The point is that by our choice of principal values we may always refer to a first quadrant angle without loss of generality.)

Then,

$$\sin 2x = 2 \sin x \cos x$$

$$= 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right)$$

$$= \frac{2u}{1+u^2}$$

(Stated analytically without reference to a diagram, we have:

$$\tan^{-1}u = x \longrightarrow \sin 2x = \sin(2\tan^{-1}u) = 2\sin(\tan^{-1}u)\cdot\cos(\tan^{-1}u)$$

$$u = \tan x \longrightarrow u^2 + 1 = \sec^2 x \longrightarrow \sec x = \pm\sqrt{1+u^2} \longrightarrow \cos x = \pm\frac{1}{\sqrt{1+u^2}}$$

and $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \longrightarrow \cos x > 0 \longrightarrow \cos x (= \cos(\tan^{-1}u)) = \dfrac{1}{\sqrt{1+u^2}}$.

Then $\sin(\tan^{-1}u) = \sin x = (\tan x)(\cos x) = u\left(\dfrac{1}{\sqrt{1+u^2}}\right) = \dfrac{u}{\sqrt{1+u^2}}$, etc.)

3.3.3 (L)

(a) We are given $y = \tan^{-1}x$ and we wish to determine $\frac{dy}{dx}$. Since we assume that we already know how to differentiate tan u with respect to u (if we didn't, it would hardly be of value to "fool" with arctan), we rewrite the given problem equivalently as

$$x = \tan y \quad , \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \quad .$$

[3.3.3 (L) cont'd]

Then,

$\frac{dx}{dy} = \sec^2 y$ (we can either invoke the "recipe" or we can write tan y $\frac{\sin y}{\cos y}$ and use the quotient rule.)

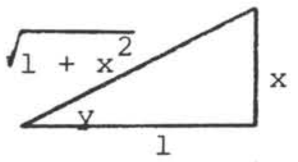
Since $\frac{dy}{dx}$ is the reciprocal of $\frac{dx}{dy}$, we can, in turn, write:

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y \quad .$$

To put our answer in terms of x (this in not necessarily required but whenever y is given in the form y = f(x) we would like to express $\frac{dy}{dx} = f'(x)$, also, explicitly as a function of x), we could write:

$$\frac{dy}{dx} = \cos^2 y \quad \text{where} \quad y = \tan^{-1} x \quad . \qquad (1)$$

This answer can be made even more explicit if we employ the techniques of the previous two exercises. Namely, $y = \tan^{-1} x \longrightarrow x = \tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2} \rightarrow$



$$\therefore \cos y = \frac{1}{\sqrt{1 + x^2}}$$

$$\text{or } \cos^2 y = \frac{1}{1 + x^2}$$

Putting this into (1), we obtain that if $y = \tan^{-1} x$ then $\frac{dy}{dx} = \frac{1}{1 + x^2}$.

(b) We are given that $\frac{dy}{dx} = \frac{1}{1 + x^2}$. Hence, by part (a), review trig substitutions

[3.3.3 (L) cont'd]

$y = \tan^{-1}x + C$. Next since $y = 3$ when $x = 0$, we see that:

$$3 = \tan^{-1}0 + C$$ arctan 0

or $$3 = 0 + C$$

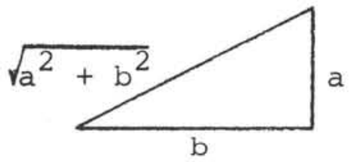
or $$3 = C \quad .$$

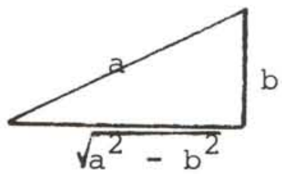
Therefore, the desired f is given by:

$$f(x) = \tan^{-1}x + 3 \quad .$$

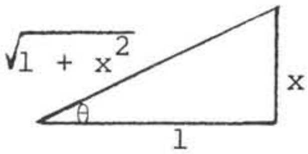
There are a few main points we would like to make with regard to this example:

(1) While it is convenient to remember that $\frac{d(\tan^{-1}x)}{dx}$ is $\frac{1}{1 + x^2}$ when we want to compute $\int \frac{dx}{1 + x^2}$, it is not necessary to memorize this result. The main idea is that expressions such as $a^2 + b^2$ and $a^2 - b^2$ should, by the Pythagorean Theorem, suggest right triangles. For example,





So given $\int \frac{dx}{1 + x^2}$, $1 + x^2$ might suggest the reference triangle:



and this, in turn might suggest the transformation $\tan\theta = x$ (or $\theta = \tan^{-1}x$). The diagram, of course, also suggests $\cos\theta = \frac{1}{\sqrt{1 + x^2}}$

[3.3.3 (L) cont'd]

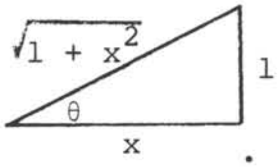
etc., but, computationally, $\tan\theta = x$ is the simplest for computing dx in terms of θ.

In any event, letting $x = \tan\theta$, we obtain: $dx = \sec^2\theta d\theta$ while $\sec\theta = \sqrt{1+x^2}$; hence, $1 + x^2 = \sec^2\theta$. We then obtain:

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2\theta d\theta}{\sec^2\theta} = \int d\theta = \theta + C \quad \text{where } \theta = \tan^{-1}x$$

$$\therefore \int \frac{dx}{1+x^2} = \tan^{-1}x + C \quad . \tag{3}$$

The diagram also could have been labeled:

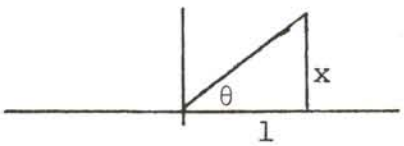


Whence the suggested substitution would have been $x = \cot\theta$. Then $dx = -\csc^2\theta d\theta$ while $\csc^2\theta = 1 + x^2$.

$$\therefore \int \frac{dx}{1+x^2} = \int \frac{-\csc^2\theta d\theta}{\csc^2\theta} = -\theta + C = -\cot^{-1}x + C \quad . \tag{4}$$

This leads to our second point.

(2) At first glance $\tan^{-1}x + C$ and $-\cot^{-1}x + C$ might not look like equivalent statements. However it should be noted that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$. Since, viewed as acute angles, $\tan^{-1}x$ and $\cot^{-1}x$ are complementary angles. Pictorially, $\tan^{-1}x = \theta \longrightarrow$



* " Knowing the sine of an angle determines the magnitude of
** the cosine " (determines the cosine "up to" the sign)

[3.3.3 (L) cont'd]

This in turn implies that $\cot(\frac{\pi}{2} - \theta) = x$ and hence that

$$\frac{\pi}{2} - \theta = \cot^{-1} x$$

$$\therefore \frac{\pi}{2} - \tan^{-1} x = \cot^{-1} x \quad (\text{since } \theta = \tan^{-1} x)$$

$$\therefore \tan^{-1} + \cot^{-1} x = \frac{\pi}{2} \quad .$$

(The fact that the trigonometric identity $\tan \theta = \cot(\frac{\pi}{2} - \theta)$ holds for any θ allows us to remove the restriction that θ be an acute angle.)

In a similar way, it can be shown that if f denotes any trigonometric function and if co-f denotes the complimentary function then

$$f^{-1}(x) + \text{co-}f^{-1}(x) = \frac{\pi}{2} \quad .$$

In any event we then have

$$\tan^{-1} x + C = \frac{\pi}{2} - \cot^{-1} x + C = -\cot^{-1} x + (\frac{\pi}{2} + C)$$

$$= -\cot^{-1} x + C_1 \quad \text{where} \quad C_1 = \frac{\pi}{2} + C \quad .$$

* differing C's

Hence, since C denotes an arbitrary constant, we may write $\tan^{-1} x + C = -\cot^{-1} x + C_1$. The point being that the inverse trigonometric identities may allow us to get equivalent answers which look very different to us.

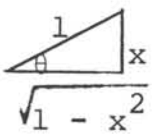
(3) It is very important to notice that in this example no knowledge of trigonometry was necessary for the statement of the problem. That is, $\frac{dy}{dx} = \frac{1}{1 + x^2}$ is an algebraic not a trigonometric relationship.

**

[3.3.3 (L) cont'd]

Yet the solutions of this exercise still required our "inventing" the inverse trigonometric functions. This is further corroboration of our claim that trigonometry transcends the mere applications to geometry. We shall write more of this in later exercises.

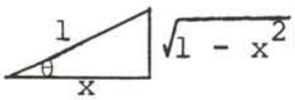
3.3.4

(a) $\int \frac{dx}{\sqrt{1 - x^2}}$ suggests 

which in turn suggests the substitution $\sin\theta = x$ (or $\theta = \sin^{-1}x$) . Then $\cos\theta\, d\theta = dx$ while $\sqrt{1 - x^2} = \cos\theta$. Hence:

$$\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta + C = \sin^{-1}x + C \quad .$$

We could also have labeled our triangle

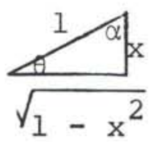


whereupon we would let $\cos\theta = x$ $(\theta = \cos^{-1}x)$.

Then $-\sin\theta\, d\theta = dx$ while $\sqrt{1 - x^2} = \sin\theta$.

Therefore

$$\int \frac{dx}{\sqrt{1 - x^2}} = \int -\frac{\sin\theta\, d\theta}{\sin\theta} = -\theta + C = -\cos^{-1}x + C \quad .$$

(b) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ since 

$\alpha + \theta = \frac{\pi}{2}$

and $\alpha = \cos^{-1}x$ while $\theta = \sin^{-1}x$.

[3.3.4 cont'd]

Hence, $\sin^{-1}x + C = \frac{\pi}{2} - \cos^{-1}x + C$

$$= -\cos^{-1}x + \left(\frac{\pi}{2} + C\right)$$

$$= -\cos^{-1}x + C_1$$ (since $\frac{\pi}{2} + C$ is still an arbitrary constant) .

(c)

$$y = \sin^{-1}\left(\frac{x-1}{x+1}\right) \longrightarrow \sin y = \frac{x-1}{x+1} \quad , \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Then, by implicit differentiation,

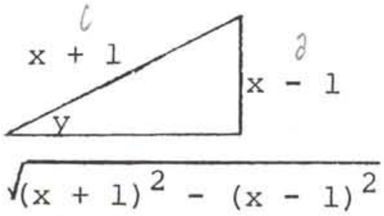
$$\cos y \frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d(x+1)}{dx}}{(x+1)^2}$$

$$= \frac{(x+1) - (x-1)}{(x+1)^2}$$

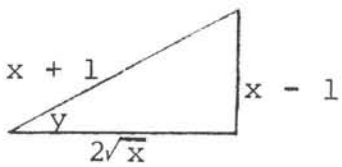
$$= \frac{2}{(x+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2 \sec y}{(x+1)^2} \quad \text{where} \quad y = \sin^{-1}\frac{x-1}{x+1} \quad . \qquad (1)$$

To make this answer into a form such that $\frac{dy}{dx}$ is explicitly a function of x, we have



or



[3.3.4 cont'd]

hence $\sec y = \dfrac{x + 1}{2\sqrt{x}}$ and putting this into (1), we obtain

$$\frac{dy}{dx} = \left[\frac{2}{(x + 1)^2}\right]\left[\frac{x + 1}{2\sqrt{x}}\right] = \frac{1}{\sqrt{x}(x + 1)} \quad .$$

(Our triangle would be in trouble if x + 1 were a shorter length than x - 1 which could happen if x were negative. In this case, $\left|\frac{x - 1}{x + 1}\right| > 1$ and this is impossible since $\left|\frac{x - 1}{x + 1}\right| = |\sin y| \leqslant 1$.)

3.3.5

$$v = \frac{dx}{dt} = \frac{1}{1 + t^2}$$

Hence, $x = \displaystyle\int \frac{dt}{1 + t^2}$

or $\quad x = \tan^{-1} t + C \quad .$ (1)

Now, when t = 0, x = 0. Therefore (1) becomes:

$$0 = \tan^{-1} 0 + C$$

$$0 = 0 + C$$

$$C = 0 \quad .$$

Thus:

$$x = \tan^{-1} t \qquad (2)$$

In particular when t = 1, $\underline{x = \frac{\pi}{4}}$.

[3.3.5 cont'd]

That is, the particle has travelled $\frac{\pi}{4}$ feet in 1 second. Note especially that the distance travelled is not an angle. In other words x is a number in (2). If we insist on an angular interpretation, then x must be in radians. It is incorrect to say that when $t = 1$, $x = 45°$.

Finally, notice again how the inverse trigonometric functions find their way into non-geometric situations.

3.3.6 (L)

(a) We have that $a = \frac{dv}{dt} = -9x$.

In the above form, we seem to have one variable too many. We, therefore, elect to write $\frac{dv}{dt}$ as $(\frac{dv}{dx})(\frac{dx}{dt}) = (\frac{dv}{dx})v$. This leads to

$$v(\frac{dv}{dx}) = -9x \quad . \qquad (1)$$

Treating (1) as a differential equation, we may separate the variables and obtain

$$vdv = -9xdx \qquad (2)$$

whereupon integration yields

$$\frac{v^2}{2} = \frac{-9x^2}{2} + c_1$$

or:

$$v^2 = -9x^2 + c \quad . \qquad (3)$$

We may then determine the constant c in (3) from the condition that when $x = 4$, $v = 0$. That is, (3) becomes

[3.3.6 (L) cont'd]

$$0 = -9(16) + c$$

whereupon c = 144, and, since c is constant, (3) says

$$v^2 = 144 - 9x^2 \qquad (4)$$

or:

$$v = \pm\sqrt{144 - 9x^2} \qquad (5)$$

and we cannot, without further specifications, select between the positive and negative root in (5) since, for example, when x = 0 the speed of the particle can be either 12 feet per second in the direction from right to left or in the direction from left to right.

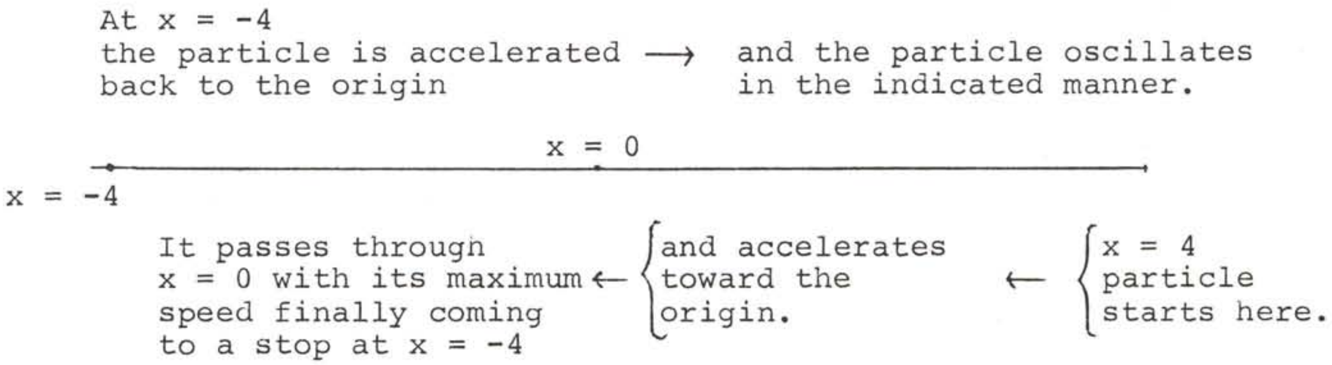
In any event (5) gives us an expression for v as a function of x and unless we wish to express the function in two parts (branches) we see that v is not a single valued function of x.

Equation (5) should be refined by the observation that it is only defined when $|x| \leq 4$, since if x exceeds four in magnitude, $144 - 9x^2$ will be negative and hence its square root will not be real.

While it is not necessary for obtaining the correct answer, a certain amount of physical feeling for this problem should give us a fair idea of what our answer means. Namely when v is 0 the particle is at rest, hence our conditions in this problem tell us that the particle is at rest when x = 4. Since the acceleration is proportional to the displacement of the particle but in the opposite direction (this is what a = -9x tells us), we conclude that the particle is always accelerated toward the origin. Thus the particle starts at x = 4 and accelerates back to x = 0. This means that the particle is gaining speed during this time but since the magnitude of x is

[3.3.6 (L) cont'd]

decreasing so is the acceleration of the particle. Once the particle passes through $x = 0$, the acceleration reverses direction and consequently the particle decelerates and finally comes to a stop, evidently at $x = -4$, since any lesser value of x makes equation (5) imaginary. The particle then repeats a similar trip. Pictorially,



(Figure 1)

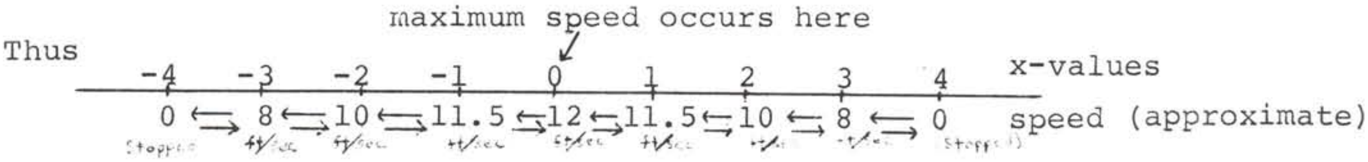
We can "sharpen" the feeling for what is happening in Figure 1 by putting in periodic "check point" speeds obtained from Equation 5.

For example equation (5) tells us that when:

$x = \pm 4$, $v = 0$; $x = \pm 3$, $v = \pm\sqrt{144 - 81} = \pm\sqrt{63} \approx \pm 8$; $x = \pm 2$, $v = \pm\sqrt{144 - 36} = \pm\sqrt{108} \approx \pm 10$

$x = \pm 1$, $v = \pm\sqrt{144 - 9} = \pm\sqrt{135} \approx 11.5$; and $x = 0$, $v = \pm 12$.

Thus



(Figure 2)

[3.3.6 (L) cont'd]

We could also, if we wished, plot v as a function of x from Equation (5). Clearly (5) is equivalent to

$$v^2 = 144 - 9x^2$$

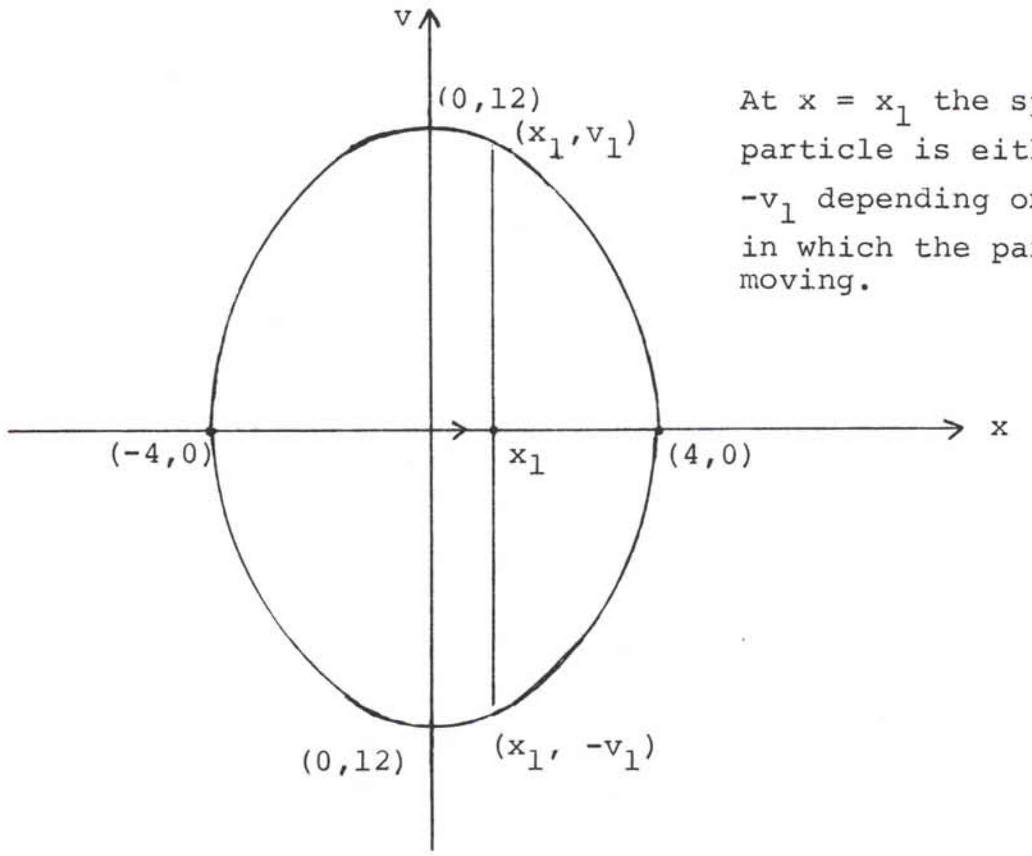
or

$$9x^2 + v^2 = 144$$

or

$$\frac{x^2}{16} + \frac{v^2}{144} = 1$$

which represents an ellipse.



At $x = x_1$ the speed of the particle is either $+v_1$ or $-v_1$ depending on the direction in which the particle is moving.

(Figure 3)

[3.3.6 (L) cont'd]

The major point of Figure 3 is that the ellipse is not the path followed by the particle. Obviously, the particle is moving along the x-axis. The ellipse allows us to see graphically the velocity of the particle at the point $x = x_1$. It is precisely the v-coordinate of the corresponding point on the curve.

(b) This part of the exercise is designed to show once again how the circular functions enter into the picture even though we seem not to be dealing with geometry and we would also like to show how calculus is used in conjunction with our intuition to help us formalize and quantify certain conjectures we might have.

To begin with, notice that our discussion of part (a) has already led us to suspect that the particle moves with oscillatory motion and we might even visualize that its graph seems to resemble a sine curve. Of course what it means to "resemble" a curve is another matter. (For example $y = x^2$ is a parabola; $y = x^4$ is not a parabola, yet, in a quick sketch, the two curves "seem to belong" to the same family.)

To get back to the specific aspects of this exercise, we return to Equation (5) and rewrite it as

$$\frac{dx}{dt} = \sqrt{144 - 9x^2} \qquad (6)$$

(where we have disregarded the negative root so that we utilize the results of calculus which require that we have single-valued functions. We are not discarding the negative root. Rather, once we have solved (6) it is simple to hangle the negative root by symmetry considerations).

At any rate, if we separate the variables in (6) we obtain

[3.3.6 (L) cont'd]

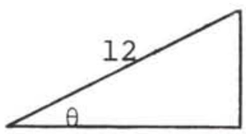
$$\frac{dx}{\sqrt{144 - 9x^2}} = dt$$

whence

$$\int \frac{dx}{\sqrt{144 - 9x^2}} = t + C_1 \quad . \qquad (7)$$

If we have not explicitly memorized how to evaluate $\int \frac{dx}{\sqrt{144 - 9x^2}}$ we can return to our reference triangle:

$\sec \theta = \frac{12}{(144 - 9x^2)^{1/2}}$



The least complicated transformation seems to be:

$$\sin \theta = \frac{3x}{12} = \frac{x}{4} \qquad \left(\text{or} \quad \theta = \sin^{-1} \frac{x}{4}\right) \quad .$$

Hence $4 \sin \theta = x$ or $dx = 4 \cos \theta \, d\theta$. Also

$$\cos \theta = \frac{\sqrt{144 - 9x^2}}{12} \longrightarrow \sqrt{144 - 9x^2} = 12 \cos \theta \quad .$$

Thus:

$$\int \frac{dx}{144 - 9x^2} = \int \frac{4 \cos \theta \, d\theta}{12 \cos \theta} = \frac{1}{3} \int d\theta = \frac{\theta}{3} + C_2 \text{ where}$$

$$\cot^{-1}\left(\frac{\sqrt{144-9x^2}}{3x}\right) = \theta = \sin^{-1} \frac{x}{4} = \tan^{-1}\left(\frac{3x}{(144-9x^2)^{1/2}}\right) = \cos^{-1}\left(\frac{\sqrt{144-9x^2}}{12}\right) \qquad (8)$$

Putting (8) into (7) and combining the arbitrary constants we obtain:

$$\sin^{-1} \frac{x}{4} = t + C_3 \qquad (9)$$

*note "placing" of "C"

[3.3.6 (L) cont'd]

or:

$$\frac{x}{4} = \sin(t + C_3) \quad \text{(by taking sine of both sides of equation(9))} \quad .$$

Hence,

$$x = 4 \sin (t + C_3) \quad . \tag{10}$$

The value of C_3 in (10) depends on the initial position of the particle. Notice that being told that the particle was stopped at x = 4 does not tell us at what time this occurs.

If, as in this exercise, we are told that x = 4 at t = 0 (that is, the particle starts at its maximum displacement), (10) yields:

$$4 = 4 \sin C_3 \quad .$$

Hence $\sin C_3 = 1$, whereupon $C_3 = \frac{\pi}{2}$. (Notice here that one-to-oneness is not necessary in the sense that $4 \sin(t + \frac{\pi}{2})$ and $4 \sin(t + \frac{5\pi}{2})$ etc. are identical.) We obtain as our answer to (b):

$$x = 4 \sin(t + \frac{\pi}{2}) \tag{11}$$

Since $\sin(t + \frac{\pi}{2}) = \sin t \cos \frac{\pi}{2} + \cos t \sin \frac{\pi}{2}$

$$= \sin t\ (0) + \cos t\ (1)$$

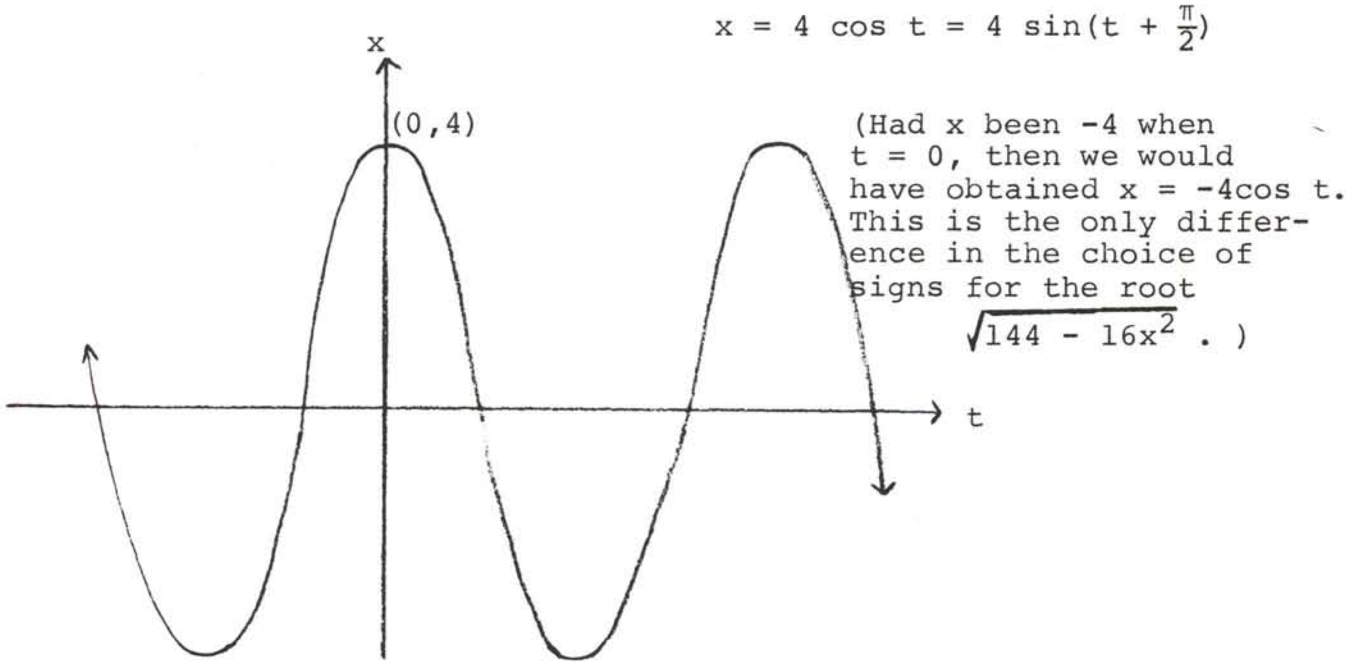
$$= \cos t,$$

equation (11) can be written as:

$$x = 4 \cos t \tag{12}$$

[3.3.6 (L) cont'd]

The graph of either (11) or (12) is given by



(Figure 4)

The result of this exercise is easily generalized. To begin with, any motion in which the acceleration is proportional to the displacement but in the opposite direction is known as <u>simple harmonic motion</u>. In the language of differential equation, we may express simple harmonic motion by:

$$\frac{d^2x}{dt^2} = -k^2x \qquad (13)$$

where we write k^2 to emphasize that $\frac{d^2x}{dt^2}$ and x have different signs.

[3.3.6 (L) cont'd]

Recall $\frac{d^2x}{dt^2} = -kx$ allows both x and $\frac{d^2x}{dt^2}$ to have the same sign if k is negative. In no case can k^2 be negative if k is real.

We then write $\frac{d^2x}{dt^2}$ as $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ to obtain from (13):

$$v\frac{dv}{dx} = -k^2x$$

and separating variables yields:

$$v\,dv = -k^2x\,dx$$

whereupon integration yields:

$$v^2 = -k^2x^2 + C_1 \quad . \qquad (14)$$

We can conclude form (14) that C_1 is non-negative since it is the sum of two squares. That is, $C_1 = (v)^2 + (kx)^2 \geqslant 0$ since a real square cannot be negative. To emphasize that C_1 is positive (non-negative) we will write it as $C_1 = C_2{}^2$, whence (14) becomes:

$$v^2 = C_2{}^2 - k^2x^2$$

or

$$v = \pm\sqrt{C_2{}^2 - k^2x^2} \qquad (15)$$

Equation (15) is the generalization of Equation (3) and $C_2{}^2$ cannot be determined unless we have some specific value for v at a given value for x.

We can then write (15) as

[3.3.6 (L) cont'd]

$$\frac{dx}{dt} = \pm\sqrt{C_2^{\,2} - k^2x^2}$$

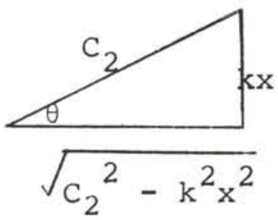
and separate the variables to obtain

$$\frac{dx}{\sqrt{C_2^{\,2} - k^2x^2}} = \pm dt$$

whereupon

$$\int \frac{dx}{\sqrt{C_2^{\,2} - k^2x^2}} = \pm(t + C_3) \qquad (16)$$

Switching to our reference triangle,



$\sin\theta = \frac{kx}{C_2} \qquad (\theta = \sin^{-1}\frac{kx}{C_2})$

$$C_2\sin\theta = kx \longrightarrow C_2\cos\theta\, d\theta = k\,dx$$

$$\cos\theta = \frac{\sqrt{C_2^{\,2} - k^2x^2}}{C_2} \longrightarrow \sqrt{C_2^{\,2} - k^2x^2} = C_2\cos\theta \quad .$$

Hence,

$$\int \frac{dx}{\sqrt{C_2^{\,2} - k^2x^2}} = \int \frac{\frac{C_2}{k}\cos\theta\, d\theta}{C_2\cos\theta} = \frac{1}{k}\theta + C_4 = \frac{1}{k}\sin^{-1}\frac{kx}{C_2} + C_4$$

whereupon (16) becomes:

$$\frac{1}{k}\sin^{-1}\frac{kx}{C_2} = \pm(t + C_5)$$

[3.3.6 (L) cont'd]

or $$\sin^{-1} \frac{kx}{C_2} = \pm k[t + C_5]$$

or $$\frac{kx}{C_2} = \sin [\pm k(t + C_5)]$$

$$= \pm\sin (kt + kC_2).$$

Hence,

$$x = \pm \frac{C_2}{k} \sin (kt + kC_5). \qquad (17)$$

The constants C_2 and C_5 can then be determined from specific given conditions.

But the major point is that simple harmonic motion conceptually exists without specific reference to classical trigonometry, but the explicit representation of x as a function of t does invoke the circular functions.

In still other words, this exercise affords us another example of a non-trigonometric differential equation which has a trigonometric solution.

3.3.7 (L)

Since both C and S are given to be differentiable functions of x, so also is $C^2(x) + S^2(x)$. Now one of the most convenient ways of proving that a function of x is a constant with respect to x is to show that its derivative with respect to x is identically zero.

The derivative of $C^2(x)$ with respect to x is $2C(x)C'(x)$ while the derivative of $S^2(x)$ is $2S(x)S'(x)$. (Remember, we must use the chain rule here). In any event, this means that the derivative of

[3.3.7 (L) cont'd]

$C^2(x) + S^2(x)$ is given by

$$2C(x)C'(x) + 2S(x)S'(x) \quad . \qquad (1)$$

We now put into (1) the information that $C'(x) = -S(x)$ and $S'(x) = C(x)$ to obtain that the derivative of $C^2(x) + S^2(x)$ with respect to x is given by:

$$2C(x)[-S(x)] + 2S(x)C(x)$$

which is, clearly, identically zero (that is, it is $-2CS + 2CS$) .

Quite possibly one has already noticed more than a fancy coincidence in our choice of C and S as symbols in this exercise. It is obvious that if C(x) is set equal to cos x and S(x) to sin x we get true facts about these functions. However, our exercise shows us that much more than this is true. Indeed, we have just shown that no matter how C and S are chosen, provided only that they have the given properties, that C and S must be circular functions in the sense that $C^2(t) + S^2(t) = k^2$ implies that $(C(t),S(t))$ is a point on the circle $x^2 + y^2 = k^2$. The choice of constant is determined by specifying a value for $C^2 + S^2$ for a given value of t. For example, if $C^2(0) + S^2(0) = 4$, then the circle is given by $x^2 + y^2 = 4$. Thus the particular trigonometric functions sin x and cos x are special cases wherein $C^2(0) + S^2(0) = 1$ and in this case we obtain the "usual" circular functions.

From still another point of view if we let $x = S(t)$ we see that $x' = C(t)$ and hence that $x'' = -S(t)$. This in turn says that $x'' = -x$ or, in the language of differentials,

$$\frac{d^2x}{dt^2} = -x$$

and this has the form of simple harmonic motion mentioned in our discussion of Exercise 3.3.6 (L) .

In other words, aside from further experience with the circular functions, the main aim of this exercise was to provide us with

[3.3.7 (L) cont'd]

additional insight to what is meant by the circular functions from a non-trigonometric point of view. Roughly speaking, except for the name we give the functions, C and S are pretty uniquely determined as circular functions form the conditions that $C'(x) = -S(x)$ and $S'(x) = C(x)$. The beauty of this approach is that there is nothing in either the statement of the problem or its solution that requires any a priori understanding of trigonometry. Indeed the specific exercise itself could have been solved prior to our study of this Block. Except for our explanatory commentary, no reference to trigonometry was made in the solution.

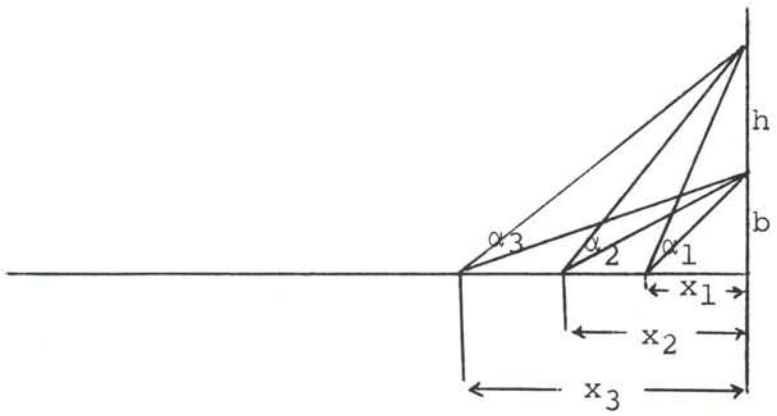
3.3.8 (L)

There is now a danger that because of our de-emphasis of traditional trigonometry, we may begin to believe that one has no great use for non-modern trigonometry. Such a view point would be unfortunate. Indeed, it should be obvious that we stressed the modern aspects of trigonometry because they have not been stressed enough in the ordinary curriculum; and we did not stress the traditional aspects of trigonometry because these have already been stressed too much in the curriculum.

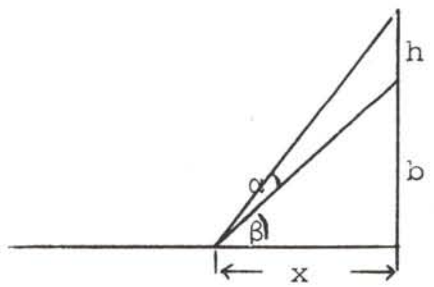
In any event, in order to instill a bit of balance to our presentation of trigonometry, we conclude our discussion with this interesting problem concerned with an angle of vision. We have a picture of height, h placed with its base b feet above the eye of an observer. Clearly, the angle of vision depends on the location of the observer. For example, if he stands directly beneath the picture, he has a zero angle of vision as he tries to view the picture. As he moves back from the picture, Figure 4 shows that the angle of vision subtended by the picture increases up to a point,

[3.3.8 (L) cont'd]

after which the angle again diminishes and approaches zero as the observer moves sufficiently far from the picture. Denoting the angle of vision by α and the distance of the observer from the picture by x, the problem is to determine the value of x for which α is maximum.



(Figure 4)



(Figure 5)

We depict the general situation in Figure 5, where we also introduce the angle, β, to ease our quest for a functional relationship between α and x. In particular, we observe that:

$$\text{ctn}(\alpha + \beta) = x/(b + h)$$

and

[3.3.8 (L) cont'd]

$$\mathrm{ctn}\beta = x/b \quad .$$

Then switching to the language of inverse functions, we obtain $\alpha + \beta = \mathrm{ctn}^{-1}x/(b + h)$, and $\beta = \mathrm{ctn}^{-1}x/b$. Since $\alpha = (\alpha + \beta) - \beta$, we obtain the required relationship, namely:

$$\alpha = \mathrm{ctn}^{-1}x/(b + h) - \mathrm{ctn}^{-1}x/b \quad . \qquad (1)$$

Since for physical reasons, α must exceed 0 radians but be less than $\pi/2$ radians, we see that α is a well-defined, continuous function of x in the required interval. Thus, the answer to our question involves nothing more than using equation (1) to find $d\alpha/dx$, and then setting $d\alpha/dx = 0$. Recalling that $d(\mathrm{ctn}^{-1}u)/dx = (-1/1 + u^2))(du/dx)$, we obtain:

$$\frac{d\alpha}{dx} = \left[\frac{-1}{1 + \left(\frac{x}{b+h}\right)^2}\right]\left(\frac{1}{b+h}\right) + \left[\frac{1}{1 + \left(\frac{x}{b}\right)^2}\right]\left(\frac{1}{b}\right) \qquad (2)$$

Equating $d\alpha/dx$ to 0, and simplifying (2), we obtain:

$$0 = -(b + h)/[x^2 + (b + h)^2] + b/(x^2 + b^2) \quad . \qquad (3)$$

Transposing, and clearing demonimators, we obtain:

$$(b + h)(x^2 + b^2) = b[x^2 + (b + h)^2] \quad . \qquad (4)$$

Simplifying (4) results in:

$$x^2 = b(b + h) \quad . \qquad (5)$$

Thus, we find that α is maximum when $x = \sqrt{b(b + h)}$. (6)

[3.3.8 (L) cont'd]

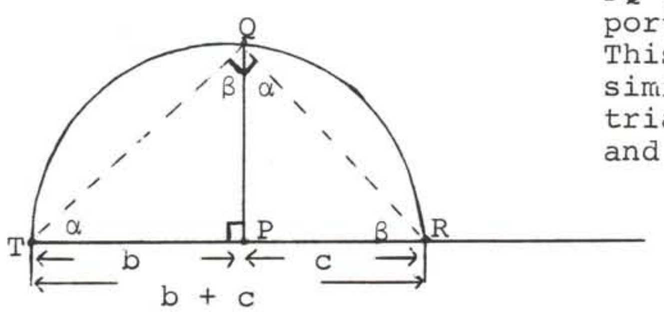
We could stop at this point, smug in the knowledge that we've shown a use for classical trigonometry in the context of calculus. But we would be at least somewhat remiss now if we didn't make some geometric comments about (6). Recall that the ancient Greek had defined the concept of a mean proportional, and that he could construct it by Euclidean means.

To refresh our memories, recall that the mean proportional, m, of b and c, is defined by $m = bc$. That is, m is defined by the proportion:

$$b/m = \sqrt{m/c} \quad .$$

Geometrically, if we view b and c as lengths, we may construct m as follows:

On the same straight line, mark off lengths b and c successively, thus forming b + c. With b + c as a diameter, construct a semi-circle. At the point where b joins c erect a perpendicular which shall extend from the point of juncture to the circle. The length of this segment is the mean proportional between b and c. (See Figure 6)

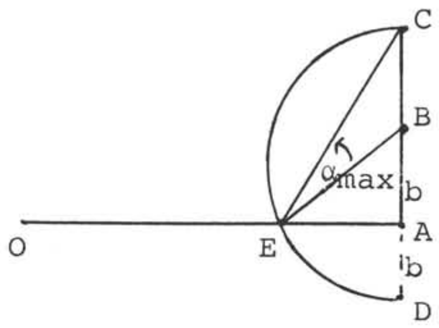


$\overline{PQ}$ denotes the mean proportional of b and c. This follows from the similarity of the three triangles: PQT, PQR, and QTR.

(Figure 6)

[3.3.8 (L) cont'd]

At any rate, if we now apply this result to (6), we obtain the very interesting result that the maximum value of α occurs when x is chosen to be the mean proportional between b and (b + h). This, in turn, leads to a particularly simple geometrical solution to our problem. Namely, (see Figure 7), we use the same diagram as in Figure 2, only now we lable certain key points for expository clarity. We merely extend BA its own length to D, and then construct a semi-circle on CD as diameter. Letting E denote the point at which the semi-circle intersects OA, we have, quite simply, that E is the point at which the observer should be stationed if he wishes the maximum angle of vision (for by our construction, AE is the mean proportional between b + h and b, and by (5), this determines the position at which the maximum angle of vision occurs).



(Figure 7)

QUIZ

1. (a) $$\lim_{x\to 0} \frac{\sin 4x}{3x} = \lim_{x\to 0} \left[\frac{4x}{3x}\left(\frac{\sin 4x}{4x}\right)\right] = \lim_{x\to 0}\left[\frac{4}{3}\left(\frac{\sin 4x}{4x}\right)\right]$$

$$= \frac{4}{3}\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right) = \frac{4}{3}\lim_{4x\to 0}\left(\frac{\sin 4x}{4x}\right)$$

$$= \frac{4}{3}$$

(The key point here is that $\lim_{(\ )\to 0} \frac{\sin(\ )}{(\ )} = 1.$)

(b) $\lim_{x\to 0} \frac{\cos 4x}{3x} = \underline{\infty}$ since $\cos 4x \to 1$ as $x\to 0$ while $3x\to 0$ as $x\to 0$. That is, our denominator gets arbitrarily small in magnitude while our numerator stays near 1 as x gets small.

(Notice in this problem, unlike in (a), the numbers 4 and 3 have no bearing on the answer.)

(c) $$\lim_{x\to 0} \frac{\tan 4x}{3x} = \lim_{x\to 0} \frac{\sin 4x}{3x \cos 4x} = \lim_{x\to 0} \left[\left(\frac{\sin 4x}{3x}\right) \frac{1}{\cos 4x}\right]$$

$$= \lim_{x\to 0}\left(\frac{\sin 4x}{3x}\right) \lim_{x\to 0}\left(\frac{1}{\cos 4x}\right)$$

But by (a), $\lim_{x\to 0}\left(\frac{\sin 4x}{3x}\right) = \frac{4}{3}$, while $\lim_{x\to 0} \frac{1}{\cos 4x} = \frac{1}{1} = 1$. Hence:

$$\lim_{x\to 0} \frac{\tan 4x}{3x} = \left(\frac{4}{3}\right)(1) = \frac{4}{3}$$

(d) $$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} = \lim_{x\to 3}\left[\frac{(x+3)\sin(x^2-9)}{(x+3)(x-3)}\right] = \lim_{x\to 3}\left[(x+3)\ \frac{\sin(x^2-9)}{x^2-9}\right]$$

[1. cont'd]

$$= \lim_{x\to 3} (x+3) \quad \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9}$$

$$= \lim_{x\to 3} (x+3) \quad \lim_{x^2-9\to 0} \frac{\sin(x^2-9)}{x^2-9}$$

$$= 6 \quad (1)$$

$$= 6$$

(e) Observe that $\lim_{h\to 0} \left[\frac{\tan(x+h) - \tan x}{h}\right]$ is the <u>definition</u> of $\frac{d(\tan x)}{dx}$

$$\therefore \lim_{h\to 0} \left[\frac{\tan(x+h) - \tan x}{h}\right] = \sec^2 x$$

provided, of course, that $\sec^2 x \neq \infty$. In other words

$$\lim_{h\to 0} \left[\frac{\tan(x+h) - \tan x}{h}\right] = \begin{cases} \sec^2 x, \text{ if } x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots. \\ \infty \quad\quad , \text{ otherwise} \end{cases}$$

2. (a) $\frac{d(\sin^4 x)}{dx} = 4\sin^3 x \frac{d(\sin x)}{dx} = 4 \sin^3 x \cos x$

$$\frac{d(\cos^4 x)}{dx} = 4 \cos^3 x \frac{d(\cos x)}{dx} = -4 \cos^3 x \sin x$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sin^4 x - \cos^4 x) = \frac{d}{dx} \sin^4 x - \frac{d}{dx} \cos^4 x$$

[2. cont'd]

$$= \underline{4 \sin^3 x \cos x + 4 \cos^3 x \sin x}$$

$$= 4 \sin x \cos x\ (\sin^2 x + \cos^2 x)$$

$$= 4 \sin x \cos x$$

$$= 2 \sin 2x$$

(In fact, you may have noticed that

$$y = \sin^4 x - \cos^4 x = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$$

$$= \sin^2 x - \cos^2 x = -\ (\cos^2 x - \sin^2 x) = -\cos 2x$$

$$\therefore \frac{dy}{dx} = 2 \sin 2x.)$$

(b) If $y = (\tan u)^n$ then $\frac{dy}{dx} = n(\tan u)^{n-1} \frac{d(\tan u)}{dx}$, but $\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}$. Hence, if $y = \tan^3 2x = (\tan 2x)^3$, we have $\frac{dy}{dx} = 3 \tan^2 2x(\sec^2 2x)2 = 6 \tan^2 2x \sec^2 2x$. (Notice that the key to this problem lies in the <u>chain rule</u> and the basic formulas for differentiation.)

3. (a) If $y = \sin^4 x + \cos^4 x$ then:

$$\begin{aligned}\frac{dy}{dx} &= 4 \sin^3 x \cos x - 4 \cos^3 x \sin x \\ &= 4 \sin x \cos x(\sin^2 x - \cos^2 x) \\ &= -4 \sin x \cos x\ (\cos^2 x - \sin^2 x) \\ &= -2(2 \sin x \cos x)(\cos^2 x - \sin^2 x) \\ &= -2 \sin 2x \cos 2x \\ &= \underline{-\sin 4x}\end{aligned}$$

[3. cont'd]

$\therefore \frac{dy}{dx} = 0 \leftrightarrow \sin 4x = 0 \leftrightarrow 4x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi, \ldots$

$\leftrightarrow x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

(Notice that since $0 \leq x \leq 2\pi$, we must have $0 \leq 4x \leq 8\pi$.)

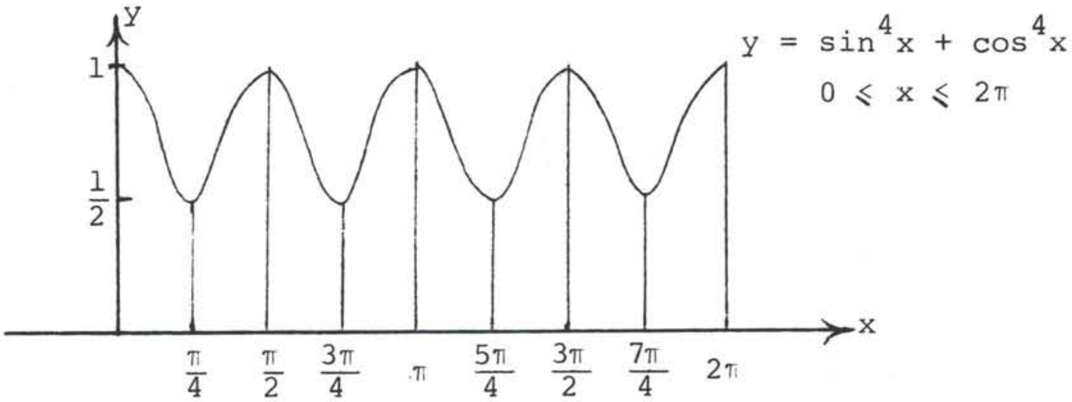
(b) In (a) we found that $\frac{dy}{dx} = -\sin 4x$. Hence,

$$\frac{d^2y}{dx^2} = -4\cos 4x$$

$\therefore$ At $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π, $\frac{d^2y}{dx^2} = -4$ $\therefore$ Curve has a maximum at these values of x. In other words, $(0,1)$, $(\frac{\pi}{2},1)$, $(\pi,1)$, $(\frac{3\pi}{2},1)$, and $(2\pi,1)$ are high points.

Similarly $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ give rise to low points. Since $\sin^4\frac{\pi}{4} + \cos^4\frac{\pi}{4} = (\frac{1}{\sqrt{2}})^4 + (\frac{1}{\sqrt{2}})^4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, we have that $(\frac{\pi}{4},\frac{1}{2})$, $(\frac{3\pi}{4},\frac{1}{2})$, $(\frac{5\pi}{4},\frac{1}{2})$, and $(\frac{7\pi}{4},\frac{1}{2})$ are low points.

Leaving the other details for you to check by any method you prefer, we obtain



[3. cont'd]

(An alternative method which requires less calculus but more algebra and trig identities is

$$y = \sin^4 x + \cos^4 x = \frac{1}{2}\left[(\sin^2 x + \cos^2 x)^2 + (\sin^2 x - \cos^2 x)^2\right]^*$$

$$= \frac{1}{2}\left[(1)^2 + (-\cos 2x)^2\right] = \frac{1}{2}[1 + \cos^2 2x]$$

$$= \frac{1}{2}\left[1 + \left(\frac{1 + \cos 4x}{2}\right)\right] + \frac{1}{2} + \frac{1}{4} + \cos\frac{4x}{4}$$

$$\therefore y = \frac{3 + \cos 4x}{4}.)$$

4. (a) $\int \tan^5 x \sec^2 x dx = \int \tan^5 x \, d(\tan x) = \frac{1}{6}\tan^6 x + c$

$$\therefore \int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x dx = \frac{1}{6}\tan^6 x \Big|_{x=0}^{\frac{\pi}{3}} = \frac{1}{6}\tan^6\frac{\pi}{3} - \frac{1}{6}\tan^6 0$$

$$= \frac{1}{6}\tan^6\frac{\pi}{3}$$

Now, $\tan\frac{\pi}{3} = \sqrt{3}$, hence $\tan^6\frac{\pi}{3} = (\sqrt{3})^6 = 3^3 = 27$

$$\therefore \int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x dx = \frac{27}{6}$$

---

*Quite in general $(a^2 + b^2)^2 + (a^2 - b^2)^2 = a^4 + 2a^2b^2 + b^4 + a^4 - 2a^2b^2 + b^4 = 2a^4 + 2b^4$. Therefore $a^4 + b^4 = \frac{1}{2}\left[(a^2 + b^2)^2 + (a^2 - b^2)^2\right]$.

[4. cont'd]

(b) $\int \cos^2 x dx = \int \frac{1 + \cos 2x}{2} dx = \int (\frac{1}{2} + \frac{1}{2} \cos 2x) dx$

$$= \frac{x}{2} + \frac{1}{4} \sin 2x + c$$

$$\therefore \int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{x}{2} + \frac{1}{4} \sin 2x \Bigg|_{x=0}^{\frac{\pi}{4}}$$

$$= \left[\frac{1}{2}(\frac{\pi}{4}) + \frac{1}{4} \sin \frac{2\pi}{4}\right] - 0$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

5. Here we apply the discussion of total and net distance in Block II to trigonometric functions.

For part (a), we recall that

$$\Delta x = \int_0^{\frac{\pi}{3}} \cos 2t\, dt = \frac{1}{2} \sin 2t \Bigg|_0^{\frac{\pi}{3}} = \frac{1}{2} \sin \frac{2\pi}{3} = \frac{1}{2} (\frac{1}{2} \sqrt{3})$$

$$= \frac{\sqrt{3}}{4} (\approx 0.433) \text{ feet}$$

Thus the <u>displacement</u> of the particle is <u>0.433 ft.</u> ($\approx$ 5.2 inches) to the right of the starting point.

(b) We observe that $\cos 2t \geqslant 0$ if $0 \leqslant 2t \leqslant \frac{\pi}{2}$ or $0 \leqslant t \leqslant \frac{\pi}{4}$. Hence the particle moves from left-to-right if $0 \leqslant t \leqslant \frac{\pi}{4}$ and from right-to-left if $\frac{\pi}{4} < t \leqslant \frac{\pi}{3}$. Hence the total distance travelled by the particle is

[5. cont'd]

$$\left|\int_0^{\frac{\pi}{4}} \cos 2t\,dt\right| + \left|\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos 2t\,dt\right| =$$

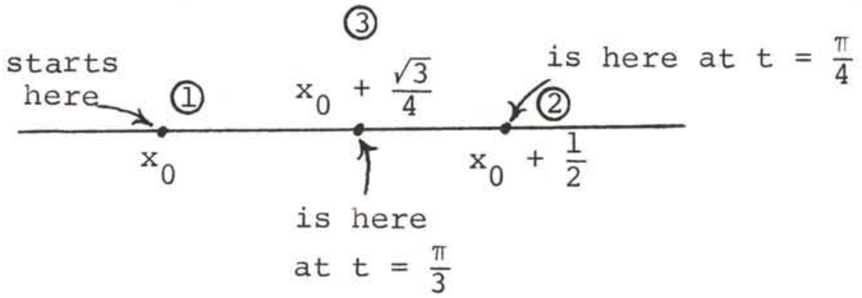
$$\left|\frac{1}{2}\sin 2t\,\Big|_{t=0}^{\frac{\pi}{4}}\right| + \left|\frac{1}{2}\sin 2t\,\Big|_{t=\frac{\pi}{4}}^{\frac{\pi}{3}}\right| =$$

$$\left|\frac{1}{2}\sin\frac{\pi}{2}\right| + \left|\frac{1}{2}\sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{\pi}{2}\right| =$$

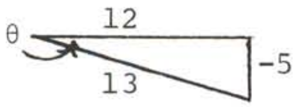
$$\left|\frac{1}{2}\right| + \left|\frac{\sqrt{3}}{4} - \frac{1}{2}\right| = \frac{1}{2} + \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right) =$$

$$\underline{1 - \frac{\sqrt{3}}{4}} \approx \underline{0.567\text{ ft.}} \approx \underline{6.8\text{ inches}}$$

Pictorially



6. Let $\theta = \sin^{-1}\left(\frac{-5}{13}\right)$. Therefore $\sin\theta = \frac{-5}{13}$. Moreover by principal values $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$. Hence $\cos\theta \geqslant 0$. In terms of our reference triangle:



[6. cont'd]

Therefore

(a) $\cos\left[\sin^{-1}\left(\frac{-5}{13}\right)\right] = \cos\theta = \frac{12}{13}$

(b) $$\sin\left[2\sin^{-1}\left(\frac{-5}{13}\right)\right] = \sin 2\theta$$

$$= 2\sin\theta \cos\theta$$

$$= 2\left(\frac{-5}{13}\right)\left(\frac{12}{13}\right)$$

$$= \frac{-120}{169}$$

7. (a) $y = \cos^{-1}u \rightarrow u = \cos y,\ 0 \leqslant y \leqslant \pi$

$$\therefore \frac{du}{dy} = -\sin y$$

$\therefore$ By inverse functions, $\frac{dy}{du} = \frac{-1}{\sin y}$, and by the chain rule

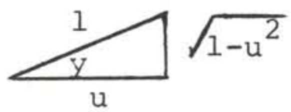
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-1}{\sin y}\frac{du}{dx}$$

Finally, $u = \cos y \rightarrow u^2 = \cos^2 y \rightarrow 1-u^2 = 1-\cos^2 y = \sin^2 y$

$\therefore \sin y = \pm\sqrt{1-u^2}$, but since $\sin y \geqslant 0$ if $0 \leqslant y \leqslant \pi$, we have

$$\sin y = \sqrt{1-u^2}$$

(In terms of a reference triangle $\cos y = u \rightarrow$



)

Hence

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

[7. cont'd]

(b) This is (a) with $u = 3x^2$. Hence $\frac{du}{dx} = 6x$. Therefore

$$\frac{dy}{dx} = \left[\frac{-1}{\sqrt{1-9x^4}}\right]6x = \frac{-6x}{\sqrt{1-9x^4}}, \text{ where } 3x^2 < 1^*$$

(c) If $y = \cos^{-1}(3x^2+5)$ then $\cos y = 3x^2 + 5 \geqslant 5$. But this is impossible when y is real. Namely, y real implies $|y| \leqslant 1$. Hence there are no real numbers x and y such that $y = \cos^{-1}(3x^2+5)$. That is $\frac{dy}{dx}$ does not exist!

(We included (c) only to show that numerical answers must be checked for realness. For example, had we computed the answer using (a), we would obtain

$$\frac{dy}{dx} = \frac{-6x}{\sqrt{1-(3x^2+5)^2}}$$

but $1-(3x^2+5)^2 \leqslant 1-(0+5)^2 = -24$. Therefore $\sqrt{1-(3x^2+5)^2}$ cannot be real for any real value of x.)

(d) $\displaystyle\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1}u + c \quad \text{if } |u| < 1$

$$\therefore \int \frac{du}{\sqrt{1-u^2}} = -\cos^{-1}u + c \quad \text{if } |u| < 1$$

---

*Again notice that $y = \cos^{-1}3x^2 \rightarrow |3x^2| \leqslant 1$. In other words $y = \cos^{-1}3x^2$ means $3x^2 = \cos y$ and $|\cos y| \leqslant 1$. As an arithmetic check notice that if $3x^2 \geqslant 1$, then $\frac{6x}{\sqrt{1-9x^4}}$ is not a real number.

[7. cont'd]

$$\therefore \int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} = -\cos^{-1} u \Big|_{u=0}^{u=\frac{1}{2}} \text{ (since for } u=\tfrac{1}{2} \text{ or } u = 0,\ |u| < 1)$$

$$= [-\cos^{-1} \tfrac{1}{2}] - [-\cos^{-1} 0]$$

$$= [-\tfrac{\pi}{3}] - [-\tfrac{\pi}{2}]$$

$$= -\tfrac{\pi}{3} + \tfrac{\pi}{2}$$

$$= \tfrac{\pi}{6}$$

SOLUTIONS: Calculus of a Single Variable - Block IV: The Definite Integral

PRETEST

1. (a) $\int_0^1 f(x)\,dx$ (b) $\frac{2}{\pi}$

2. $\frac{\pi}{4}$

3. $\sqrt{2} - 1$

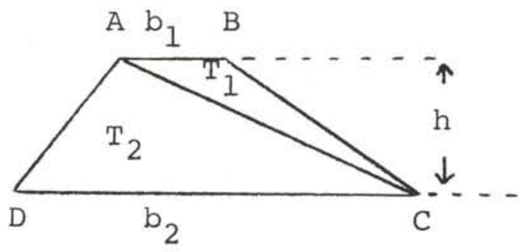
4. $\frac{18}{37}$

5. (a) $\frac{2\pi}{35}$ (b) $\frac{\pi}{10}$

6. $\frac{53}{6}$

UNIT 1: Area

4.1.1



Draw $\overline{AC}$ ($\overline{BD}$ could be used as well), thus decomposing the trapezoid into two triangles T_1 and T_2.

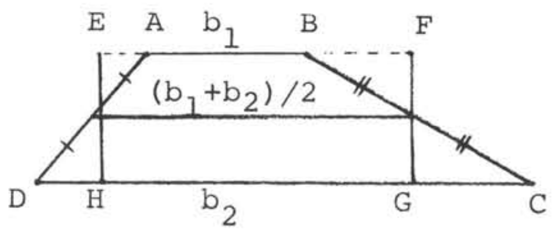
The Area of $T_1 = \frac{1}{2} b_1 h = A_{T_1}$. Similarly $A_{T_2} = \frac{1}{2} b_2 h$. Since the trapezoid is $T_1 \cup T_2$, and T_1 and T_2 share only a boundary in common, the area of the trapezoid is:

$$A_{T_1} + A_{T_2} =$$

$$\frac{1}{2} b_1 h + \frac{1}{2} b_2 h =$$

$$\left(\frac{b_1 + b_2}{2}\right) h \quad .$$

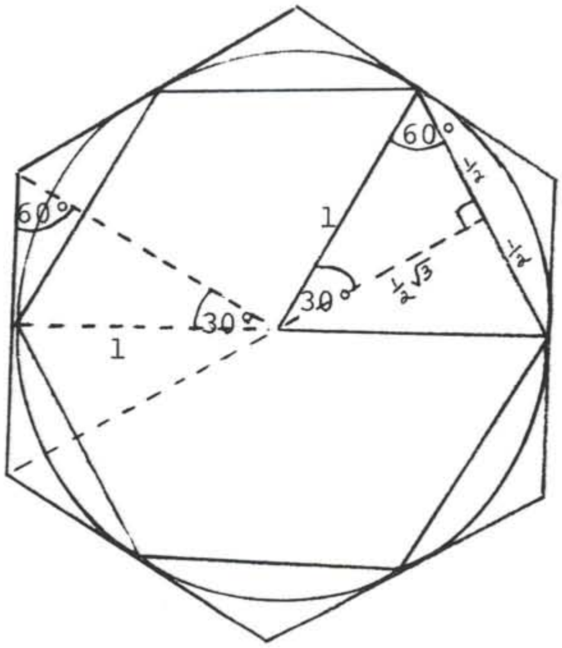
(Geometrically $\frac{b_1 + b_2}{2}$ is an "average" base. That is:



Area of rectangle EFGH = Area of trapezoid ABCD.)

4.1.2 (L)

The first word of caution here is to observe that if we are to apply the logic of this section we must use area, not perimeter (although, historically, the problem was done in terms of perimeter). The reason is, as mentioned in the supplementary notes, that we can not conclude that the larger region has the larger perimeter. In other words, the principle of "sandwiching" applies to area but not to perimeter.



The inscribed hexagon consists of six congruent equilateral triangles, each of whose sides has length 1 unit. The altitude of each triangle is $\frac{1}{2}\sqrt{3}$. Hence the area of each triangle is $\frac{1}{2}bh$ = $\frac{1}{2}(1)(\frac{1}{2}\sqrt{3}) = \frac{1}{4}\sqrt{3}$.

Since the inscribed hexagon is the union of these six triangles, we have that the area of the inscribed hexagon is $6 \times \frac{1}{4}\sqrt{3} = \frac{3}{2}\sqrt{3}$ ($> \frac{3}{2}(1.73) = 2.66$). Thus, since the circle contains this hexagon, the area of the circle, which is π (since its radius is 1), must exceed the area of the inscribed hexagon. That is:

$$(2.66 <)\ \frac{3\sqrt{3}}{2} < \pi$$

[4.1.2 (L) cont'd]

Similarly, the circumscribed hexagon is made up of six equilateral triangles, each of whose altitudes is 1 unit. Hence a side, s, of the triangle is given by: $\cos 30° = \frac{1}{s} = \frac{\sqrt{3}}{2}$. Thus $s = \frac{2}{\sqrt{3}}$.

Therefore, the area of each triangle is $\frac{1}{2}bh = \frac{1}{2}(\frac{2}{\sqrt{3}})(1) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$; and as a result of the area of the circumscribed hexagon is $6 \times \frac{\sqrt{3}}{3} = 2\sqrt{3}$. Consequently, since our circle is contained in the hexagon, we have:

$$\pi < 2\sqrt{3} \; (< 2(1.74) = 3.48) \quad .$$

In summary then:

$$\boxed{\frac{3\sqrt{3}}{2} < \pi < 2\sqrt{3}}$$

or in decimal form:

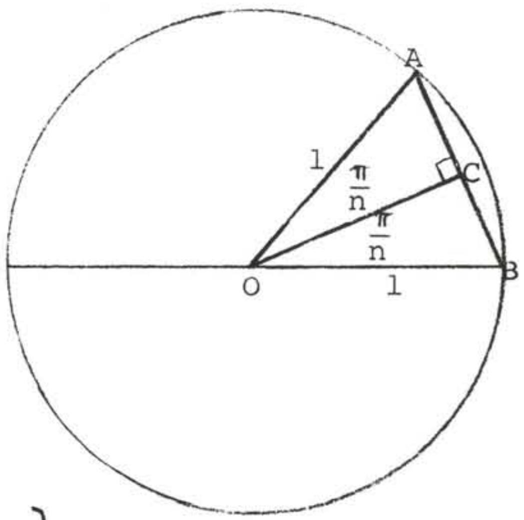
$$\boxed{2.66 < \pi < 3.48} \quad .$$

Hopefully, this example illustrates the so-called "method of exhaustion" employed by the ancient Greeks in which the region under investigation was squeezed between two other regions. Our choice of a hexagon simplified the computations by yielding equilateral triangles. The technique, however, applies to any regular inscribed and circumscribed n-sided polygons. While we needn't prove it, it should seem apparent that as n increases without bound,

[4.1.2 (L) cont'd]

the polygons serve as better and better approximations to the circle. It is for this reason that one sometimes, quite loosely, refers to a circle as an "infinitely many sided" regular polygon. In short, the "method of exhaustion" anticipates the limit concept.

If we accept the circumference of a circle as $2\pi r$, etc. we can compute π as follows. If we inscribe a regular n-sided polygon we may think of the polygon as being the union of n triangles. The central angle at each triangle measures $\frac{2\pi}{n}$ radians. Thus:



$\overline{OC} = \cos\frac{\pi}{n}$, $\overline{BC} = \sin\frac{\pi}{n}$ $\Big\}$ $\therefore$ Area of OAB $= \sin\frac{\pi}{n}\cos\frac{\pi}{n}$

$\therefore$ Area of inscribed n-sided polygon $= n\sin\frac{\pi}{n}\cos\frac{\pi}{n}$

It will be helpful if we re-write

$$n\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \pi\left[\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cos\frac{\pi}{n}\right] .$$

Assuming that the area of the circle is the limit of this expression as $n\to\infty$,

$$\therefore \text{Area} = \pi\left(\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)\left(\lim_{n\to\infty}\cos\frac{\pi}{n}\right) = \pi\left(\lim_{m\to 0}\frac{\sin m}{m}\right)\left(\lim_{m\to 0}\cos\pi m\right)$$

$$= \pi(1)(1)$$

$$= \pi$$

4.1.3 (L)

The aim of this exercise is to make sure we are familiar with some of the $\sum$-notation.

(a) $$\sum_{k=1}^{n}(a_k + b_k) = (a_1 + b_1) + \dots + (a_n + b_n) \quad \text{(by definition)}$$

$$= (a_1 + \dots + a_n) + (b_n + \dots + b_n)$$

$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \quad .$$

Note:

While (a) may "look right," it is by no means self-evident. For example, it might appear that

$$\sum_{k=1}^{n} a_k b_k = \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

but this is not true. What we can write is that

$$\sum_{k=1}^{n} a_k b_k = a_1 b_1 + \dots + a_n b_n$$

whereas

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) = (a_1 + \dots + a_n)(b_1 + \dots + b_n)$$

$$= a_1 b_1 + \dots + a_n b_n + \text{many more terms of the form } a_i b_j \quad . \qquad i \neq j$$

In essence, in the expression $\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$ each term in $\sum a_k$ multiplies each term in $\sum b_k$, whereas in $\sum_{k=1}^{n} a_k b_k$ the "kth" a multiplies only the "kth" b. As a simple example, we may let n = 2.

[4.1.3. (L) cont'd]

Then $\sum_{k=1}^{2} a_k b_k = a_1 b_1 + a_2 b_2$ while $(\sum_{k=1}^{2} a_k)(\sum_{k=1}^{2} b_k) = (a_1 + a_2)(b_1 + b_2)$

$$= a_1 b_1 + a_2 b_2 + a_2 b_1 + a_1 b_2.$$

(b) $\sum_{k=1}^{n} c$ means $\sum_{k=1}^{n} a_k$ where $a_k = c$ for each $k = 1, \ldots, n$

$$\therefore \sum_{k=1}^{n} c = \underbrace{c + \ldots + c}_{n \text{ times}} = nc$$

Note:

It is only necessary that c be constant with respect to k. For example $\sum_{k=1}^{n} x$ means $a_k = x$ for each k. Hence $\sum_{k=1}^{n} x = nx$. The point of (b) is that if c is any factor inside our summation sign which does not vary with k, we may take c outside the summation sign.

(c) $\sum_{k=1}^{n} ca_k = ca_1 + \ldots + ca_n$

$$= c(a_1 + \ldots + a_n)$$

$$= c \sum_{k=1}^{n} a_k$$

Notice that (c) is the generalization of (b). Namely, (b) is (c) with $a_k = 1$ for all k.

4.1.4

$$\sum_{k=1}^{n} (2k - 1) =$$

$$\sum_{k=1}^{n} [2k + (-1)] =$$

$$\sum_{k=1}^{n} 2k + \sum_{k=1}^{n} (-1) =$$

$$2 \sum_{k=1}^{n} k + n(-1) =$$

$$\frac{2n(n + 1)}{2} - n =$$

$$n^2 + n - n =$$

$$n^2$$

Notice that this is another proof that the sum of the first n odd integers is the n'th perfect square. That is

$$\sum_{k=1}^{n} (2k - 1) = 1 + 3 + 5 + \ldots + (2n - 1) \quad .$$

4.1.5 (L)

(a) This result is known as a telescoping sum since all but the first and last terms cancel. We may show the result in several different ways. Perhaps the most straightforward is to apply the definition directly. Thus,

[4.1.5 (L) cont'd]

$$\sum_{k=1}^{n} (a_{k+1} - a_k) = (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + \ldots + (a_{n-1} - a_{n-2}) +$$

$$(a_n - a_{n-1}) + (a_{n+1} - a_n)$$

$$= -a_1 + a_{n+1}$$

$$= a_{n+1} - a_1$$

Another way is to say

$$\sum_{k=1}^{n} (a_{k+1} - a_k) = \sum_{k=1}^{n} a_{k+1} - \sum_{k=1}^{n} a_k$$

$$= a_2 + a_3 + \ldots + a_n + a_{n+1}$$

$$\underline{-(a_1 + a_2 + a_3 + \ldots + a_n)}$$

$$= -a_1 + a_{n+1}$$

(b) $\sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1$ becomes

$$\sum_{k=1}^{n} [(k + 1)^2 - k^2] = (n + 1)^2 - 1^2 \text{ when } a_k = k^2 \quad .(1)$$

Now $(n + 1)^2 - 1^2 = n^2 + 2n$. While

$$\sum_{k=1}^{n} [(k + 1)^2 - k^2] = \sum_{k=1}^{n} (2k + 1) \quad .$$

[4.1.5 (L) cont'd]

Putting these results into (1), we obtain

$$\sum_{k=1}^{n} (2k + 1) = n^2 + 2n \quad . \tag{2}$$

But

$$\sum_{k=1}^{n} (2k + 1) = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

$$= 2 \sum_{k=1}^{n} k + n$$

and putting this into (2), we obtain

$$2 \sum_{k=1}^{n} k + n = n^2 + 2n$$

$$\therefore 2 \sum_{k=1}^{n} k = n^2 + n$$

$$\therefore \sum_{k=1}^{n} k = \frac{n^2 + n}{2} = \frac{n(n + 1)}{2} \quad .$$

Again, aside from gaining additional experience with the $\sum$-notation, notice that we have also found a rather nice way of evaluating $\sum_{k=1}^{n} k$ with a minimum of guess work. Moreover, as we shall see in the next few exercises, this technique generalizes very nicely. The key point is that we have found a way to determine that $\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$ without having to know the answer first (as is required in other mathematical induction).

4.1.6

We have

$$\sum_{k=1}^{n} [(k + 1)^3 - k^3] = (n + 1)^3 - 1$$

$$= n^3 + 3n^2 + 3n \quad . \qquad (1)$$

Moreover, $(k + 1)^3 - k^3 = 3k^2 + 3k + 1$.

Putting this into (1) we have:

$$\sum_{k=1}^{n} (3k^2 + 3k + 1) = n^3 + 3n^2 + 3n \quad . \qquad (2)$$

Now:

$$\sum_{k=1}^{n} (3k^2 + 3k + 1) = \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

$$= 3 \sum_{k=1}^{n} k^2 + 3 \sum_{k=1}^{n} k + n$$

$$= 3 \sum_{k=1}^{n} k^2 + \frac{3n(n + 1)}{2} + n \quad . \qquad (3)$$

Putting (3) into (2):

$$3 \sum_{k=1}^{n} k^2 + \frac{3n(n + 1)}{2} + n = n^3 + 3n^2 + 3n$$

$$\therefore 6 \sum_{k=1}^{n} k^2 + 3n(n + 1) + 2n = 2n^3 + 6n^2 + 6n$$

[4.1.6 cont'd]

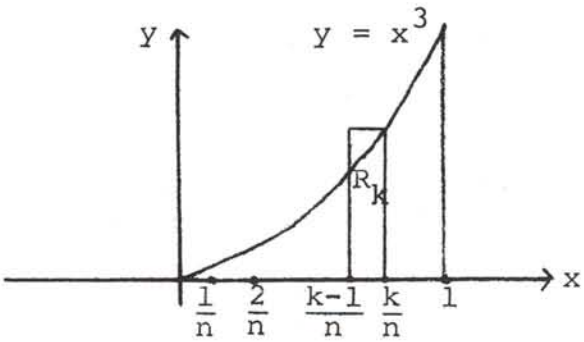
$$\therefore 6\sum_{k=1}^{n} k^2 = 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1)$$
$$= n(n + 1)(2n + 1)$$

$$\therefore \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}$$ - again notice that we arrived at the "recipe" without having had to know the answer first.

4.1.7 (L)

In terms of the central theme of this unit, we actually want to solve (b). What we find is that to evaluate U_n "conveniently" we are required to come to grips with $\sum_{k=1}^{n} k^3$. Namely,

(b)



$$A_{R_k} = \left(\frac{k}{n}\right)^3 \frac{1}{n} = \frac{k^3}{n^4}$$

$$U_n = \sum_{k=1}^{n} A_{R_k} = \sum_{k=1}^{n} \frac{k^3}{n^4} = \frac{1}{n^4}\sum_{k=1}^{n} k^3$$

$$= \frac{1^3 + 2^3 + \dots + n^3}{n^4}. \qquad (1)$$

[4.1.7 cont'd]

With the aid of modern calculators we could compute (1) for large values of n, and, in this way, we could approximate $\lim_{n\to\infty} U_n$ as closely as we wish. In fact, except in relatively simple cases, such as is this exercise, we often have to settle for such approximations.

In any event, the answer to (b) is

$$A_R = \lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3$$

and any further refinement will involve computing $\sum_{k=1}^{n} k^3$, a result that we have not yet found in this course.

With this as motivation, we tackle (a). Mimicking the approach in the previous exercises, we have

$$\sum_{k=1}^{n} ((k + 1)^4 - k^4) = (n + 1)^4 - 1^4$$

$$\therefore \sum_{k=1}^{n} [4k^3 + 6k^2 + 4k + 1] = n^4 + 4n^3 + 6n^2 + 4n$$

$$\therefore 4 \sum_{k=1}^{n} k^3 + 6 \sum_{k=1}^{n} k^2 + 4 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = n^4 + 4n^3 + 6n^2 + 4n$$

$$\therefore 4 \sum_{k=1}^{n} k^3 + \frac{6n(n + 1)(2n + 1)}{6} + \frac{4n(n + 1)}{2} + n = n^4 + 4n^3 + 6n^2 + 4n$$

$$4 \sum_{k=1}^{n} k^3 + [2n^3 + 3n^2 + n] + [2n^2 + 2n] + n = n^4 + 4n^3 + 6n^2 + 4n$$

4.1.7 cont'd

$$4\sum_{k=1}^{n} k^3 = n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1) = n^2(n + 1)^2$$

$$\therefore \sum_{k=1}^{n} k^3 = \frac{n^2(n + 1)^2}{4} \qquad \underline{\text{answer to (a)}}\ ^* \qquad (2)$$

(<u>Aside</u>: $\frac{n^2(n + 1)^2}{4} = \left[\frac{n(n + 1)}{2}\right]^2$ and $\frac{n(n + 1)}{2} = \sum_{k=1}^{n} k$.

Hence we have the interesting number theory result

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \text{ that is, } 1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2.)$$

At any rate if we put the result of (2) into (1), we see that:

$$U_n = \frac{1}{n^4}\,\frac{n^2(n + 1)^2}{4}$$

$$= \frac{1}{4}\,\frac{(n + 1)^2}{n^2} = \frac{1}{4}\left(\frac{n + 1}{n}\right)^2$$

$$= \frac{1}{4}\left(1 + \frac{1}{n}\right)^2 \qquad (3)$$

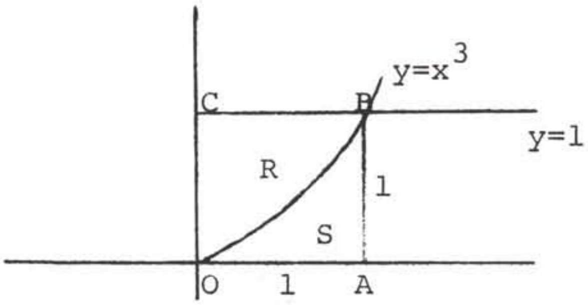
$$\therefore A_R = \lim_{n\to\infty} U_n = \frac{1}{4}(1 + 0)^2 = \frac{1}{4} \qquad (4)$$

---

*Notice now that we know $\sum_{k=1}^{n} k^3 = \frac{n^2(n + 1)^2}{4}$ we can assign it as an exercise in math induction if we so desired, but it is hardly likely that induction provided the first proof for this result.

[4.1.7 cont'd]

(c)



Area of OABC = 1 x 1 = 1

$A_S = \frac{1}{4}$ (from the previous exercise)

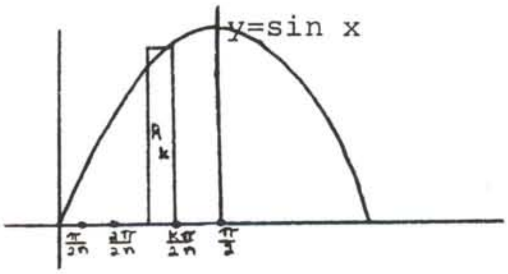
Area of OABC = $A_R + A_S$

$1 = A_R + \frac{1}{4}$

$A_R = \frac{3}{4}$

Our main aim here is to emphasize the property that the area of the whole equals the sum of the areas of the parts.

4.1.8 (L)



(a)

$$A_{R_K} = [\sin \frac{k\pi}{2n}]\frac{\pi}{2n}$$

[4.1.8 (L) cont'd]

$$\underline{U_n = \sum_{k=1}^{n} A_{R_k} = \sum_{k=1}^{n} \frac{\pi}{2n} \sin\left(\frac{k\pi}{2n}\right)}$$

(b)

$$A_R = \lim_{n\to\infty} U_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{2n} \sin\left(\frac{k\pi}{2n}\right)$$

$$= \lim_{n\to\infty} \left\{ \frac{\pi}{2n} \sum_{k=1}^{n} \sin\left(k\left[\frac{\pi}{2n}\right]\right) \right. . \qquad (1)$$

Assuming for the moment (we will have a further note at the end of this section) that we really did know that

$$\sum_{k=1}^{n} \sin k\theta = \frac{\cos\frac{\theta}{2} - \cos\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}} .$$

Let $\theta = \frac{\pi}{2n}$.

Then $\sum \sin k\left[\frac{\pi}{2n}\right] = \dfrac{\cos\frac{\pi}{4n} - \cos\left[\left(\frac{2n+1}{2}\right)\frac{\pi}{2n}\right]}{2\sin\frac{\pi}{4n}}$

$$= \frac{\cos\frac{\pi}{4n} - \cos\left(\frac{\pi}{2} + \frac{\pi}{4n}\right)}{2\sin\frac{\pi}{4n}}$$

$$= \frac{\cos\frac{\pi}{4n} + \sin\frac{\pi}{4n}}{2\sin\frac{\pi}{4n}} . \qquad (2)$$

Putting (2) into (1):

[4.1.8 (L) cont'd]

$$A_R = \lim_{n\to\infty} \frac{\pi}{2n} \left[\frac{\cos \frac{\pi}{4n} + \sin \frac{\pi}{4n}}{2 \sin \frac{\pi}{4n}}\right]$$

$$= \lim_{n\to\infty} \frac{\pi}{4n} \left[\frac{\cos \frac{\pi}{4n} + \sin \frac{\pi}{4n}}{\sin \frac{\pi}{4n}}\right]$$

$$= \lim_{m\to 0} \frac{\pi m}{4} \left[\frac{\cos \frac{\pi m}{4} + \sin \frac{\pi m}{4}}{\sin \frac{\pi m}{4}}\right]$$

$$= \lim_{m\to 0} \left[\frac{\cos \frac{\pi m}{4} + \sin \frac{\pi m}{4}}{\left(\frac{\sin \frac{\pi m}{4}}{\frac{\pi m}{4}}\right)}\right] . \qquad (3)$$

Now:

$$\lim_{m\to 0} \cos \frac{\pi m}{4} = \cos 0 = 1$$

$$\lim_{m\to 0} \sin \frac{\pi m}{4} = \sin 0 = 0$$

$$\lim_{u\to 0} \frac{\sin u}{u} = 1 .$$

Hence (3) becomes:

$$A_R = \left[\frac{1 + 0}{1}\right] = 1 . \qquad (4)$$

For better or worse, the major aim of this exercise is to show how quickly computations tend to get out of hand even for

[4.1.8 (L) cont'd]

rather elementary expression for f(x). This degree of difficulty is important for us to grasp if we are to truly appreciate the results of the next unit.

A Note on the identity:

$$\sum_{k=1}^{n} \sin k\theta = \frac{\cos \frac{\theta}{2} - \cos \frac{2n+1}{2}\theta}{2 \sin \frac{\theta}{2}}.$$

To derive this result we can put telescoping sums to good use.

$$\sum_{k=1}^{n} \left[\cos \frac{2k-1}{2}\theta - \cos \frac{2k+1}{2}\theta\right] = \cos \frac{\theta}{2} - \cos \frac{2n+1}{2}\theta$$

$$\left(a_k = \cos \frac{2k-1}{2}\theta\right) .$$

Now $\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$

$$\therefore \cos \frac{(2k-1)\theta}{2} - \cos \frac{(2k+1)\theta}{2} = -2 \sin k\theta \sin \left(\frac{-\theta}{2}\right)$$

$$= 2 \sin \frac{\theta}{2} \sin k\theta$$

$$\therefore \sum_{k=1}^{n} 2 \sin \frac{\theta}{2} \sin k\theta = \cos \frac{\theta}{2} - \cos \left(\frac{2n+1}{2}\right)\theta$$

$$\therefore 2 \sin \frac{\theta}{2} \sum_{k=1}^{n} \sin k\theta = \cos \frac{\theta}{2} - \cos \left(\frac{2n+1}{2}\right)\theta$$

$$\therefore \sum_{k=1}^{n} \sin k\theta = \frac{\cos \frac{\theta}{2} - \cos \left(\frac{2n+1}{2}\right)\theta}{2 \sin \frac{\theta}{2}} .$$

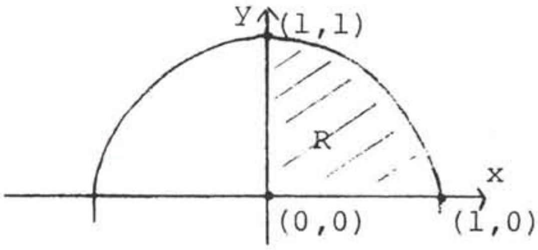
[4.1.8 (L) cont'd]

Again, while this is a tough result to derive (that's why we gave the result rather than have you derive it), it should help us capture the flavor of the computational gymnastics involved in evaluating infinite sums.

UNIT 2: Area as a Differential Equation

4.2.1 (L)

(a) Geometrically, $\int_0^1 \sqrt{1 - x^2}\, dx$ represents the area of the region R, where R is bounded above by the curve $y = \sqrt{1 - x^2}$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = 1$. Observe that $y = \sqrt{1 - x^2}$ corresponds to the upper half of the circle $x^2 + y^2 = 1$ (since our convention is that $\sqrt{1 - x^2}$ is non-negative unless otherwise stated, y is non-negative). Thus, R is



$y = \sqrt{1 - x^2} \longrightarrow$

$y^2 = 1 - x^2, \quad y \geq 0$

(Figure 1)

Then it is easy to see that A_R is exactly the area of the first quadrant of the circle whose radius is 1. That is, $A_R = \frac{\pi}{4}$. Finally, since A_R is denoted by $\int_0^1 \sqrt{1 - x^2}\, dx$, we have:

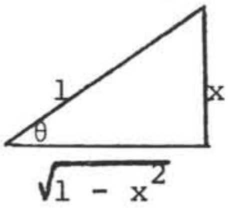
$$\int_0^1 \sqrt{1 - x^2}\, dx = \frac{\pi}{4} \quad .$$

(b) We also know that $\int_0^1 \sqrt{1 - x^2}\, dx = G(1) - G(0)$ where $G'(x) = \sqrt{1 - x^2}$. However, we may not know such a function G. (Beware not to think that $G(x) = \frac{2}{3}(1 - x^2)^{3/2}$. If $G(x) = \frac{2}{3}(1 - x^2)^{3/2}$, $G'(x) = \frac{2}{3}(\frac{3}{2})(1 - x^2)^{1/2}[\frac{d}{dx}(1 - x^2)] = [\sqrt{1 - x^2})\underline{(-2x)}$.)

If we recall our earlier remarks when we discussed the inverse circular functions, we may notice that $\sqrt{1 - x^2}$ suggests the

[4.2.1 (L) cont'd]

"reference triangle"



(Figure 2)

Hence,

$$\left.\begin{array}{r} \sin\theta = x \\ \therefore \cos\theta d\theta = dx \end{array}\right\} \text{ and } \cos\theta = \sqrt{1 - x^2}.$$

Thus, with θ as above,

$$\begin{aligned} \int \sqrt{1 - x^2}\, dx &= \int \cos\theta (\cos\theta) d\theta \\ &= \int \cos^2\theta\, d\theta \\ &= \int \frac{1 + \cos 2\theta}{2}\, d\theta \\ &= \frac{1}{2}\int (1 + \cos 2\theta) d\theta \\ &= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + c \quad . \qquad (1) \end{aligned}$$

Since we would like our answer in terms of x, we may use Figure 2 to "convert" (1) into a suitable expression involving x.

$$\frac{\theta}{2} + \frac{1}{4}\sin 2\theta =$$

$$\frac{\theta}{2} + \frac{1}{4}[2\sin\theta\cos\theta] =$$

$$\frac{\sin^{-1}x}{2} + \frac{1}{2}(x)(\sqrt{1 - x^2})$$

[4.2.1 (L) cont'd]

$$\therefore \int \sqrt{1 - x^2}\, dx = \frac{1}{2}[\sin^{-1}x + x\sqrt{1 - x^2}] + c$$

$$\therefore \int_0^1 \sqrt{1 - x^2}\, dx = \left(\frac{1}{2}[\sin^{-1}x + x\sqrt{1 - x^2}]\right)\Big|_0^1 \qquad (2)$$

$$= [\tfrac{1}{2}\sin^{-1}1 + 0] - [\tfrac{1}{2}\sin^{-1}0 + 0]$$

$$= \tfrac{1}{2}\sin^{-1}1$$

$$= \tfrac{1}{2}(\tfrac{\pi}{2})$$

$$= \frac{\pi}{4} \qquad (3)$$

In this case, we were able to find the required G(x), but it was not too convenient. In other cases, the desired G(x) may not exist at all, explicitly. In any event, our hope is that (a) and (b) emphasize the difference between areas and derivatives.

Note:

There is an alternative way for determining $\int_0^1 \sqrt{1 - x^2}\, dx$ making use of Figure 2. "Mechanically," the approach is as follows:

We let $\sin\theta = x$ and obtain as before that $dx = \cos\theta\, d\theta$, $\sqrt{1 - x^2} = \cos\theta$. We also observe (using principle values) that when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$. Finally observing that $\int_a^b f(x)dx$ really means $\int_{x=0}^{x=b} f(x)dx$, we obtain:

$$\int_{x=0}^{x=1} \sqrt{1 - x^2}\, dx = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta(\cos\theta\, d\theta)$$

[4.2.1 (L) cont'd]

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1 + \cos 2\theta)}{2} d\theta$$

$$= [\frac{\theta}{2} + \frac{1}{4} \sin 2\theta] \Big|_0^{\frac{\pi}{2}}$$

$$= (\frac{\pi}{4} + 0) - (0 + 0)$$

$$= \frac{\pi}{4}$$

and we arrive at the same result as in (3) but in a somewhat less cumbersome manner than the result in equation (2) leads to.

The validity of this approach may be generalized as follows:

Suppose we have $\int_a^b f(x)dx$ and we know that $x = h(u)$ where h is a continuous 1-1 function (so h^{-1} exists). Then

$$\int_{x=a}^{x-b} f(x)dx = \int_{u=h^{-1}(a)}^{u=h^{-1}(b)} f(h(u))d(h(u)).$$ That is, if we consistently make the substitution $x = h(u)$, we do not change the definite integral.

To demonstrate this result, suppose

$$\int_a^b f(x)dx = G(b) - G(a) \text{ where } G' = f \quad . \qquad (4)$$

[4.2.1 (L) cont'd]

Notice in (4) that x is a "dummy variable" and that more generally, (4) can be represented as

$$\int_{[\ ]=a}^{[\ ]=b} f([\ ])d([\ ]) = G([\ ])\Big|_a^b = G(b) - G(a) \quad . \tag{5}$$

In particular with [ ] = h(u), the left side of (5) becomes $\int_{h(u)=a}^{h(u)=b} f(h(u)d(h(u)$, <u>but</u> if h is 1-1, $h(u) = b \longleftrightarrow u = h^{-1}(b)$, and $h(u) = a \longleftrightarrow u = h^{-1}(a)$. Thus (5) becomes

$$\int_{u=h^{-1}(a)}^{u=h^{-1}(b)} f(h(u))d(h(u)) = G(h(u))\Big|_{u=h^{-1}(a)}^{u=h^{-1}(b)}$$

$$= G(h(h^{-1}(b))) - G(h(h^{-1}(a)))$$

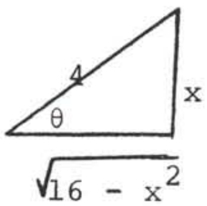
$$= G(b) - G(a)$$

$$= \int_a^b f(x)dx$$

which is the desired result.

<u>4.2.2</u>

(a)



[4.2.2 cont'd]

$\sin\theta = \frac{x}{4}$ or $x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$\cos\theta = \frac{\sqrt{16 - x^2}}{4}$ or $\sqrt{16 - x^2} = 4\cos\theta$

$x = 2 \longrightarrow \sin\theta = \frac{1}{2} \longrightarrow \theta = \frac{\pi}{6}$

$x = 4 \longrightarrow \sin\theta = 1 \longrightarrow \theta = \frac{\pi}{2}$

$$\therefore \int_{x=2}^{x=4} \sqrt{16 - x^2}\, dx = \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}} 4\cos\theta\,(4\cos\theta\, d\theta)$$

$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 8\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$

$$= 8(\theta + \tfrac{1}{2}\sin 2\theta)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 8[(\tfrac{\pi}{2} + 0) - (\tfrac{\pi}{6} + \tfrac{1}{2}\sin\tfrac{\pi}{3})]$$

[4.2.2 cont'd]

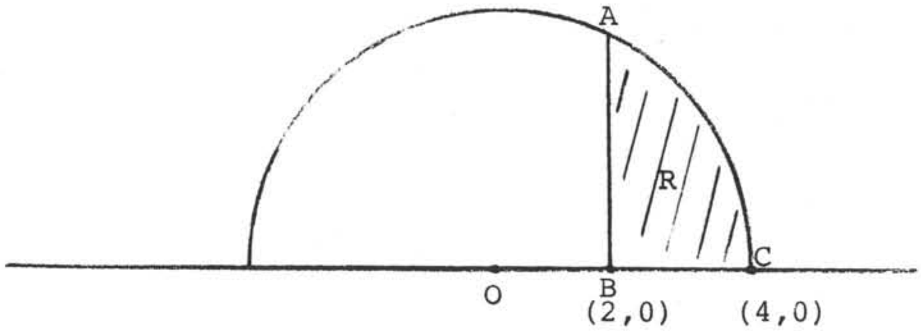
$$= 8\left[\frac{\pi}{2} - \left(\frac{\pi}{6} + \frac{1}{2}\,\frac{\sqrt{3}}{2}\right)\right]$$

$$= 8\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \frac{8[4\pi - 3\sqrt{3}]}{12} = \frac{8\pi}{3} - 2\sqrt{3} \quad .$$

(b) $y = +\sqrt{16 - x^2} \longrightarrow x^2 + y^2 = 16 \quad , \quad y \geqslant 0$

Hence, R is given by

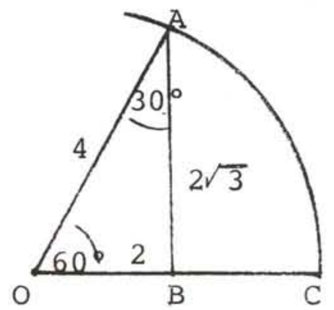
$x^2 + y^2 = 16 \quad , \quad y > 0$



That is,

$$A_R = \int_2^4 \sqrt{16 - x^2}\, dx \quad .$$

The key, geometrically, is to draw $\overline{OA}$ and observe $\measuredangle$ AOC = 60° ($\frac{\pi}{3}$ radians).



$\sin \measuredangle OAB = \frac{2}{4} = \frac{1}{2}$

$\therefore \measuredangle OAB = 30°$

$\overline{AB}^2 = 16 - 4 = 12$

$\overline{AB} = 2\sqrt{3}$

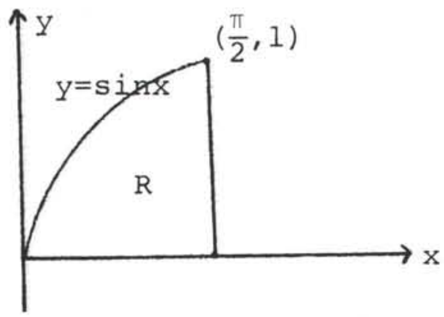
[4.2.2 cont'd]

Sector AOC(=S) is $\frac{1}{6}$ of the circle, hence its area is $\frac{1}{6}$ that of the circle. The area of the circle is 16π. Hence, the area of the sector OAC $= \frac{1}{6}(16\pi) = \frac{8\pi}{3}$. The area of ΔOBA $= \frac{2(2\sqrt{3})}{2} = 2\sqrt{3}$.

$A_R = A_S - A_T$ where T is ΔOBA

$= \frac{8\pi}{3} - 2\sqrt{3}$

4.2.3



$$A_R = \int_0^{\frac{\pi}{2}} \sin x \, dx = G(\tfrac{\pi}{2}) - G(0) \text{ where } G'(x) = \sin x$$

$\therefore$ We may let $G(x) = -\cos x$

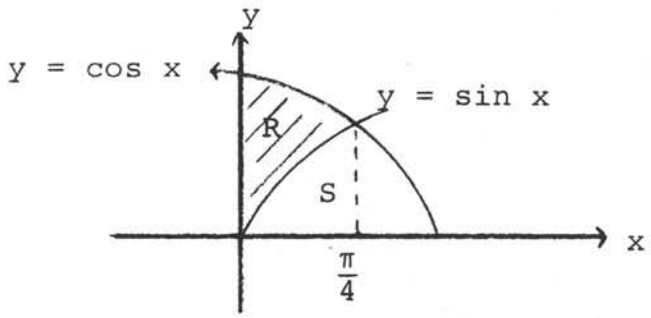
$\therefore A_R = (-\cos\frac{\pi}{2}) - (-\cos 0)$

$= -0 - (-1)$

$= 1$

We get the same answer as we did in Exercise 4.1.8 (L) but the ease of the anti-derivative technique obviously speaks for itself.

4.2.4



$$A_{R\cup S} = \int_0^{\frac{\pi}{4}} \cos x \, dx$$

$$A_S = \int_0^{\frac{\pi}{4}} \sin x \, dx$$

$$\therefore A_R = A_{R\cup S} - A_S = \int_0^{\frac{\pi}{4}} \cos x \, dx - \int_0^{\frac{\pi}{4}} \sin x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx$$

$$= (\sin x + \cos x) \Big|_0^{\frac{\pi}{4}}$$

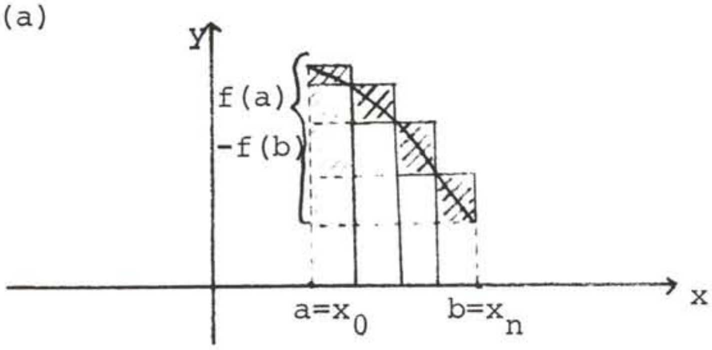
$$= (\sin \tfrac{\pi}{4} + \cos \tfrac{\pi}{4}) - (\sin 0 + \cos 0)$$

$$= (\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}) - (0 + 1)$$

$$= \sqrt{2} - 1$$

4.2.5 (L)

(a)



Since the highest point now occurs at the left end point we have:

$$U_n = f(x_0)\Delta x + \ldots + f(x_{n-1})\Delta x$$

while

$$L_n = f(x_1)\Delta x + \ldots + f(x_n)\Delta x \quad .$$

Hence $U_n - L_n = f(x_0)\Delta x - f(x_n)\Delta x$, since all other terms cancel

$$= [f(a) - f(b)]\Delta x$$

$$= [f(a) - f(b)]\left(\frac{b - a}{n}\right) \quad .$$

(b)

$$y = \frac{1}{1 + x^2} = (1 + x^2)^{-1}$$

Hence:

$$y' = -(1 + x^2)^{-2}2x = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{(1 + x^2)^2(-2) - (-2x)[2(1 + x^2)2x]}{(1 + x^2)^4}$$

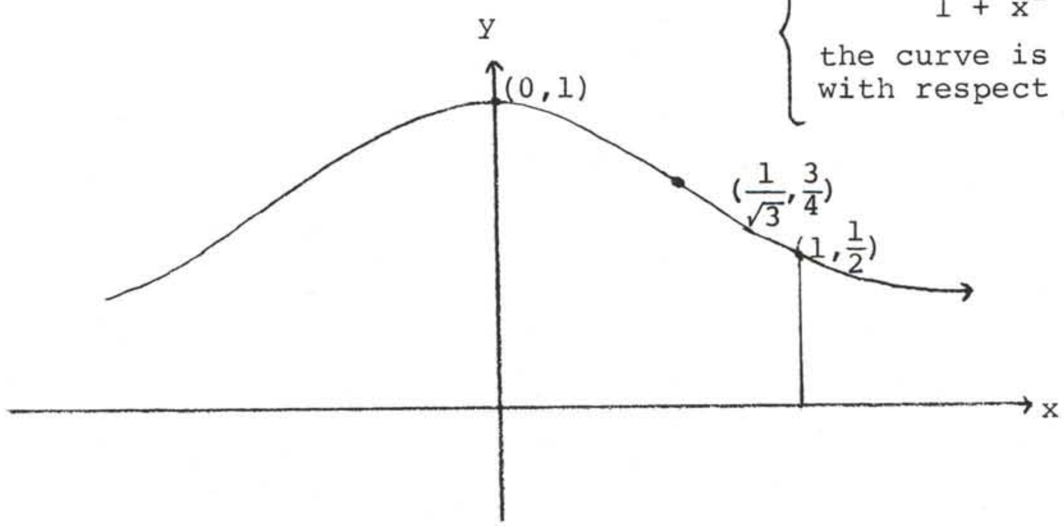
[4.2.5 (L) cont'd]

$$= \frac{2(1 + x^2)[-(1 + x^2) + 4x^2]}{(1 + x^2)^4}$$

$$= \frac{2(3x^2 - 1)}{(1 + x^2)^3} \quad .$$

Since $1 + x^2 \geqslant 1$ for all x, the denominators in y, y', and y" are all positive; hence, the signs are determined by the numerators. We then see that the curve is always above the x-axis. It rises for negative values of x, falls for positive values of x and attains a maximum when x = 0, or at the point (0,1). Moreover the curve holds water for $3x^2 - 1 > 0$ (or when $|x| > \sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$), spills water when $|x| < \sqrt{\frac{1}{3}}$, and changes concavity when $x = \sqrt{\frac{1}{3}} \approx 0.6$.

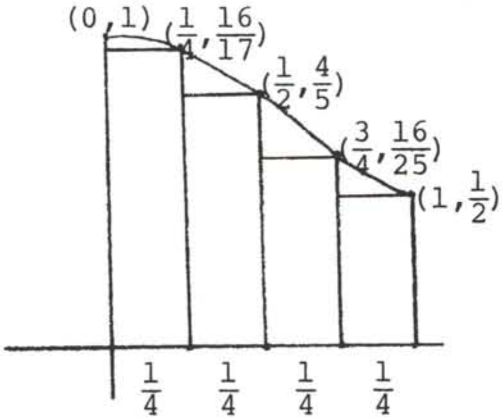
Hence, the required graph is



Since $\frac{1}{1 + x^2} = \frac{1}{1 + (-x)^2}$ the curve is symmetric with respect to the y-axis.

[4.2.5 (L) cont'd]

(c)



Since the curve is always falling in the given interval the mimimum height of each inscribed rectangle occurs at the <u>right</u> point of each interval.

Hence:

$$L_4 = \frac{1}{4}\left(\frac{16}{17}\right) + \frac{1}{4}\left(\frac{4}{5}\right) + \frac{1}{4}\left(\frac{16}{25}\right) + \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \frac{4}{17} + \frac{1}{5} + \frac{4}{25} + \frac{1}{8}$$

$$= \frac{2449}{3400} \quad \text{(In decimal form } 0.71 < L_4 < 0.72 \text{ .)}$$

(d) Our curve obeys the conditions of part (a); hence,

$$U_4 - L_4 = \frac{b-a}{n}[f(a) - f(b)] = \frac{1}{4}[f(0) - f(1)] = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} \ .$$

$\therefore U_4 = L_4 + \frac{1}{8} = \frac{2449}{3400} + \frac{1}{8} = \frac{2874}{3400}$. (Again in decimal form: $0.84 < U_4 < 0.85$.)

Since $L_4 < A_R < U_4$, parts (b) and (c) tell us that

$$0.71 < A_R < 0.85 \ .$$

[4.2.5(L) cont'd]

(e) Now $3.14 < \pi < 3.15 \longrightarrow$

$$0.78 < \frac{\pi}{4} < 0.79 \quad .$$

Since $0.72 < A_R < 0.85$ it is possible that $A_R = \frac{\pi}{4}$.

(f) While it is possible that $A_R = \frac{\pi}{4}$, such a result is not at all obvious (or even plausible) from our previous considerations.

Now, we do know that

$$A_R = \int_0^1 \frac{dx}{1 + x^2}$$

and this means:

$$A_R = G(1) - G(0) \text{ where } G'(x) = \frac{1}{1 + x^2} \quad .$$

But $[\arctan x]' = \frac{1}{1 + x^2}$.

Hence $A_R = \arctan x \Big|_{x=0}^{x=1} = \arctan 1 - \arctan 0$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} = \underline{\underline{\text{exact}}} \text{ value of } A_R \quad .$$

As a final remark, we are not so much interested in how one "recalls" that $(\arctan x)' = \frac{1}{1 + x^2}$, for example, we may have memorized this result or we may have derived it by use of differential

[4.2.5 (L) cont'd]

calculus, as how inverse trig functions can supply an answer to a problem which seems to require no knowledge of trigonometry.

4.2.6 (L)

The main aim of this problem is to show how we may evaluate certain infinite sums by expressing them as definite integrals and then evaluating the definite integral by differential calculus.

(a) $\lim_{n\to\infty} \frac{1}{n}[f(\frac{1}{n}) + \dots + f(\frac{n}{n})]$ is just another way (although it may take a while to get used to it) of writing $\int_0^1 f(x)\,dx$.

Namely, recall, in general, that if f is continuous on [a,b] and $a < x_1 < \dots < x_n = b$ then, among other things,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx \quad . \qquad (1)$$

Using this notation, if we let a = 0, b = 1 then $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ while $x_k = \frac{k}{n}$. Thus (1) becomes

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\tfrac{k}{n})\tfrac{1}{n} = \int_0^1 f(x)\,dx$$

or:

$$\lim_{n\to\infty} [\tfrac{1}{n} \sum_{k=1}^{n} f(\tfrac{k}{n})] = \lim_{n\to\infty} [\tfrac{1}{n}\{f(\tfrac{1}{n}) + \dots + f(\tfrac{n}{n})\}] = \int_0^1 f(x)\,dx \quad .$$

[4.2.6 (L) cont'd]

(b) This is an application of (a). Namely, if we let f(x) = sinπx, the result of (a) says

$$\lim_{n\to\alpha} \frac{1}{n} \sum_{k=1}^{n} \sin \frac{\pi k}{n} = \int_0^1 \sin\pi x \, dx$$

$$= -\frac{1}{\pi} \cos\pi x \Big|_0^1$$

$$= (-\frac{1}{\pi} \cos\pi) - (-\frac{1}{\pi} \cos 0)$$

$$= \frac{1}{\pi} + \frac{1}{\pi}$$

$$= \frac{2}{\pi} \quad .$$

4.2.7

In the previous exercise, we saw that

$$\lim_{n\to\infty} \frac{1}{n}[f(\frac{1}{n}) + f(\frac{2}{n}) + \ldots + f(\frac{n}{n})] = \int_0^1 f(x)\, dx \quad . \quad (1)$$

Let us take $f(x) = x^6$. Then (1) becomes:

$$\lim_{n\to\infty} \frac{1}{n}[\frac{1^6}{n^6} + \frac{2^6}{n^6} + \ldots + \frac{n^6}{n^6}] = \int_0^1 x^6 \, dx$$

or:

$$\lim_{n\to\infty} [\frac{1^6 + 2^6 + \ldots + n^6}{n^7}] = \frac{1}{7}x^7 \Big|_0^1 = \frac{1}{7} \quad .$$

[4.2.7 cont'd]

Notice that while the notation is new to us (relatively speaking) the concept is precisely as it was before. For example, had we been asked to find the area of the region R where R is bounded above by $y = x^6$, below by the x-axis, on the left by $x = 0$, and on the right by $x = 1$, then on the one hand this area is given by

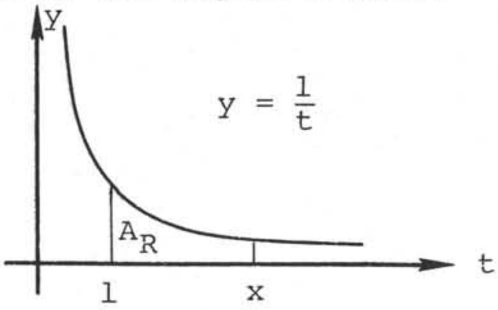
$$\int_0^1 x^6\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^6 \frac{1}{n}$$

and this is precisely the same expression that we are dealing with in this exercise.

UNIT 3: The Second Fundamental Theorem of Integral Calculus

4.3.1(L)

a. The key point here is that G(x) is precisely the area of the region R where



That is $G(x) = A_R(x)$ and, hence, $G'(x) = \frac{1}{x}$

What is really important is that our procedure in (a) gives us G(x) rather explicitly (as opposed to the implicit idea that G(x) is a function such that $G'(x) = \frac{1}{x}$) in the sense that we can compute G(x) for each $x \geqslant 1$.

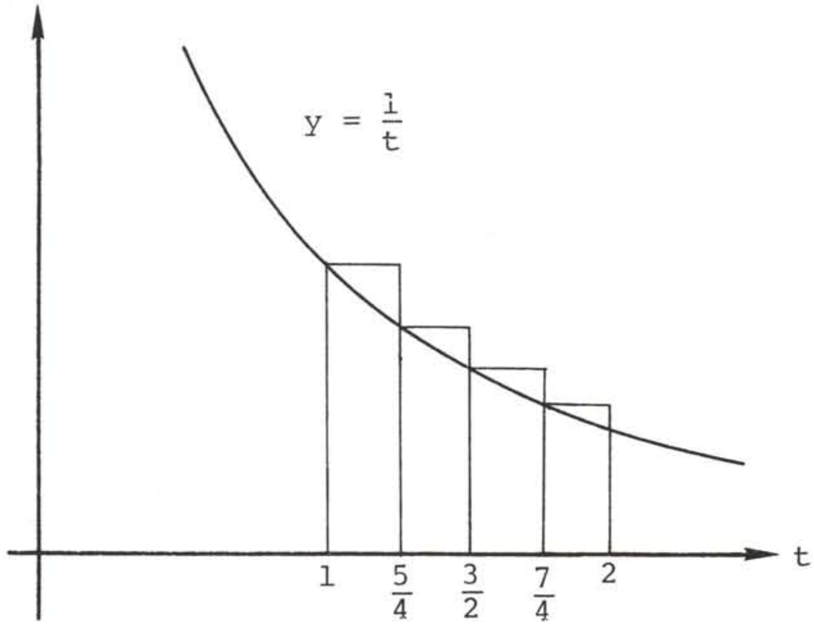
For example, $G(2) = A_R(2) = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\frac{1}{c_k}\right)\Delta x\right]$ etc.

To be sure, we might have difficulty evaluating the limit, but the point is that either we can or else we can approximate it to as great a degree of accuracy as we desire.

Specifically, in this exercise we are going to illustrate this idea by "locating" G(2) between 0.6 and 0.8, the idea being that the technique is easily generalized and/or refined.

[4.3.1 (L) cont'd]

b.



$$u_4 = \left(\frac{1}{1}\right)\frac{1}{4} + \left(\frac{1}{\frac{5}{4}}\right)\frac{1}{4} + \left(\frac{1}{\frac{3}{2}}\right)\frac{1}{4} + \left(\frac{1}{\frac{7}{4}}\right)\frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{319}{420}$$

$$L_4 = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840}$$

[4.3.1(L) cont'd]

$\therefore$ Since $A_R = G(2)$, $\frac{533}{840} < G(2) < \frac{319}{420}$

$\frac{319}{420} < 0.76$; $\frac{533}{840} > 0.63$; $\therefore$ $\underline{0.63 < G(2) < 0.76}$

$\therefore 0.6 < G(2) < 0.8$

4.3.2(L)

$$x = \int_0^y \frac{du}{\sqrt{1 - u^2}}$$

is known as an integral equation. An integral equation is one in which our "unknown" appears as one of the limits of a definite integral. The computational key lies in the second fundamental theorem. Namely,

$$\frac{d}{dy}\int_a^y f(u)\,du = f(y)$$

(In terms of a quick recipe, we replace u by y in the integral to obtain the derivative.)

In any event, from $x = \int_0^y \frac{du}{\sqrt{1-u^2}}$,we obtain

$$\frac{dx}{dy} = \frac{d}{dy}\left[\int_0^y \frac{du}{\sqrt{1-u^2}}\right]$$

$$= \frac{1}{\sqrt{1-y^2}} \qquad (1)$$

From (1) we now proceed as usual (the main idea was that we used the fundamental theorem to obtain (1)). Namely,

[4.3.2(L) cont'd]

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \sqrt{1-y^2} \qquad (2)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d\sqrt{1-y^2}}{dy}\frac{dy}{dx}$$

$$= \left[\frac{-2y}{2\sqrt{1-y^2}}\right]\frac{dy}{dx}$$

$$= \frac{-y}{\sqrt{1-y^2}}\frac{dy}{dx} \qquad (3)$$

Putting the value of $\frac{dy}{dx}$ from (2) into (3), we obtain

$$\frac{d^2y}{dx^2} = \frac{-y}{\sqrt{1-y^2}}(\sqrt{1-y^2}) \qquad (|y| < 1)$$

$$= -y$$

which is the desired result.

[Note: In order to exhibit an integral equation that could be solved without too much difficulty we "cheated" just a little. Namely, recall that

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u \quad \therefore \int_0^y \frac{du}{\sqrt{1-u^2}} = \sin^{-1}y \qquad .$$

Thus $x = \int_0^y \frac{du}{\sqrt{1-u^2}}$ was a "fancy way" of saying $x = \sin^{-1}y$ or $y = \sin x$. Now it should be clear that if $y = \sin x$, $\frac{d^2y}{dx^2} = -y$. In all fairness, however, $\int_0^y \frac{du}{\sqrt{1-u^2}}$ is more profound than it may seem to be at first glance. In particular, $\int_0^y \frac{du}{\sqrt{1-u^2}}$ is an analytic representation which does not depend on geometry. In other words,

[4.3.2(L) cont'd]

$\sin^{-1} y = \int_0^y \frac{du}{\sqrt{1-u^2}}$ gives us a non-geometric way for defining the (inverse) circular functions.]

4.3.3 Again, all we are doing here is emphasizing that if f is continuous

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

In this example $f(t) = \frac{\sin \pi t}{1+t^2}$

Thus:

$$G'(x) = \frac{d}{dx}\int_0^x \frac{\sin \pi t}{1+t^2}\,dt = \frac{\sin \pi x}{1+x^2}$$

$$\therefore\ G'(\tfrac{1}{6}) = \frac{\sin \frac{\pi}{6}}{1+(\frac{1}{6})^2} = \frac{\frac{1}{2}}{1+\frac{1}{36}} = \frac{\frac{1}{2}}{\frac{36+1}{36}} = \frac{1}{2}\times\frac{36}{36+1}$$

$$= \frac{18}{1+36} = \frac{18}{37}$$

4.3.4(L) Stripped of embellishment, this exercise is another example of an integral equation. Namely, if we let A(t) denote the area under the graph of y = f(x) between x = 0 and x = t, we have

$$A(t) = [f(t)]^3 \qquad (1)$$

Since $A(t) = \int_0^t f(x)\,dx$, we have that

$$A'(t) = f(t) \qquad (2)$$

Putting this into (1), we obtain

$$\left.\begin{aligned} A'(t) &= [f(t)^3]' \\ f(t) &= [f(t)^3]' \end{aligned}\right\} \qquad (3)$$

[4.3.4(L) cont'd]

Now $[f^3(t)]' = 3f^2(t)f'(t)$ * (4)

Coupling (3) and (4), we have

$$f(t) = 3f^2(t)f'(t) \tag{5}$$

$$\therefore 1 = 3f(t)f'(t) \quad ** \tag{5'}$$

$$1 = \frac{3}{2}[2f(t)f'(t)]$$

$$= \frac{3}{2}[f^2(t)]' \tag{6}$$

And, "integrating" (6), we obtain

$$t + c = \frac{3}{2}f^2(t)$$

$$\therefore \quad f(t) = \sqrt{\frac{2}{3}t + c} \tag{7}$$

Putting $f(0) = 0$ into (7) we have $c = 0$

$$\therefore f(t) = \sqrt{\frac{2}{3}t} \qquad \text{or} \qquad f(x) = \sqrt{\frac{2x}{3}} \tag{8}$$

---

*This is just the chain rule. Let $u = f(t)$ then $u^3 = [f(t)]^3$. Thus $[f^3(t)]' = \frac{d(u^3)}{dt} = 3u^2\frac{du}{dt} = 3[f(t)]^2 f'(t)$

**We cancelled $f(t)$ from both sides of (5) which is okay provided $f(t) \equiv 0$. If $f(t) \equiv 0$, we have a "trivial" solution to the exercise. Thus to insure an "interesting" problem, we assume $f(t)$ is not indentically zero.

[4.3.4(L) cont'd]

(An alternative way of obtaining (8) is to let $u = f(t)$ in (5'). We then obtain

$$1 = 3u\frac{du}{dt}$$

or
$$\frac{dt}{3} = udu$$

$$\therefore \quad \frac{1}{3}t + c_1 = \frac{1}{2}u^2$$

$$\therefore \quad u^2 = \frac{2}{3}t + c$$

$$\therefore \quad u = \sqrt{\frac{2}{3}t + c}$$

As a check:

$$\int_0^t \sqrt{\frac{2}{3}x}\ dx = \int_0^t \sqrt{\frac{2}{3}}\ x^{\frac{1}{2}}\ dx$$

$$= \sqrt{\frac{2}{3}}\ \frac{2}{3}\ x^{\frac{3}{2}}\Big|_0^t$$

$$= \sqrt{\frac{2}{3}}\ \frac{2}{3}\ t^{\frac{3}{2}}$$

$$= \left(\sqrt{\frac{2}{3}\ t}\ \right)^3$$

$$= f^3(t)$$

4.3.5 Here we only want to stress the chain rule again. Namely $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ requires that our upper limit and the variable of differentiation be the same. In this sense

$$\frac{d}{d(x^2)}\int_0^{x^2}\frac{\sin \pi t}{1+t^2}dt = \frac{\sin \pi(x^2)}{1+(x^2)^2} = \frac{\sin(\pi x^2)}{1+x^4} \qquad (1)$$

However, we want

$$\frac{d}{dx}\int_0^{x^2}\frac{\sin \pi t}{1+t^2}dt$$ which is equal to

$$\left[\frac{d}{d(x^2)}\int_0^{x^2}\frac{\sin \pi t}{1+t^2}dt\right]\frac{dx^2}{dx}$$

Using (1) and the fact that $\frac{dx^2}{dx} = 2x$, we obtain

$$\frac{d}{dx}\int_0^{x^2}\frac{\sin \pi t \ dt}{1+t^2} = 2x\left[\frac{\sin(\pi x^2)}{1+x^4}\right]$$

i.e.,

$$H'(x) = \frac{2x\sin(\pi x^2)}{1+x^4}$$

4.3.6

$$G(t) = \int_{t^2}^{t^4}\frac{\sin x \ dx}{1+x^4} = \int_0^{t^4}\frac{\sin x \ dx}{1+x^4} - \int_0^{t^2}\frac{\sin x \ dx}{1+x^4}$$

[4.3.6 cont'd]

$$\therefore\ G'(t) = \frac{d}{dt}\left[\int_0^{t^4} \frac{\sin x\,dx}{1+x^4}\right] - \frac{d}{dt}\left[\int_0^{t^2} \frac{\sin x\,dx}{1+x^4}\right]$$

$$= \frac{d}{dt^4}\left[\int_0^{t^4} \frac{\sin x\,dx}{1+x^4}\right]\frac{dt^4}{dt} - \frac{d}{dt^2}\left[\int_0^{t^2} \frac{\sin x\,dx}{1+x^4}\right]\frac{dt^2}{dt}$$

$$= \left[\frac{\sin (t^4)}{1+(t^4)^4}\right]4t^3 - \left[\frac{\sin (t^2)}{1+(t^2)^4}\right]2t$$

$$= \frac{4t^3 \sin (t^4)}{1+t^{16}} - \frac{2t \sin (t^2)}{1+t^8}$$

4.3.7(L)

a. Consider $\int_a^b f(x)dx$

By the first Fundamental Theorem, we have:

$$\int_a^b f(x)dx = G(b) - G(a) \tag{1}$$

But, $G(b) - G(a) = (b-a)\ G'(c)$ (2)

some $c \in (a,b)$ by mean value theorem.

Combining (1) and (2) we obtain

$$\int_a^b f(x)dx = (b-a)\ G'(c) \tag{3}$$

some $c \in (a,b)$

[4.3.7(L) cont'd]

Finally, since G' = f, we have

$$G'(c) = f(c) \tag{4}$$

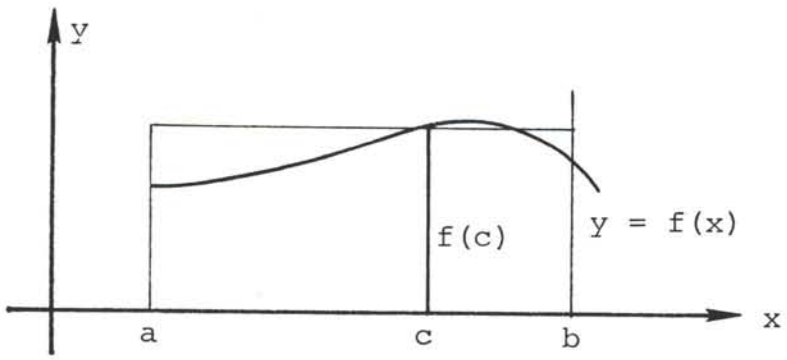
Putting (4) into (3):

$$\int_a^b f(x)\,dx = (b-a)\ f(c) \tag{5}$$

$$\text{some } c \in (a,b)$$

Because of this rather strong connection with the mean value theorem of differential calculus, (5) is usually called the mean value theorem of integral calculus.

For a given $c \in [a,b]$, $(b-a)\ f(c)$ denotes the area of the rectangle whose height is f(c) and whose base is (b-a)

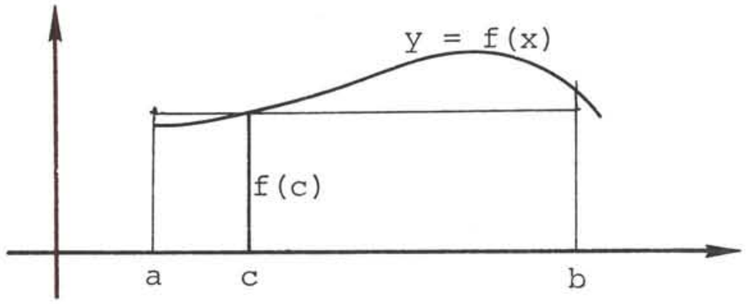


(Figure 1)

As drawn is Figure 1, (b-a) f(c) is apparently greater than the area under the graph of y = f(x) from a to b (A_R).

On the other hand, as drawn in Figure 2, (b-a)f(c) is less than A_R.

[4.3.7(L) cont'd]



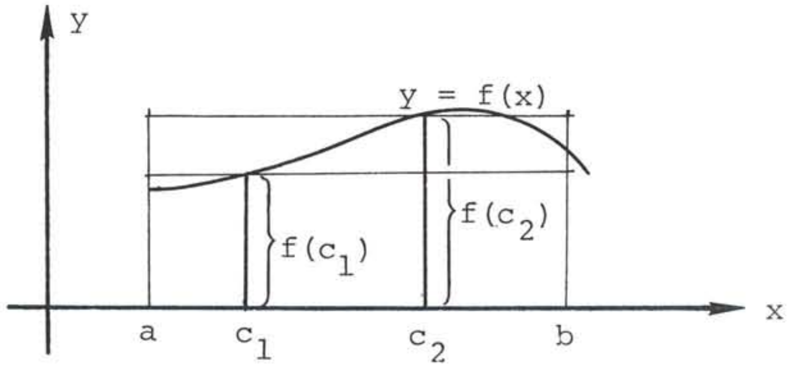
(Figure 2)

The mean value theorem [part (a)] says that c may be chosen such that (b-a) f(c) is exactly equal to A_R. Intuitively, the result may be arrived at by successive approximations. For example, if we combine our results of Figures 1 and 2, we see that

$$f(c_1)(b-a) < A_R < f(c_2)(b-a) \qquad \text{(see Figure 3)} \qquad (6)$$

Since f(x)(b-a) is continuous, (6) tells us that the desired c lies between c_1 and c_2. We then pick $c_3 \varepsilon (c_1, c_2)$ and look at $f(c_3)(b-a)$.

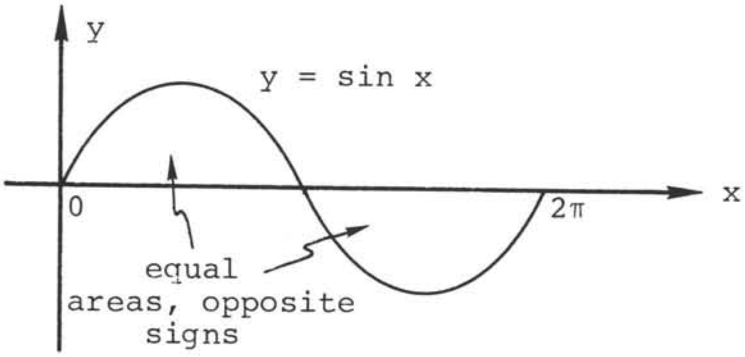
If $f(c_3)(b-a) > A_R$ then c lies between c_1 and c_3 while if $f(c_3)(b-a) < A_R$, c lies between c_2 and c_3. We may continue in this way, each time getting better approximations. In this sense, we arrive at c by a version of the limit process.



(Figure 3)

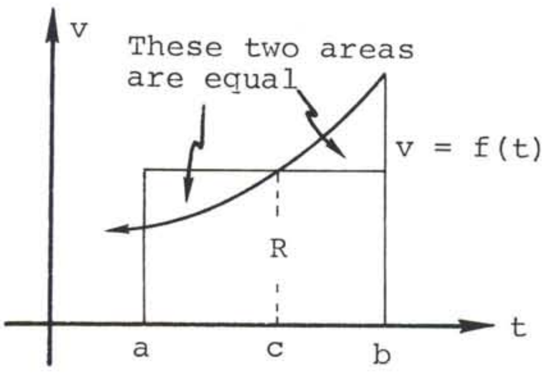
[4.3.7(L) cont'd]

Note: The restriction that f is non-negative is not crucial if we consider net area rather than total area. For example, if $f(x) = \sin x$ $0 \leqslant x \leqslant 2\pi$ we find that $\int_0^{2\pi} \sin x\, dx =$ $-\cos 2\pi - (-\cos 0) = 0$. Namely,



In this case $c = \pi$, $f(c) = 0$

It is also interesting to notice that there is good reason, at least in terms of our geometric interpretation, to define f(c) [where c is as in (5)] to be the average value of f(x) on [a,b]. A rather nice interpretation of this idea is in terms of velocity-time graphs. Namely,



$A_R = f(c)(b-a)$

[4.3.7(L) cont'd]

Now, A_R denotes the distance travelled by the particle if it moves according to $v = f(t)$ $a \leq t \leq b$. On the other hand, $f(c)(b-a)$ is the distance travelled by a particle moving at the constant speed of $f(c)$ between $t = a$ and $t = b$. This is clearly the meaning of average speed since $f(c)(b-a)$ = distance travelled ,etc.

b. We have $\int_a^b x^2 dx = A_R = \frac{1}{3} x^3 \Big|_a^b = \frac{1}{3}(b^3 - a^3)$

Now c must satisfy

$$(b-a)\ f(c) = \frac{1}{3}(b^3 - a^3) \tag{1}$$

Since $b^3 - a^3 = (b-a)(b^2+ab+a^2)$, (1) becomes:

$$(b-a)\ f(c) = \frac{1}{3}(b-a)(b^2+ab+a^2)$$

and since $b \neq a$,

$$f(c) = \frac{b^2+ab+a^2}{3} \tag{2}$$

Since $f(c) = c^2$, (2) becomes:

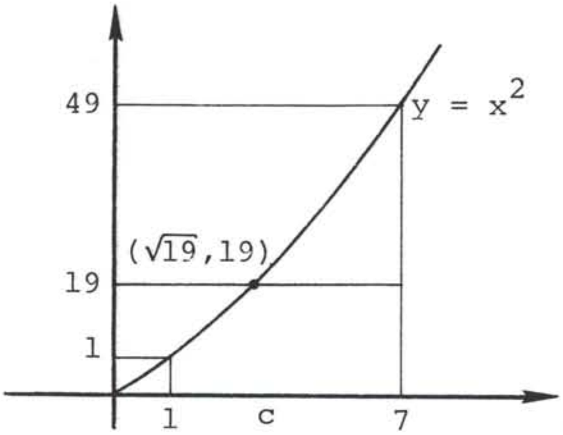
$$c = \sqrt{\frac{b^2 + ab + a^2}{3}}$$

c. This is part (b) with $a = 1$ and $b = 7$

Then $\frac{b^2 + ab + a^2}{3} = \frac{49 + 7 + 1}{3} = 19$

[4.3.7(L) cont'd]

∴ Height of rectangle = 19 and it meets $y = x^2$ at $(\sqrt{19}, 19)$



4.3.8(L) At first glance, this problem may seem superfluous. Don't we already know that G' = f? The point is that this exercise is concerned with actually differentiating the "concrete" G(x) which can be constructed (as in Exercise 4.3.1) from the given f(x).

In any event, we have

$$G(x + \Delta x) = \int_a^{x+\Delta x} f(t)\,dt$$

$$G(x) = \int_a^x f(t)\,dt$$

$$\therefore G(x + \Delta x) - G(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+\Delta x} f(t)\,dt$$

$$= (x + \Delta x - x)\ f(c),$$

some c between x and $x + \Delta x$

4.3.8(L)

$$= \Delta x\, f(c)$$

$$\therefore \quad \frac{G(x + \Delta x) - G(x)}{\Delta x} = f(c)$$

$$\therefore \quad \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} f(c) = f(x),$$

since f is continuous and $c \to x$ as $\Delta x \to 0$.

4.3.9 a. By mean value theorem of integral calculus

$$\int_{x_1}^{x} f(t)\,dt = f(c)[x - x_1] \quad \text{where c is between x and } x_1 \qquad (1)$$

$$\therefore \quad \frac{x^2}{x - x_1} \int_{x_1}^{x} f(t)\,dt = \frac{x^2}{x - x_1} f(c)(x - x_1) \qquad (2)$$

and since we may assume $x \neq x_1$ (otherwise $\int_{x_1}^{x} f(t)\,dt \equiv 0$),
(2) becomes:

$$\frac{x^2}{x - x_1} \int_{x_1}^{x} f(t)\,dt = x^2 f(c)$$

$$\therefore \quad \lim_{x \to x_1} \left[\frac{x^2}{x - x_1} \int_{x_1}^{x} f(t)\,dt \right] = \lim_{x \to x_1} x^2 f(c)$$

$$= \lim_{x \to x_1} (x^2) \bullet \lim_{x \to x_1} f(c) \qquad (3)$$

Clearly $\lim_{x \to x_1} x^2 = x_1^2$; and since c is between x and x_1,

[4.3.9 cont'd]

$\lim_{x \to x_1} f(c) = \lim_{c \to x_1} f(c) = f(x_1)$, since f is continuous.

Putting these results into (3) yields:

$$\lim_{x \to x_1} \left[\frac{x^2}{x - x_1} \int_{x_1}^{x} f(t)\,dt\right] = {x_1}^2 f(x_1)$$

(The key here is that $\lim_{x \to x_1} \left\{ \dfrac{\int_{x_1}^{x} f(t)\,dt}{x - x_1} \right\} = f(x_1)$,

since $\int_{x_1}^{x} f(t)\,dt = f(c)(x-x_1)$ and c is between x and x_1).

b.

$\lim_{x \to 2} \left[\frac{4}{x-2} \int_2^x \frac{\cos^2 t}{1 + t}\,dt\right]$ is the special case of part (a) with $x_1 = 2$ and $f(t) = \frac{\cos^2 t}{1 + t}$

Hence:

$$\lim_{x \to 2} \frac{4}{x-2} \int_2^x \frac{\cos^2 t}{1 + t}\,dt = \frac{(2)^2 \cos^2 2}{1 + 2} = \frac{4}{3} \cos^2 2$$

(By the way in $\cos^2 2$, 2 is either a pure number or else it is an angle measured in radians not degrees.)

UNIT 4: Some Simple Applications of the Definite Integral

4.4.1(L)

a. Given $y = 12x(x-1)^2$, we see at once that the curve meets the x-axis at (0,0) and (1,0).

Moreover, $y = 12x(x-1)^2 = 12x^3 - 24x^2 + 12x$ (1)

Hence, $y' = 36x^2 - 48x + 12$

$= 12(3x^2 - 4x + 1)$

$= 12(3x - 1)(x - 1)$ (2)

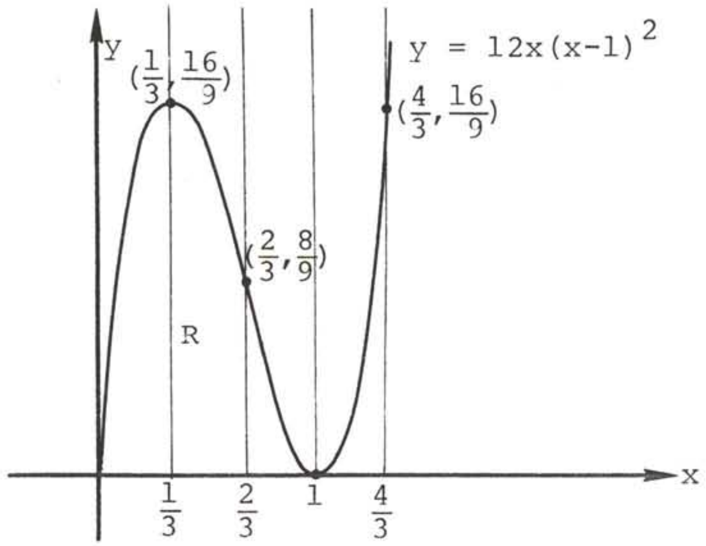
and $y'' = 72x - 48$

$= 24(3x - 2)$ (3)

From (2) we see that $y' = 0 \leftrightarrow x = 1$ or $x = \frac{1}{3}$; and from (3) we see that $y''\big|_{x=1} > 0$ while $y''\big|_{x=\frac{1}{3}} < 0$. Thus (1,0) is a low point of the curve while $(\frac{1}{3}, \frac{16}{9})$* is a high point.

Moreover, (3) also tells us that the curve changes concavity when $x = \frac{2}{3}$, or at the point $(\frac{2}{3}, \frac{8}{9})$.

Thus, our curve has the form:



*Recall, for a given x, y is determined by $y = 12x(x-1)^2$.

[4.4.1(L) cont'd]

b. $A_R = \int_0^1 12x\,(x-1)^2\,dx$

$$= \int_0^1 (12x^3 - 24x^2 + 12x)\,dx$$

$$= 3x^4 - 8x^3 + 6x^2 \Big|_{x=0}^{x=1}$$

$$= 3 - 8 + 6$$

$$= 1$$

c. We first find where the two curves intersect. Solving

$$\left.\begin{array}{l} y = 12x\,(x-1)^2 \\ y = \frac{4}{3}x \end{array}\right\} \text{ we have: } \begin{array}{l} 12x\,(x-1)^2 = \frac{4}{3}x \\ 36x\,(x-1)^2 = 4x \end{array} \text{ or: }$$

$$\therefore\ 9x\,(x-1)^2 = x \qquad (4)$$

If $x \neq 0$ in (4) we have $9(x-1)^2 = 1$

or $\quad 3(x-1) = \pm 1 \qquad (5)$

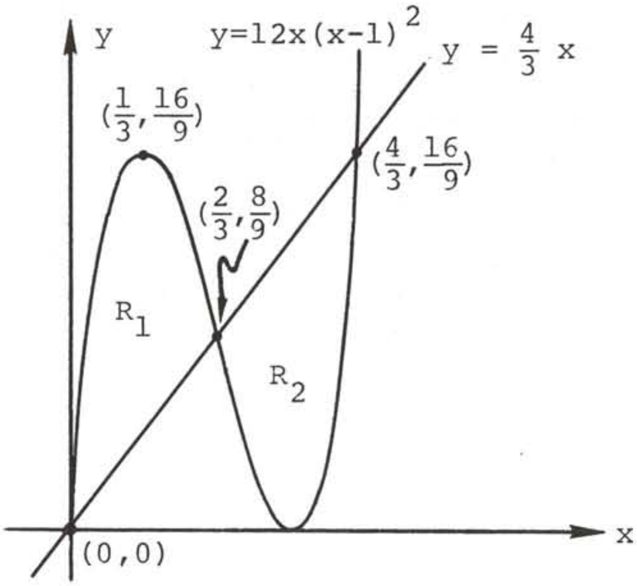
$$\therefore\ x - 1 = \pm\frac{1}{3}$$

$$\therefore\ x = 1 \pm \frac{1}{3} = \frac{2}{3} \text{ or } \frac{4}{3} \qquad (6)$$

Since these points must belong to $y = \frac{4}{3}x$ we see that if $x = \frac{2}{3}$, $y = \frac{8}{9}$ while if $x = \frac{4}{3}$, $y = \frac{16}{9}$ (We could have used $y = 12x\,(x-1)^2$, but $y=\frac{4}{3}x$ is easier to handle. As a check against errors, we can see if our answers satisfy $y = 12x\,(x-1)^2$.) Thus, from (6), we see that the curves intersect at $(\frac{2}{3},\frac{8}{9})$ and $(\frac{4}{3},\frac{16}{9})$. Moreover since the validity of (5) required that $x \neq 0$, we see that $x = 0$ is another candidate for a solution to (4). Since $x = 0$ satisfies (4), we have that (0,0) is also a point of intersection.

[4.4.1(L) cont'd]

Pictorially,



Now, for R_1, we have

$$A_{R_1} = \int_0^{\frac{2}{3}} \left[12x\,(x-1)^2 - \frac{4}{3}x\right]\,dx \qquad (7)$$

while for R_2

$$A_{R_2} = \int_{\frac{2}{3}}^{\frac{4}{3}} \left[\frac{4}{3}\,x - 12x\,(x-1)^2\right]\,dx^{*} \qquad (8)$$

---

*Notice that while we did not need a graph as accurate as that obtained in part (a), we do need enough accuracy to determine which curve is "on top". That is, if $f(x)>g(x)$ on $[a,b]$, $\int_a^b [f(x)-g(x)]dx>0$ while if $f(x)<g(x)$ on $[a,b]$ then $\int_a^b [f(x)-g(x)] <0$. Unless we keep track of how the curves "crisscross", we will find the net area rather than the total area.

[4.4.1(L) cont'd]

Now $12x\,(x-1)^2 - \frac{4}{3}x = 12x^3 - 24x^2 + 12x - \frac{4}{3}x$

$$= 12x^3 - 24x^2 + \frac{32}{3}x\ ,$$

Hence $\frac{4}{3}x - 12x(x-1)^2 = -12x^3 + 24x^2 - \frac{32}{3}x$

Putting these results into (7) and (8), we obtain

$$A_{R_1} = \int_0^{\frac{2}{3}} (12x^3 - 24x^2 + \frac{32x}{3})\,dx$$

$$= 3x^4 - 8x^3 + \frac{16x^2}{3}\Bigg|_{x=0}^{x=\frac{2}{3}}$$

$$= 3(\tfrac{2}{3})^4 - 8(\tfrac{2}{3})^3 + \tfrac{16}{3}(\tfrac{2}{3})^2$$

$$= \frac{16}{27} - \frac{64}{27} + \frac{64}{27} = \frac{16}{27} \qquad (9)$$

$$A_{R_2} = \int_{\frac{2}{3}}^{\frac{4}{3}} [-12x^3 + 24x^2 - \frac{32x}{3}]\,dx$$

$$= -3x^4 + 8x^3 - \frac{16x^2}{3}\Bigg|_{x=\frac{2}{3}}^{x=\frac{4}{3}}$$

[4.4.1(L) cont'd]

$$= \left[-3\left(\tfrac{4}{3}\right)^4 + 8\left(\tfrac{4}{3}\right)^3 - \tfrac{16}{3}\left(\tfrac{4}{3}\right)^2\right] - \left[-3\left(\tfrac{2}{3}\right)^4 + 8\left(\tfrac{2}{3}\right)^3 - \tfrac{16}{3}\left(\tfrac{2}{3}\right)^2\right]$$

$$= \left[\tfrac{-256}{27} + \tfrac{512}{27} - \tfrac{256}{27}\right] + \left[3\left(\tfrac{2}{3}\right)^4 - 8\left(\tfrac{2}{3}\right)^3 + \tfrac{16}{3}\left(\tfrac{2}{3}\right)^2\right]$$

$$= 0 + \tfrac{16}{27}$$

$$= \tfrac{16}{27} \qquad (10)$$

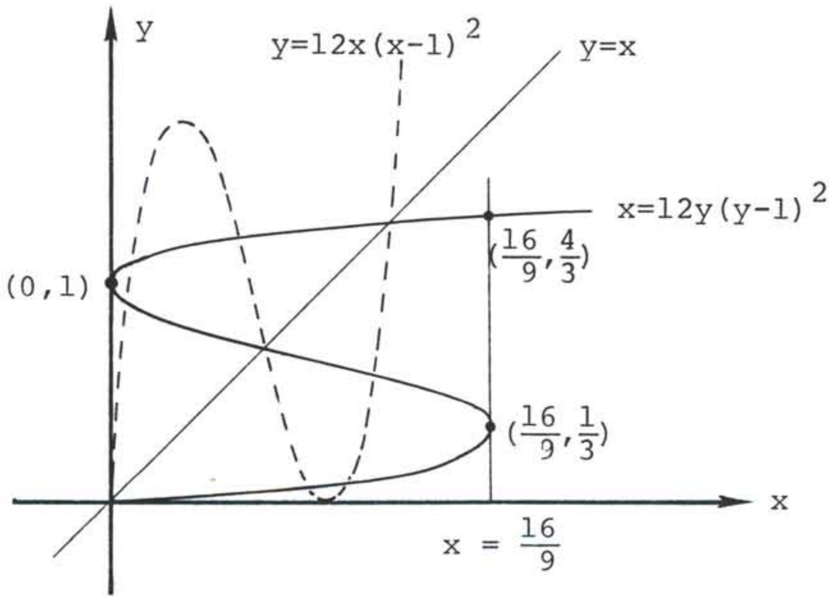
Combining (9) and (10) we obtain

$$A = A_{R_1} + A_{R_2} = \tfrac{16}{27} + \tfrac{16}{27} = \tfrac{32}{27} \qquad (11)$$

(Notice that $\int_0^{\frac{4}{3}} [12x\ (x-1)^2 - \tfrac{4}{3}x]\ dx = 0$ and this means our net area is zero. This is confirmed by the fact that $|A_{R_1}| = |A_{R_2}|$.)

4.4.2(L) This exercise gives us additional practice with the idea of inverse functions. We begin by observing that the curve $x = 12y(y-1)^2$ is the reflection of the curve $y = 12x(x-1)^2$ with respect to the line $y = x$.

We have, quite conveniently, in the previous exercise, sketched the curve $y = 12x(x-1)^2$. We have, therefore:*



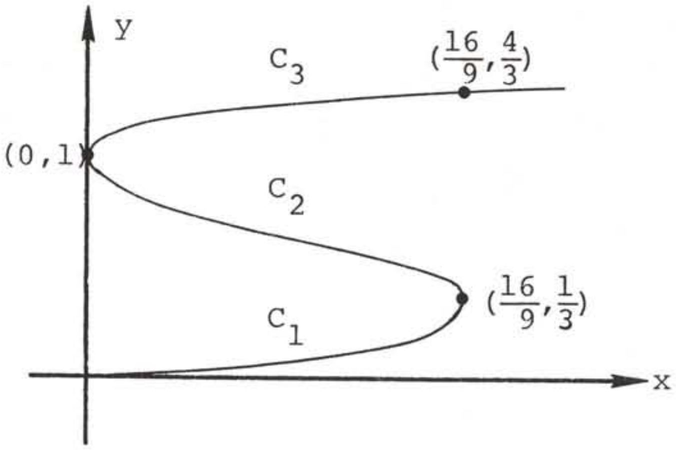
(Figure 1)

*we can, of course, obtain the graph directly by computing $\frac{dx}{dy}$ etc., but since we are used to working with $y = f(x)$ it might be easier, conceptually, to graph $y = 12x(x-1)^2$ first, rotate this through ninety degrees and the "flip it over", which is what we have done to arrive at Figure 1.

[4.4.2(L) cont'd]

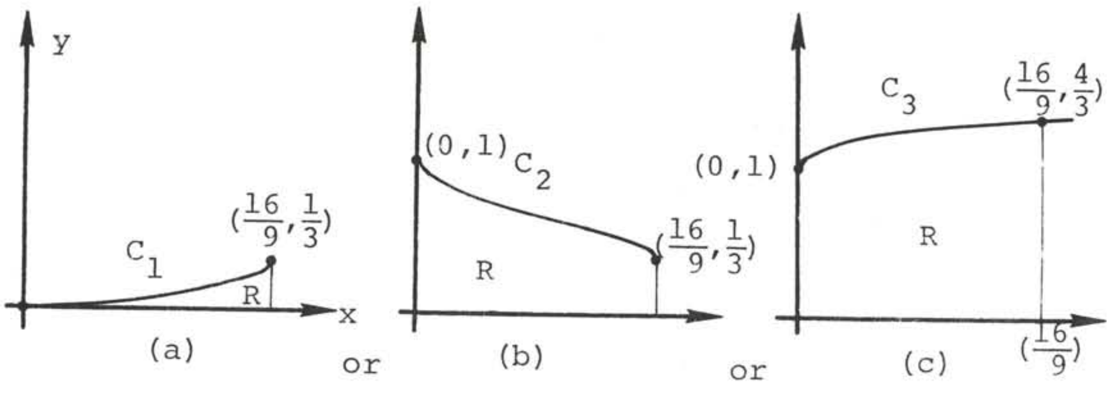
Since $y = 12x(x-1)^2$ was not one-to-one, $x = 12y(y-1)^2$ is not single valued.

We thus arrive at one awkward stage since there are now three possible tops for our region. That is, let's view $x = 12y(y-1)^2$ as $C_1 \cup C_2 \cup C_3$ where:



(Figure 2)

Notice, then, that either C_1 or C_2 or C_3 could be the upper boundary of our region. That is, R could look like



(Figure 3)

[4.4.2(L) cont'd]

In evaluating A_R in the form $\lim_{\max \Delta x_k \to 0} \left[\sum_{k=1}^{n} f(c_k) \Delta x_k \right]$

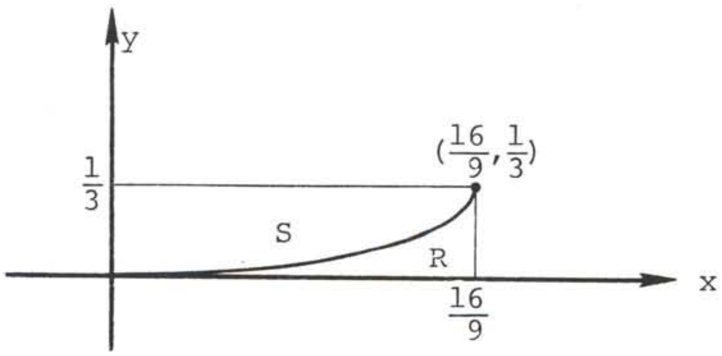
we would have the definite integral $\int_0^{\frac{16}{9}} f(x)\,dx$

where now f(x) can be related to either C_1 or C_2 or C_3. More explicitly, with enough "know how" we could take $x = 12y(y-1)^2$ and solve for y in terms of x. We could get three different solutions to correspond to the three single-valued branches.

It is not the aim of this course to teach one how to solve cubic equations. Of even more importance, this exercise could be generalized into one for which it was impossible to solve for y explicitly in terms of x.

The procedure for solving this problem hinges on thinking of the area in the form $\int_{y=a}^{y=b} x\,dy$, for in this example x is given explicitly as a function of y. Say $x = g(y)$, then $\int_a^b x\,dy = \int_a^b g(y)\,dy$ and our integrand is in "good shape".

For example, if we pick C_1 to be our top, we have



[4.4.2(L) cont'd]

$$A_R = (\frac{16}{9})(\frac{1}{3}) - A_s = \frac{16}{27} - A_s$$

But

$$A_s = \int_0^{\frac{1}{3}} x dy = \int_0^{\frac{1}{3}} 12y(y-1)^2\, dy$$

$$= \int_0^{\frac{1}{3}} (12y^3 - 24y^2 + 12y)\, dy$$

$$= 3y^4 - 8y^3 + 6y^2 \Big|_{y=0}^{y=\frac{1}{3}} = \frac{1}{27} - \frac{8}{27} + \frac{2}{3} = \frac{11}{27}$$

$$\therefore\ A_R = \frac{16}{27} - \frac{11}{27} = \frac{5}{27}$$

If we choose C_2 then

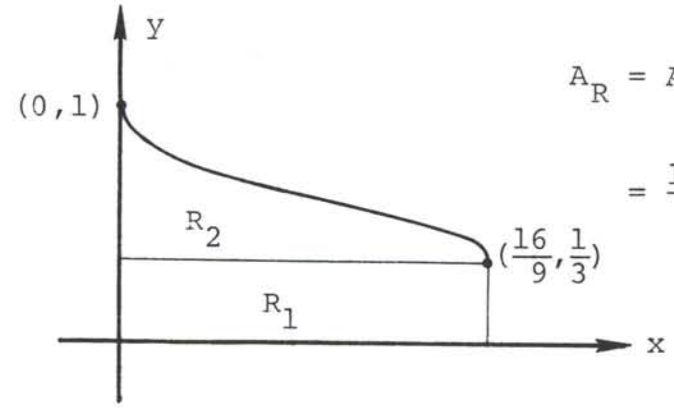
$$A_R = (\frac{16}{9})\ \frac{1}{3} + \int_{\frac{1}{3}}^{1} x dy = \frac{16}{27} + \left[3y^4 - 8y^3 + 6y^2\right|_{\frac{1}{3}}^{1}$$

$$= \frac{16}{27} + [1 - \frac{11}{27}]$$

$$\therefore\ A_R = \frac{32}{27}$$

[4.4.2(L) cont'd]

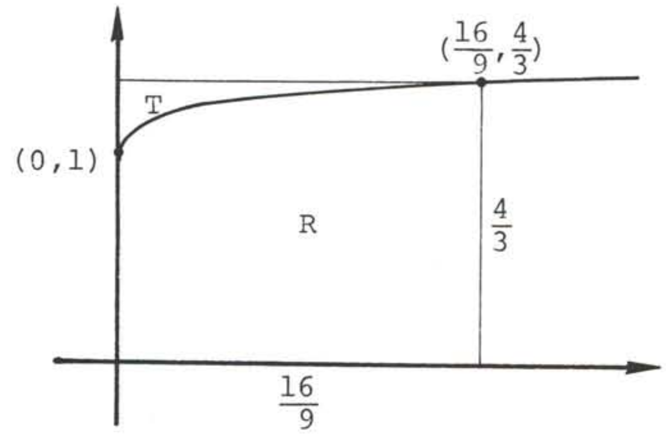
Pictorially;



$$A_R = A_{R_1} + A_{R_2}$$

$$= \frac{16}{9}\left(\frac{1}{3}\right) + \int_{\frac{1}{3}}^{1} x\,dy$$

Finally, if we let C_3 be our top, we have:



[4.4.2(L) cont'd]

$$A_R = \left(\frac{16}{9}\right)\frac{4}{3} - A_T = \frac{64}{27} - \int_1^{\frac{4}{3}} x\,dy$$

$$= \frac{64}{27} - \int_1^{\frac{4}{3}} 12y(y-1)^2\,dy$$

$$= \frac{64}{27} - \left[3y^4 - 8y^3 + 6y^2 \right|_1^{\frac{4}{3}}$$

$$= \frac{64}{27} - \left[\left\{3\left(\frac{4}{3}\right)^4 - 8\left(\frac{4}{3}\right)^3 + 6\left(\frac{4}{3}\right)^2\right\} - 1\right]$$

$$= \frac{64}{27} - \left[\left\{\frac{256}{27} - \frac{512}{27} + \frac{96}{9}\right\} - 1\right]$$

$$= \frac{64}{27} - \left[\frac{32}{27} - 1\right] = \frac{64}{27} - \frac{5}{27}$$

$$\therefore A_R = \frac{59}{27}$$

Notice how the correct answer requires that we be able to isolate, or identify, the various single-valued branches of our curve.

4.4.3 We first solve the pair of simultaneous equations

$$\left.\begin{aligned} y &= x^3 \\ y &= 7x - 6 \end{aligned}\right\} \text{ to find where the curves intersect}$$

We obtain

$$x^3 = 7x - 6$$

or:

$$x^3 - 7x + 6 = 0 \qquad (1)$$

Letting $P(x) = x^3 - 7x + 6$, we see that $P(1) = 0$.
Hence $(x - 1)$ is a factor of $x^3 - 7x + 6$.
In fact:

$$\begin{array}{r|l} x^3 - 7x + 6 & x - 1 \\ \cline{2-2} \underline{x^3 - x^2} & x^2 + x - 6 \\ x^2 - 7x + 6 & \\ \underline{x^2 - x} & \\ -6x + 6 & \\ \underline{-6x + 6} & \\ 0 & \end{array}$$

$$\therefore\ x^3 - 7x + 6 = (x - 1)(x^2 + x - 6)$$

$$= (x - 1)(x - 2)(x + 3) \qquad (2)$$

Putting (2) into (1), we see

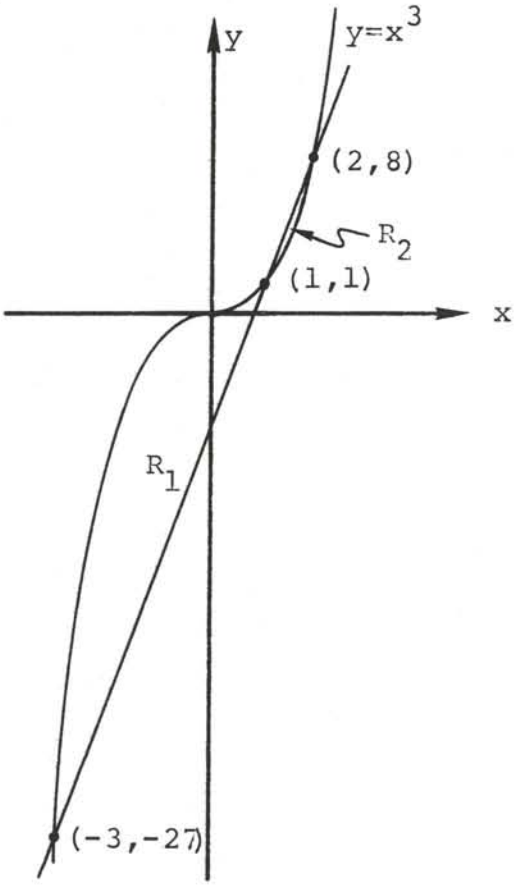
$$(x - 1)(x - 2)(x + 3) = 0$$

$$\therefore\ x = 1, \text{ or } x = 2, \text{ or } x = -3$$

[4.4.3 cont'd]

Substituting these values of x into $y = x^3$, we see that the two curves intersect at (1,1),(2,8), and (-3,-27)

Pictorially,



Now $A_R = A_{R_1} + A_{R_2}$ and:

$$A_{R_1} = \int_{-3}^{1} x^3 - (7x - 6)\, dx \quad \text{and}$$

$$A_{R_2} = \int_{1}^{2} [(7x - 6) - x^3]\, dx$$

[4.4.3 cont'd]

$$\therefore A_{R_1} = \int_{-3}^{1} (x^3 - 7x + 6)\, dx$$

$$= \frac{1}{4}x^4 - \frac{7}{2}x^2 + 6x \Big|_{-3}^{1}$$

$$= \left(\frac{1}{4} - \frac{7}{2} + 6\right) - \left[\frac{1}{4}(-3)^4 - \frac{7}{2}(-3)^2 + 6(-3)\right]$$

$$= \frac{11}{4} - \left[\frac{81}{4} - \frac{63}{2} - 18\right]$$

$$= \frac{11}{4} - \left[\frac{81 - 126 - 72}{4}\right] = \frac{11}{4} - \frac{-117}{4} = \frac{128}{4}$$

$$= 32$$

$$A_{R_2} = \int_{1}^{2} (7x - 6 - x^3)\, dx$$

$$= \frac{7}{2}x^2 - 6x - \frac{1}{4}x^4 \Big|_{1}^{2}$$

$$= \left[\frac{7}{2}(2)^2 - 6(2) - \frac{1}{4}(2)^4\right] - \left[\frac{7}{2} - 6 - \frac{1}{4}\right]$$

$$= -2 - \left(-\frac{11}{4}\right) = -2 + \frac{11}{4}$$

$$= \frac{3}{4}$$

$$A_R = A_{R_1} + A_{R_2} = 32 + \frac{3}{4} \text{ or } \frac{131}{4}$$

[4.4.3 cont'd]

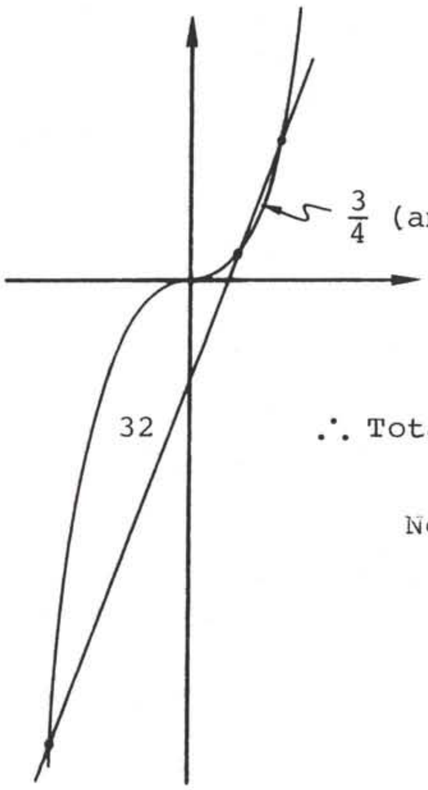
(As a check observe that $\int_{-3}^{2}[x^3 - (7x - 6)]\,dx =$

$$\frac{1}{4}x^4 - \frac{7}{2}x^2 + 6x \Big|_{-3}^{2}$$

$$= 2 - \left(-\frac{117}{4}\right)$$

$$= \frac{125}{4} = 31\frac{1}{4}$$

which is the correct net area, since:



$\frac{3}{4}$ (and the negative sign since $x^3-(7x-6)$ is negative here.)

$\therefore$ Total Area $= 32 + \frac{3}{4} = 32\frac{3}{4}$

Net Area $= 32 - \frac{3}{4} = 31\frac{1}{4}$

4.4.4

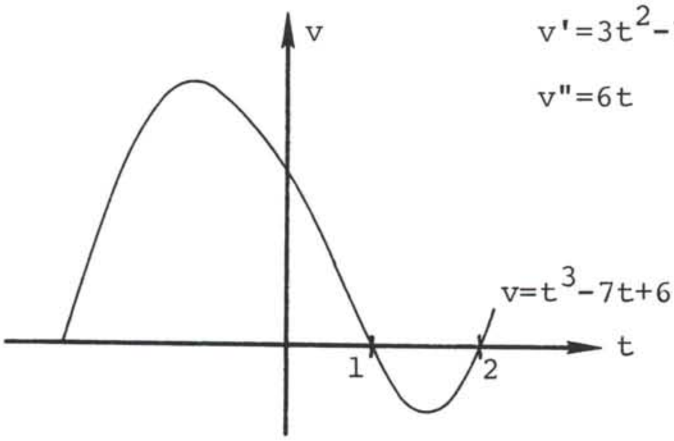
$$v = t^3 - 7t + 6$$

$$v = (t+3)(t-1)(t-2) \quad \text{(from previous exercise)}$$

Sketching v versus t we find

$v' = 3t^2 - 7;\ v' = 0 \leftrightarrow t = \pm\sqrt{\frac{7}{3}} \approx \pm 1.5$

$v'' = 6t$



$$\Delta x = \int_0^2 (t^3 - 7t + 6)dt = \frac{1}{4}t^4 - \frac{7t^2}{2} + 6t \Big|_0^2$$

$$= 4 - 14 + 12 = 2\text{(ft)} \quad \text{displacement}$$

$$\text{Distance travelled} = \int_0^1 (t^3 - 7t + 6)dt$$

$$+ \int_1^2 (-t^3 + 7t - 6)dt$$

[4.4.4 cont'd]

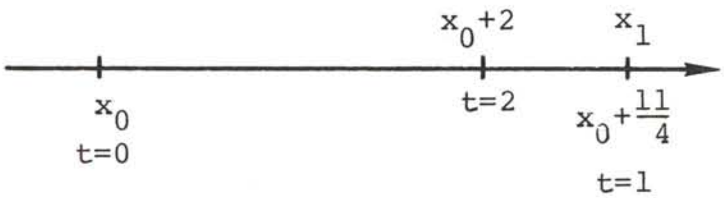
$$= \frac{1}{4}t^4 - \frac{7}{2}t^2 + 6t \Big|_0^1 + \left[-\frac{1}{4}t^4 + \frac{7}{2}t^2 - 6t \Big|_1^2\right.$$

$$= \left[\frac{1}{4} - \frac{7}{2} + 6\right] + \left[(-2) - (-\frac{1}{4} + \frac{7}{2} - 6)\right]$$

$$= \frac{11}{4} + \left[-2 + \frac{11}{4}\right]$$

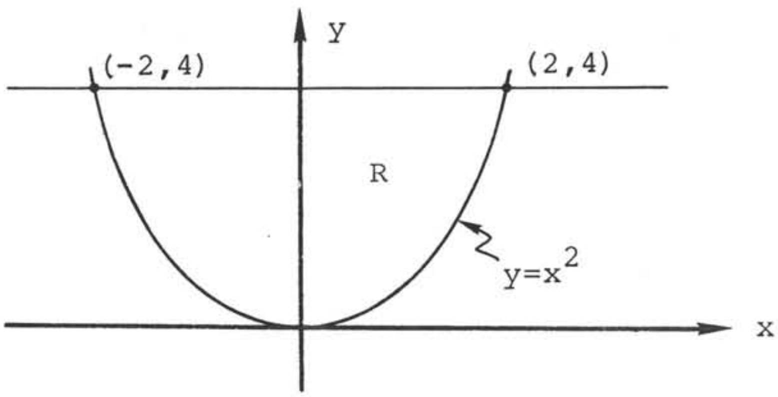
$$= \frac{11}{4} + \frac{3}{4} = \frac{14}{4} = 3\frac{1}{2} \text{ ft.}$$

Pictorially,



Travels $2\frac{3}{4}$ ft during 1st second and returns $\frac{3}{4}$ ft during next second.

4.4.5

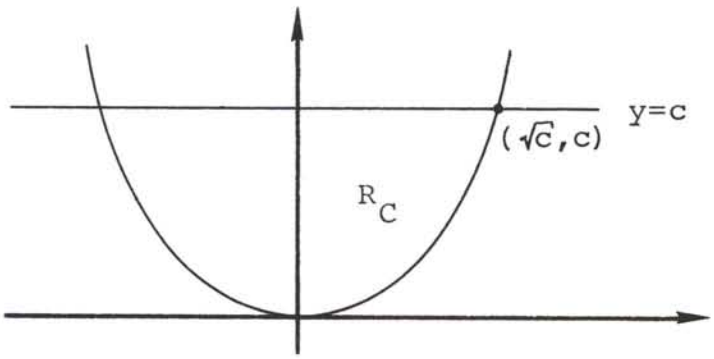


[4.4.5 cont'd]

$$A_R = \int_{-2}^{2} (4 - x^2)\ dx = 4x - \frac{1}{3}x^3 \Big|_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$$

Now



$$A_{R_c} = \int_{-\sqrt{c}}^{\sqrt{c}} (c - x^2)\ dx = 2\int_{0}^{\sqrt{c}} (c - x^2)\ dx$$

(since $c - x^2$ is an even function)

$$= 2\left[cx - \frac{1}{3}x^3 \Big|_{0}^{\sqrt{c}}\right]$$

$$= 2\left[c^{\frac{3}{2}} - \frac{1}{3}c^{\frac{3}{2}}\right]$$

$$= \frac{4}{3}c^{\frac{3}{2}}$$

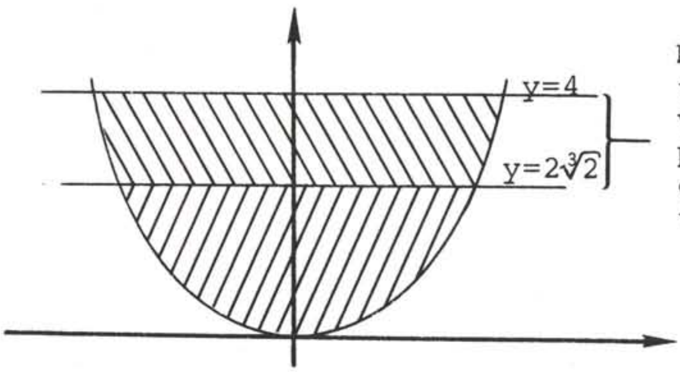
[4.4.5 cont'd]

Now, by hypothesis,

$$A_{R_c} = \frac{1}{2} A_R$$

$$\therefore \frac{4}{3} c^{\frac{3}{2}} = \frac{16}{3} \qquad \therefore c^{\frac{3}{2}} = 4$$

$$c = 4^{\frac{2}{3}} = \sqrt[3]{16} = 2\sqrt[3]{2}$$



Notice that $y=2\sqrt[3]{2}$ is above the midpoint of the interval [0,4]. This seems plausible since the area of R seems to be concentrated near the "top".

4.4.6

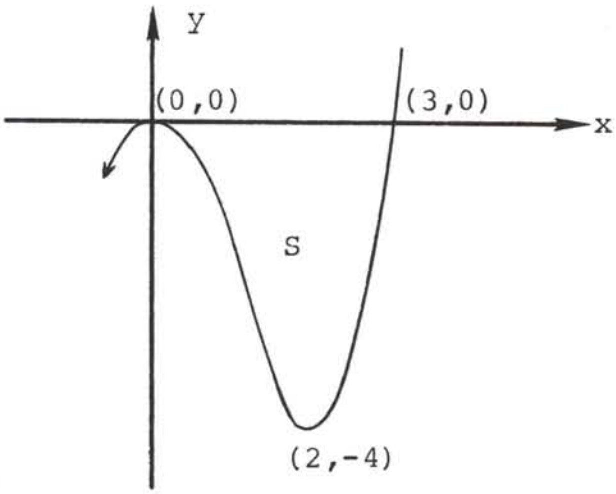
a. We can graph $y = x^2(x - 3)$.
Namely

$$y = x^2(x - 3) = x^3 - 3x^2$$

$$y' = 3x^2 - 6x$$

$$y'' = 6x - 6$$

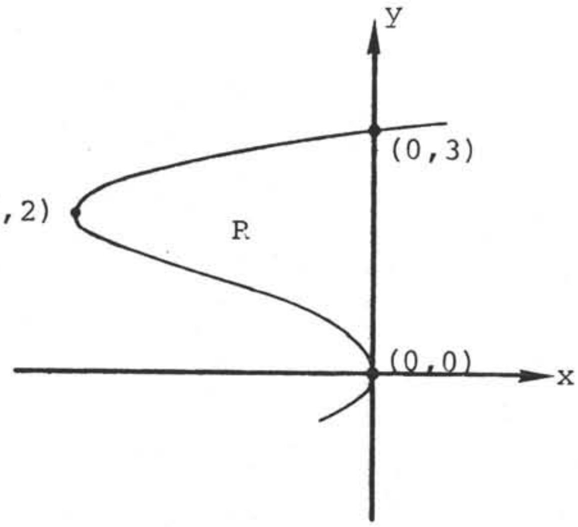
Putting these facts together yields:



(Figure 1)

Hence the curve $x = y^2(y - 3)$ is given by

[4.4.6 cont'd]



(Figure 2)

$$A_R = \int_0^3 (-x)\ dy \qquad \text{(since } x < 0 \text{ for } 0 < y < 3\text{)}$$

$$= \int_0^3 (-y^3 + 3y^2)\ dy$$

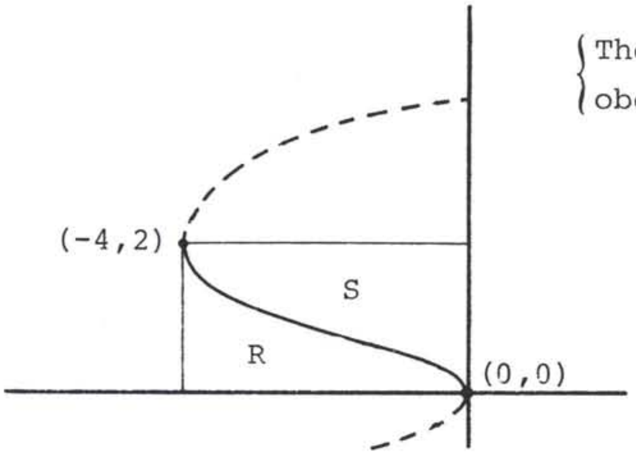
$$A_R = -\frac{1}{4} y^4 + y^3 \Big|_0^3$$

$$= -\frac{81}{4} + 27$$

$$= \frac{27}{4}$$

(As a check, this is the same area as that of region S in Figure 1.)

4.4.7 From the previous exercise, we have



{The dotted portion fails to obey $0 \leqslant y \leqslant 2$.

$R \cup S$ is a 4 by 2 rectangle, hence $A_{R \cup S} = 8$.

Now $A_R = A_{R \cup S} - A_S$

$$= 8 - \int_0^2 -x\, dy$$

$$= 8 - \int_0^2 (-y^3 + 3y^2)\, dy$$

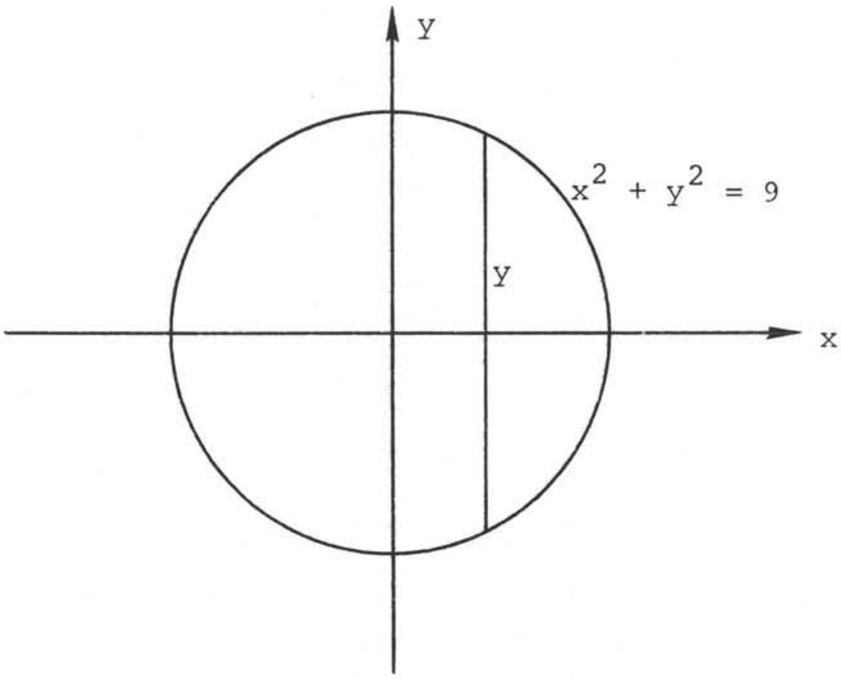
$$= 8 - \left[-\frac{1}{4} y^4 + y^3 \right|_0^2$$

$$= 8 - [-4 + 8]$$

$$= 4$$

UNIT 5: Volume

4.5.1



$$x^2 + y^2 = 9$$
$$y = \sqrt{9 - x^2}$$

$$\therefore \text{ side of square} = 2y = 2\sqrt{9 - x^2}$$
$$\therefore A(x) = \left(2\sqrt{9 - x^2}\right)^2 = 4\left(9 - x^2\right) = 36 - 4x^2$$

$$\text{In general } V = \int_a^b A(x)\,dx$$

[4.5.1 cont'd]

Thus, in our case,

$$V = \int_{-3}^{3} (36 - 4x^2)\,dx$$

$$= 2\int_{0}^{3} (36 - 4x^2)\,dx \quad \text{(because } 36 - 4x^2 \text{ is an even function)}$$

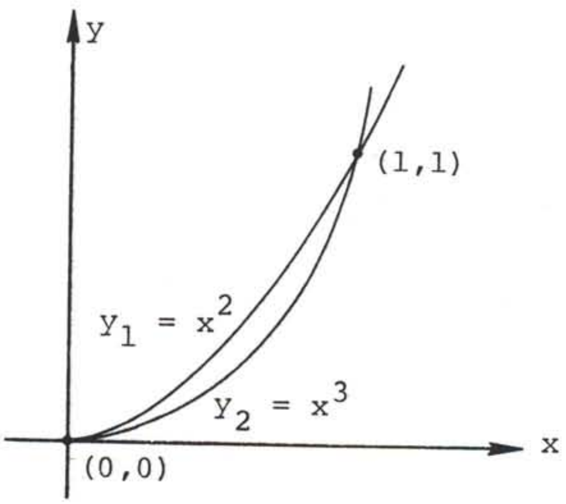
$$= 2\left[36x - \frac{4}{3}x^3 \,\right|_0^3$$

$$= 2\left[108 - \frac{4}{3}(27)\right]$$

$$= 2[108 - 36]$$

$$= 144$$

4.5.2



[4.5.2 cont'd]

a. $$V_x = \pi \int_0^1 [(x^2)^2 - (x^3)^2]\, dx$$

$$= \pi \int_0^1 (x^4 - x^6)\, dx$$

$$= \pi \left[\frac{1}{5} x^5 - \frac{1}{7} x^7 \right|_0^1$$

$$= \pi \left(\frac{1}{5} - \frac{1}{7}\right)$$

$$= \frac{2\pi}{35}$$

b. $$V_y = 2\pi \int_0^1 x(y_1 - y_2)\, dx$$

$$= 2\pi \int_0^1 (x^3 - x^4)\, dx$$

$$= 2\pi \left[\frac{1}{4} x^4 - \frac{1}{5} x^5 \right|_0^1$$

$$= 2\pi \left(\frac{1}{4} - \frac{1}{5}\right) = 2\pi \left(\frac{1}{20}\right)$$

$$= \frac{\pi}{10}$$

[4.5.2 cont'd]

Alternate way: $V_y = \pi \int_0^1 [(y^{\frac{1}{3}})^2 - (y^{\frac{1}{2}})^2]\, dy$

$$= \pi \int_0^1 (y^{\frac{2}{3}} - y)\, dy$$

$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}} - \frac{1}{2} y^2 \right|_0^1$$

$$= \pi\left(\frac{3}{5} - \frac{1}{2}\right)$$

$$= \frac{\pi}{10}$$

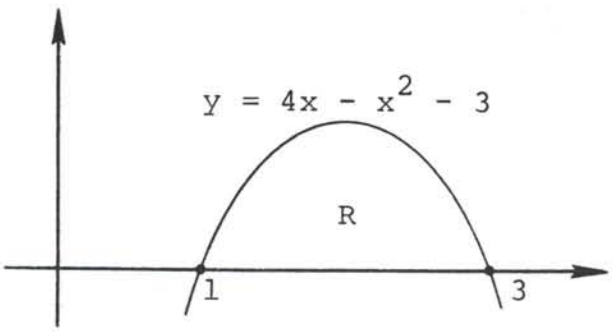
<u>4.5.3</u>

$$y = 4x - x^2 - 3$$

$$y' = 4 - 2x$$

$$y'' = -2$$

Hence R is given by:



[4.5.3 cont'd]

a. $V_x = \int_1^3 \pi y^2 dx = \int_1^3 \pi(4x - x^2 - 3)^2 dx$

$= \pi \int_1^3 (16x^2 + x^4 + 9 - 8x^3 - 24x + 6x^2)\ dx$

$= \pi \int_1^3 (x^4 - 8x^3 + 22x^2 - 24x + 9)\ dx$

$= \pi\left[\frac{1}{5} x^5 - 2x^4 + \frac{22}{3} x^3 - 12x^2 + 9x \right|_1^3$

$= \pi\left[(\frac{243}{5} - 162 + 198 - 108 + 27) - (\frac{1}{5} - 2 + \frac{22}{3} - 12 + 9)\right]$

$= \pi\left(\frac{242}{5} - \frac{22}{3} - 40\right) = \frac{\pi}{15}\left(726 - 110 - 600\right)$

$= \frac{16\pi}{15}$

b. $V_y = 2\pi \int_1^3 xydx = 2\pi \int_1^3 (4x^2 - x^3 - 3x)\ dx$

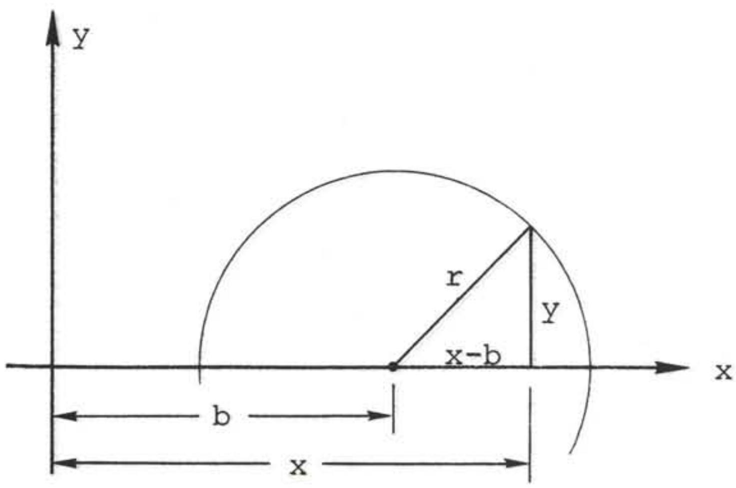
$= 2\pi\left[\frac{4}{3} x^3 - \frac{1}{4} x^4 - \frac{3}{2} x^2 \right|_1^3$

[4.5.3 cont'd]

$$= 2\pi\left[\left(36 - \frac{81}{4} - \frac{27}{2}\right) - \left(\frac{4}{3} - \frac{1}{4} - \frac{3}{2}\right)\right]$$

$$= 2\pi\left[\frac{9}{4} - \left(-\frac{5}{12}\right)\right] = 2\pi\left[\frac{9}{4} + \frac{5}{12}\right] = \frac{16\pi}{3}$$

4.5.4(L) First of all the equation of the circle is $(x - b)^2 + y^2 = r^2$ since

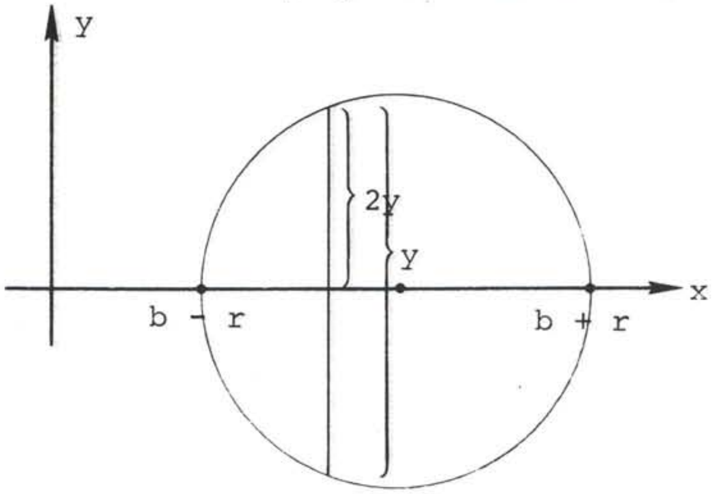


By Pythagorean Theorem,

$$(x - b)^2 + y^2 = r^2$$

$$y = \pm\sqrt{r^2 - (x - b)^2}$$

Now, by the shell-method we have:



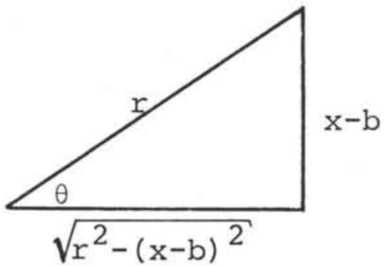
[4.5.4(L) cont'd]

$$V_y = 2\pi \int_{b-r}^{b+r} x(2y)\,dx = 4\pi \int_{b-r}^{b+r} x\sqrt{r^2 - (x-b)^2}\,dx \qquad (1)$$

Notice that obtaining (1) was essentially no more difficult than any of our exercises to date. The work comes in when we try to evaluate

$$\int x\sqrt{r^2 - (x-b)^2}\,dx$$

To this end, we may invoke the reference triangle



$$\sin\theta = \frac{x-b}{r}$$

$$r\sin\theta = x-b;\ x = r\sin\theta + b$$

$$r\cos\theta\,d\theta = dx$$

$$r\cos\theta = \sqrt{r^2 - (x-b)^2}$$

When $x = b + r$, $\sin\theta = \dfrac{(b+r) - b}{r} = 1 \quad \therefore \theta = \dfrac{\pi}{2}$

When $x = b - r$, $\sin\theta = \dfrac{(b-r) - b}{r} = -1 \quad \therefore \theta = -\dfrac{\pi}{2}$

$$\therefore 4\pi \int_{x=b-r}^{x=b+r} x\sqrt{r^2 - (x-b)^2}\,dx$$

$$= 4\pi \int_{=-\frac{\pi}{2}}^{\frac{\pi}{2}} (r\sin\theta + b)(r\cos\theta)(r\cos\theta)\,d\theta$$

[4.5.4(L) cont'd]

$$= 4\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (r \sin \theta + b) \cos^2 \theta \, d\theta$$

$$= 4\pi r^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \sin \theta \, d\theta + 4\pi r^2 b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta ,$$

and since $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$, we have

$$V_y = 4\pi r^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \sin \theta \, d\theta + 4\pi r^2 b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} \, d\theta$$

$$= 4\pi r^3 \left[-\frac{1}{3} \cos^3 \theta \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 4\pi r^2 b \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\pi r^3 [o - o] + 4\pi r^2 b \left\{ \left(\frac{\pi}{4} + \frac{1}{4} \sin \pi\right) - \left[-\frac{\pi}{4} + \frac{1}{4} \sin(-\pi)\right] \right\}$$

$$= 4\pi r^2 b \left[\frac{\pi}{2}\right]$$

$$= 2\pi^2 r^2 b$$

(Aside: The area of the circle is πr^2 and its center moves through a distance of $2\pi b$. Notice that the answer to this problem is the area of the circle multiplied by the distance through which its center travels)

4.5.5(L) Here we have a nice application of the second fundamental theorem to a volume of revolution. We have:

$$V_x = \pi \int_0^a f^2(x)\,dx = a^5 \quad (a > 0) \tag{1}$$

Now, while we may not be used to viewing a as a variable, the fact remains that (1) implies:

$$\frac{d}{da}\left[\pi \int_0^a f^2(x)\,dx\right] = \frac{d(a^5)}{da} \tag{2}$$

Now $\frac{d}{da}\int_0^a f^2(x)\,dx = f^2(a)$

Hence (2) becomes

$$\pi f^2(a) = 5a^4$$

$$\therefore f^2(a) = \frac{5a^4}{\pi} \qquad \text{or}$$

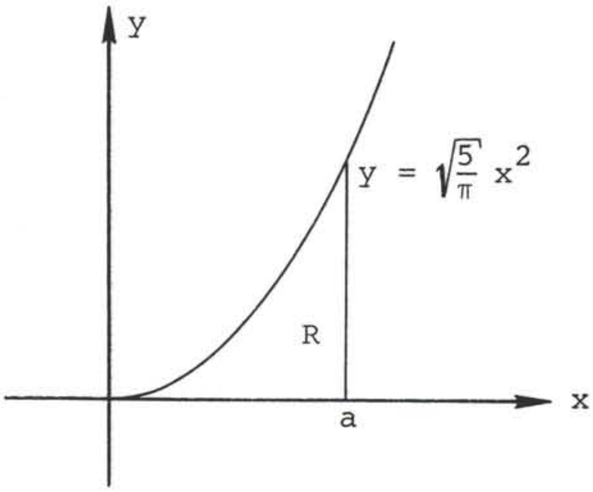
$$f(a) = \sqrt{\frac{5a^4}{\pi}} = a^2\sqrt{\frac{5}{\pi}} \tag{3}$$

Replacing a by x in (3), we obtain:

$$f(x) = \sqrt{\frac{5}{\pi}}\; x^2$$

[4.5.5(L) cont'd]

As a check:



$$A_R = \pi \int_0^a y^2 dx$$

$$= \pi \int_0^a \frac{5x^4}{\pi}\, dx$$

$$= \int_0^a 5x^4\, dx$$

$$= x^5 \Big|_0^a$$

$$= a^5$$

UNIT 6: Arcs and Approximations

4.6.1(L) The main aim of this exercise is to emphasize the fact that arc length is dependent only on the path and not on the equation which represents the path.

For example, when we developed the "recipe" that:

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (1)$$

we were assuming that our curve was given in the form $y = f(x)$, which made it convenient to compute $\frac{dy}{dx}$.

Had the equation been given in the form $x = g(y)$ it would probably have been more convenient to compute $\frac{dx}{dy}$. In this case, reasoning similar to that which led to (1), would yield:

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad (2)$$

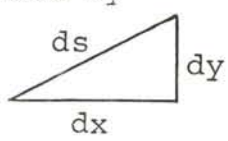
Other times, the equation of the curve is given parametrically in the form

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

in which case it is most convenient to compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. It can then be shown that

$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (3)$$

(while (1), (2), and (3) are all proven rigorously, assuming the proper degree of continuity and differentiability, once proven, they can be memorized by thinking of the "pseudo-reference triangle"



[4.6.1(L) cont'd]

wherein $ds^2 = dx^2 + dy^2$, from which we can find

$ds = \sqrt{1 + (\frac{dy}{dx})^2}\,dx$ or $ds = \sqrt{1 + (\frac{dx}{dy})^2}\,dy$ or

$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$ by dividing through by dx,

dy, or dt respectively. This, of course, is not a proof but rather an easy way of recalling the correct result once it has been proved.)

The key idea is that (1), (2), and (3) are equivalent and so, in a given problem, we invoke the one that is the most convenient for us to use.

In this exercise, the fact that $x = \frac{y^3}{3} + \frac{1}{4y}$ makes it most convenient (at least compared with finding $\frac{dy}{dx}$) to compute $\frac{dx}{dy}$. Thus,

$$x = \frac{y^3}{3} + \frac{1}{4y} = \frac{y^3}{3} + \frac{1}{4}y^{-1} \quad (1 \leqslant y \leqslant 3).$$ Therefore,

$$\frac{dx}{dy} = y^2 - \frac{1}{4}y^{-2} = y^2 - \frac{1}{4y^2}$$

We then invoke (2) to obtain:

$$s = \int_1^3 \sqrt{1 + (\frac{dx}{dy})^2}\,dy \qquad (4)$$

where $1 + (\frac{dx}{dy})^2 = 1 + (y^2 - \frac{1}{4y^2})^2$

$$= 1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}$$

[4.6.1(L) cont'd]

$$= y^4 + \frac{1}{2} + \frac{1}{16y^4}$$

$$= (y^2 + \frac{1}{4y^2})^2 \quad *$$

Thus (4) becomes

$$s = \int_1^3 (y^2 + \frac{1}{4y^2})\,dy = \int_1^3 (y^2 + \tfrac{1}{4}y^{-2})\,dy$$

$$\text{or} \quad s = \frac{1}{3}y^3 - \frac{1}{4}y^{-1}\Big|_1^3$$

$$= \frac{1}{3}y^3 - \frac{1}{4y}\Big|_1^3$$

$$= (9 - \frac{1}{12}) - (\frac{1}{3} - \frac{1}{4}) = 9 - \frac{1}{6}$$

$$= \frac{53}{6}$$

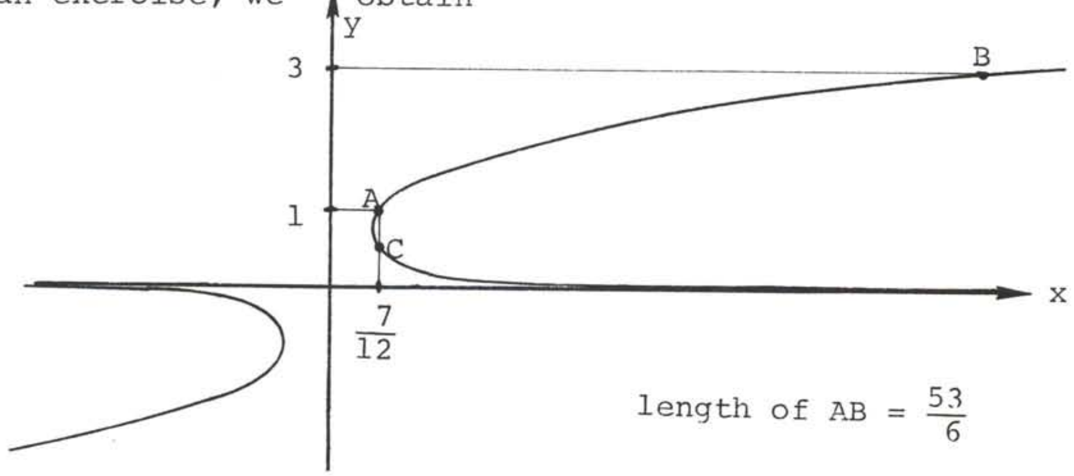
It might be helpful to look at the graph of

---

*Notice how contrived our equations have to be if we are to be able to handle ds. In most real life situations $\sqrt{1 + (\frac{dx}{dy})^2}$ would be extremely difficult to integrate and it is often likely that we will have to resort to some sort of numerical approximation.

[4.6.1(L) cont'd]

$x = \frac{y^3}{3} + \frac{1}{4y}$ so that we can get a better picture of the problem we've solved. To this end, we reflect the graph of $y = \frac{x^3}{3} + \frac{1}{4x}$ with respect to the line $y = x$, and, leaving the details as an exercise, we obtain



(Figure 1)

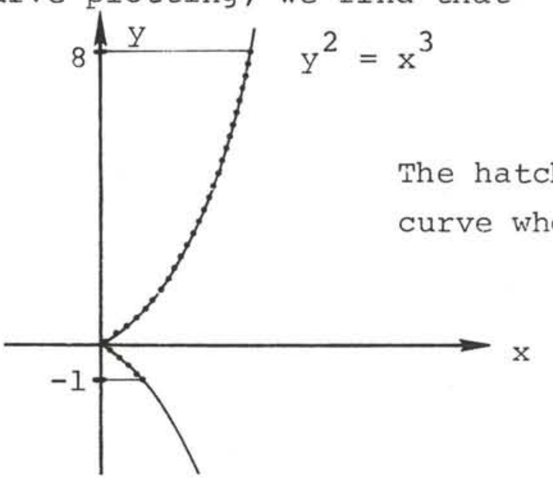
Notice that our curve is single-valued as a function of y but double-valued as a function of x. For example, when $y = 1$, $x = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ while when $y = 3$, $x = 9 + \frac{1}{12}$. Had we been asked, however, to find s if $\frac{7}{12} \leq x \leq 9\frac{1}{12}$ we would have had "trouble" since when $x = \frac{7}{12}$ y need not equal 1. That is, the line $x = \frac{7}{12}$ cuts our curve in Figure 1 at A and C. In essence, had our limits been given in terms of x, we would have had to break the curve into single-valued branches; algebraically, this is not always easy. (For example, given that $x = \frac{7}{12}$, we have $\frac{7}{12} = \frac{y^3}{3} + \frac{1}{4y}$ or $7y = 4y^4 + 3$, from which it is

[4.6.1(L) cont'd]

easy to check that $y = 1$ is a root, but what is the other real root, which we know exists from Figure 1?)

Finally let us observe that we would also have had big trouble if the range of y included $y = 0$ since our curve is not continuous (it isn't even defined) when $y = 0$.

4.6.2(L) Given that $y^2 = x^3$ we have a choice of expressing y as a function of x or x as a function of y. Before exercising our option, let us graph $y^2 = x^3$ so that we may anticipate any trouble spots. Since $y^2 \geqslant 0$, we have that x^3, hence x, cannot be negative. Making use of our usual techniques of curve plotting, we find that



The hatched portion represents the curve whose length we are seeking.

(Figure 1)

The graph tells us that we have a problem regardless of the choice we make! Namely, viewed as a function of x the curve is double-valued and, hence, we would have to treat it as two separarate branches (which in

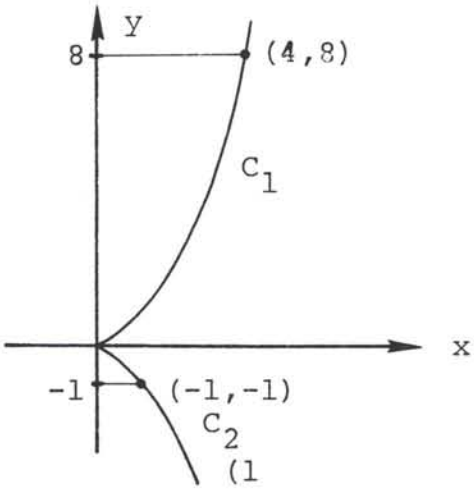
[4.6.2 (L) cont'd]

this case, at least, is not very complicated); while viewed as a function of y the curve is single-valued but is not differentiable* at $y = 0$, which is a point in the given range of y. Thus, in this case, we would have to view the curve in two segments $-1 \leqslant y \leqslant 0$ and $0 \leqslant y \leqslant 8$.

For the sake of experience, we will tackle the problem from both points of view.

Suppose first that we elect to express y as a function of x. We have, since $y^2 = x^3$, that $y = \pm x^{\frac{3}{2}}$ (Notice again that $y^2 = x$ and $y = \sqrt{x}$ are not the same. $y^2 = x$ means $y = \pm\sqrt{x}$).

We let C_1 denote $y = x^{\frac{3}{2}}$ and C_2 denote $y = -x^{\frac{3}{2}}$



(Figure 2)

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*Recall that the formula $s = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2}\, dx$ required that $\frac{dx}{dy}$ exist at all points in the interval so that we might invoke the mean value theorem, etc.

[4.6.2 (L) cont'd]

We then observe that for C_1 y varies from 0 to 8, hence x varies from 0 to 4; while for C_2 y varies from -1 to 0; hence, x varies from 1 to 0. In other words,

$$\left.\begin{aligned} \text{length of } C_1 &= \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ \text{length of } C_2 &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}\right\} \qquad (1)$$

Now, $y = \pm x^{\frac{3}{2}}$ implies $\frac{dy}{dx} = \pm\frac{3}{2}\, x^{\frac{1}{2}}$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{9}{4}\, x \therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{9}{4}\, x}$$

$$= \frac{1}{2}\sqrt{4 + 9x}$$

Putting this result into (1), we find

$$\left.\begin{aligned} \text{length of } C_1 &= \frac{1}{2} \int_0^4 \sqrt{4 + 9x}\, dx \\ \text{length of } C_2 &= \frac{1}{2} \int_0^1 \sqrt{4 + 9x}\, dx \end{aligned}\right\} \qquad (2)$$

Letting $u = 4 + 9x$, we see that $du = 9dx$ or $dx = \frac{du}{9}$.

Therefore:

$$\int \sqrt{4 + 9x}\, dx = \int u^{\frac{1}{2}}\, \frac{du}{9} = \frac{2}{3}\, \frac{u^{\frac{3}{2}}}{9} + C$$

[4.6.2 (L) cont'd]

$$= \frac{2}{27} u^{\frac{3}{2}}$$

$$= \frac{2}{27} (4 + 9x)^{\frac{3}{2}} + C$$

$$\frac{1}{2} \int \sqrt{4 + 9x}\, dx = \frac{1}{27} (4 + 9x)^{\frac{3}{2}} + C_1 \qquad (3)$$

Putting the result of (3) into (2) we obtain:

$$\text{length of } C_1 = \frac{1}{27} (4 + 9x)^{\frac{3}{2}} \Big|_0^4$$

$$= \frac{1}{27} \left[40^{\frac{3}{2}} - 4^{\frac{3}{2}} \right] \qquad (4)$$

$$\text{length of } C_2 = \frac{1}{27} (4 + 9x)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{1}{27} \left[13^{\frac{3}{2}} - 4^{\frac{3}{2}} \right] \qquad (5)$$

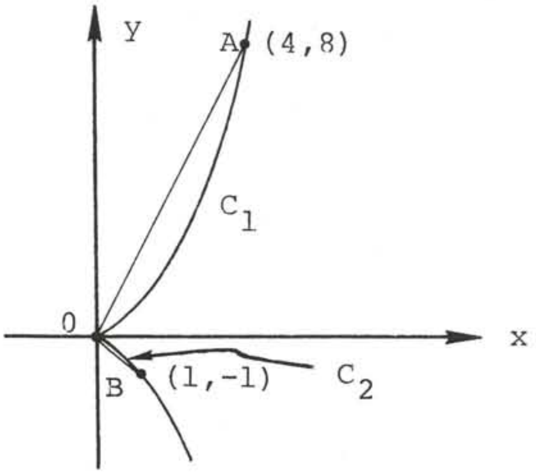
Hence, the desired arclength is the sum of (4) and (5); or

$$\frac{1}{27} \left[40^{\frac{3}{2}} + 13^{\frac{3}{2}} - 2(4)^{\frac{3}{2}} \right] =$$

$$\frac{1}{27} \left[40\sqrt{40} + 13\sqrt{13} - 16 \right] \approx 10.5$$

[4.6.2 (L) cont'd]

Note: While the exact answer may be difficult to obtain, we still have a relatively simple way of obtaining a rough check. Namely, we may think of the length as a sum of lengths of straight line segments, which , of course, is in accord with the definition of arc length. For example, as a very quick check we have



$$\overline{OA} = \sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$$

$$\overline{OB} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\overline{OA} + \overline{OB} \approx 8.9 + 1.4 \approx 10.3$$

which shows that our answer is at least in the "right ball park". By the way, also notice that our approximation should be less than the correct answer since we have replaced the curves OA and OB by the lines OA and OB.

As for the second method, we would write $x = y^{\frac{2}{3}}$, whereupon $\frac{dx}{dy} = \frac{2}{3} y^{-\frac{1}{3}}$ (and this is infinite at $y = 0$, thus giving us an additional clue to be wary of $y = 0$).

Then $(\frac{dx}{dy})^2 = \frac{4}{9} y^{-\frac{2}{3}}$ and $\sqrt{1 + (\frac{dx}{dy})^2} = \sqrt{1 + \frac{4}{9} y^{-\frac{2}{3}}}$

[4.6.2 (L) cont'd]

$$\left.\begin{aligned} \text{length of } C_1 &= \int_0^8 \sqrt{1 + \tfrac{4}{9} y^{-\frac{2}{3}}}\, dy \\ \text{length of } C_2 &= \int_{-1}^0 \sqrt{1 + \tfrac{4}{9} y^{-\frac{2}{3}}}\, dy \end{aligned}\right\} \qquad (6)$$

(Notice that while (1) and (6) are equivalent, the resulting integrals are not of equal difficulty. In other words, the choice of whether we write $y = f(x)$ or $x = g(y)$ might depend on the difficulty in evaluating the resulting integral.)

To evaluate

$$\int \sqrt{1 + \tfrac{4}{9} y^{-\frac{2}{3}}}\, dy$$

we might proceed by writing $\int \sqrt{1 + \frac{4}{9} y^{-\frac{2}{3}}}\, dy$

$$= \int \sqrt{1 + \frac{4}{9y^{\frac{2}{3}}}}\, dy$$

$$= \int \frac{1}{3y^{\frac{1}{3}}} \sqrt{9y^{\frac{2}{3}} + 4}\, dy$$

$$= \frac{1}{3} \int y^{-\frac{1}{3}} \sqrt{9y^{\frac{2}{3}} + 4}^{\,*}\, dy$$

*See footnote next page.

[4.6.2 (L) cont'd]

We could then let $u = 9y^{\frac{2}{3}} + 4$, whence $du = 6y^{-\frac{1}{3}}dy$ or $y^{-\frac{1}{3}}dy = \frac{du}{6}$. Hence,

$$\frac{1}{3}\int y^{-\frac{1}{3}}\sqrt{9y^{\frac{2}{3}} + 4}\ dy = \frac{1}{18}\int u^{\frac{1}{2}}\ du = \frac{1}{18}[\tfrac{2}{3}\ u^{\frac{3}{2}}] + C$$

$$= \frac{1}{27}(9y^{\frac{2}{3}} + 4)^{\frac{3}{2}} + C \qquad (7)$$

Putting (7) into (6), we find:

$$\text{length of } C_1 = \frac{1}{27}(9y^{\frac{2}{3}} + 4)^{\frac{3}{2}}\Bigg|_0^8$$

$$= \frac{1}{27}[40^{\frac{3}{2}} - 4^{\frac{3}{2}}] \quad \text{(which checks with (4))}$$

$$\text{length of } C_2 = \frac{1}{27}(9y^{\frac{3}{2}} + 4)^{\frac{3}{2}}\Bigg|_{-1}^0$$

$$= \frac{1}{27}[4^{\frac{3}{2}} - 13^{\frac{3}{2}}]^{*} \quad \text{(which except for sign checks with (5))}$$

---

*We must be careful with algebraic signs. For example, $y^{-\frac{1}{3}}\sqrt{9y^{\frac{2}{3}} + 4}$ is negative when y is negative. Yet we think of length as being positive. If we want our convention of length being positive to be obeyed we should write:

$$\int_{-1}^{0}\sqrt{1 + \frac{4}{9}\ y^{-\frac{2}{3}}}\ dy = -\frac{1}{3}\int y^{-\frac{1}{3}}\sqrt{9y^{\frac{2}{3}} + 4}\ dy$$

[4.6.2 (L) cont'd]

As a final note, observe what would have happened had we failed to observe that y = 0 was a trouble spot. Namely, we would have blithely computed

$$\int_{-1}^{8}\sqrt{1+\frac{4}{9}y^{-\frac{2}{3}}}\,dy = \frac{1}{27}(9y^{\frac{2}{3}}+4)^{\frac{3}{2}}\Bigg|_{-1}^{8}$$

$$= \frac{1}{27}[40^{\frac{3}{2}} - 13^{\frac{3}{2}}] \text{ etc}$$

which would not have been the correct answer.

4.6.3 (L)

a. Things look fine for a start. Afterall, $\frac{dy}{dx} = \cos x$. Hence $1 + (\frac{dy}{dx})^2 = 1 + \cos^2 x$. Hence the answer to this exercise is:

$$L = \int_0^{\frac{\pi}{2}}\sqrt{1+\cos^2 x}\,dx \qquad (1)$$

b. Technically speaking, (1) is the solution to the exercise, but we might desire a more specific numerical form for the answer.

To be sure, (1) can be re-written as:

$$\left.\begin{array}{l} G(\frac{\pi}{2}) - G(0) \\ \text{where } G'(x) = \sqrt{1+\cos^2 x} \end{array}\right\} \qquad (2)$$

The trouble with (2) is that there is no elementary (familiar) function G(x) such that $G'(x) = \sqrt{1+\cos^2 x}$.

However, (1) does name the exact area of a particular

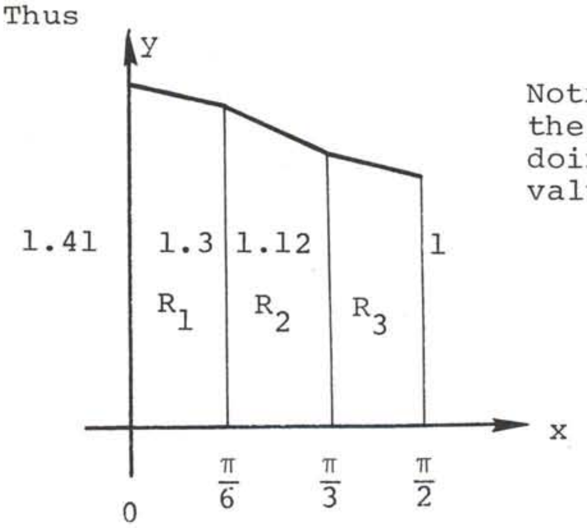
[4.6.3 (L) cont'd]

region; namely the region bounded above by $y = \sqrt{1 + \cos^2 x}$, below by the x-axis, on the left by the y-axis, and on the right by the line $x = \frac{\pi}{2}$. (Notice that to work <u>analytically</u> with this region it is <u>not</u> important to have an accurate plot of $y = \sqrt{1 + \cos^2 x}$. What is important is that we know $y = \sqrt{1 + \cos^2 x}$ is continuous.)

Now, to approximate (1) by the use of trapezoids with n = 3 we have:

x	$\cos x$	$\cos^2 x$	$\sqrt{1 + \cos^2 x}$
0	1	1	$\sqrt{2} \approx 1.41$
$\frac{\pi}{6}$	$\frac{1}{2}\sqrt{3}$	$\frac{3}{4}$	$\sqrt{\frac{7}{4}} \approx 1.30$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\sqrt{\frac{5}{4}} \approx 1.12$
$\frac{\pi}{2}$	0	0	$1 = 1.00$

Thus



Notice why we don't need the actual curve.All we are doing is using the tabulated value to form trapezoids.

(Figure 1)

[4.6.3 (L) cont'd]

$$A_{R_1} = \left[\frac{1.41 + 1.3}{2}\right]\frac{\pi}{6} = \frac{\pi}{12}\ (2.71)$$

$$A_{R_2} = \left[\frac{1.3 + 1.12}{2}\right]\frac{\pi}{6} = \frac{\pi}{12}\ (2.42)$$

$$A_{R_3} = \left[\frac{1.12 + 1}{2}\right]\frac{\pi}{6} = \frac{\pi}{12}\ (2.12)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x}\ dx \approx A_{R_1} + A_{R_2} + A_{R_3} = \frac{\pi}{12}\ (7.25)$$

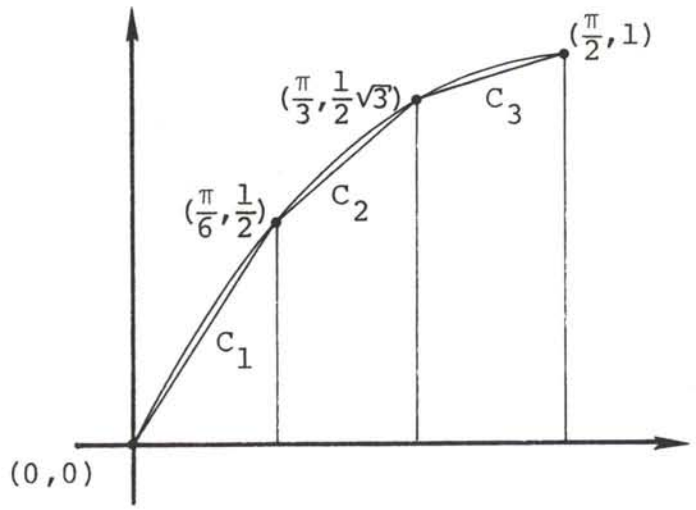
$$\approx .604\pi \approx 1.96$$

(If we let T = 1.96 be the trapezoidal approximation, we can use the result

$$\int_a^b f(x)dx = T - \frac{b - a}{12} f''(e)\overline{\Delta x}^2 \qquad a \leqslant e \leqslant b$$

to estimate our error. We will not take the time to do this here.)

c.



(Figure 2)

[4.6.3 (L) cont'd]

$$\text{length of } C_1 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\pi}{6}\right)^2} = \sqrt{\frac{\pi^2}{36} + \frac{1}{4}} \approx \sqrt{.28 + .25}$$

$$\approx 0.73$$

$$\text{length of } C_2 = \sqrt{\left(\frac{\pi}{3} - \frac{\pi}{6}\right)^2 + \left(\frac{1}{2}\sqrt{3} - \frac{1}{2}\right)^2} = \sqrt{\frac{\pi^2}{36} + \left(1 - \frac{1}{2}\sqrt{3}\right)}$$

$$= \sqrt{.28 + .14} \approx 0.65$$

$$\text{length of } C_3 = \sqrt{\left(\frac{\pi}{2} - \frac{\pi}{3}\right)^2 + \left(1 - \frac{1}{2}\sqrt{3}\right)^2} = \sqrt{\frac{\pi^2}{36} + \frac{7}{4} - \sqrt{3}}$$

$$= \sqrt{.28 + .02} \approx 0.55$$

$0.73 + 0.65 + 0.55 = 1.93$

Thus, 1.93 is an underestimate of the arc length, and a glance at Figure 2 seems to indicate that the error in our approximation is not too great.

Coupling this result with (b) we have:

$$1.93 < \int_0^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x}\, dx \approx 1.96$$

4.6.4 (L) The aim of this exercise is to emphasize our remarks concerning infinitesimals and "squeezing out" errors when we take limits.

We have that $\frac{\Delta y}{\Delta x} = m$ and $\overline{\Delta s}^2 = \overline{\Delta x}^2 + \overline{\Delta y}^2$

$$\therefore\ \overline{\Delta s}^2 = \overline{\Delta x}^2 + m^2\overline{\Delta x}^2 = (1 + m^2)\overline{\Delta x}^2$$

$$\therefore\ \Delta s = \sqrt{1 + m^2}$$

[4.6.4 (L) cont'd]

$$= \Delta x - \Delta x + \sqrt{1 + m} \;\Delta x$$

$$= \Delta x + (\sqrt{1 + m^2} - 1)\Delta x$$

$$\therefore \alpha = \sqrt{1 + m^2} - 1 \qquad (1)$$

Since m is a constant, so is α. Now the only way for a constant to be an infinitesimal is if the constant is zero. From (1) this means that $\sqrt{1 + m^2} - 1 = 0$, whence $\underline{m = 0}$.

Thus unless m = 0, we do not "squeeze out" the error in our approximation.

On the other hand, when m = 0 we have a line parallel to the x-axis, and in this case Δs and Δx are identical.

4.6.5

$$\left.\begin{aligned} y &= \tfrac{1}{3}(2t + 1)^{\frac{3}{2}} \\ x &= \tfrac{1}{2}t^2 \end{aligned}\right\}$$

$$\therefore \frac{dy}{dt} = \frac{1}{3}\left[\frac{3}{2}(2t + 1)^{\frac{1}{2}}2\right] = \sqrt{2t + 1} \qquad \therefore \left(\frac{dy}{dt}\right)^2 = 2t + 1$$

$$\frac{dx}{dt} = t \qquad \therefore \left(\frac{dx}{dt}\right)^2 = t^2$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2 + 2t + 1 = (t + 1)^2$$

$$\therefore \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t + 1$$

[4.6.5 cont'd]

$$s = \int_2^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_2^6 (t + 1)\, dt$$

$$= \frac{1}{2} t^2 + t \Big|_2^6$$

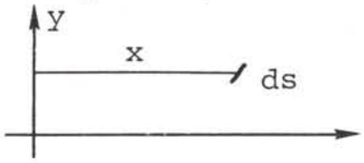
$$= 24 - 4$$

$$= 20$$

4.6.6 (L) Quite mechanically, we can apply our recipe to obtain

$$S = \int_{s_1}^{s_2} 2\pi x\, dx \qquad (1)$$

Notice that we write $2\pi x$ when the rotation is with respect to the y-axis, since



Since x and y are both functions of t, it is most convenient to express ds in the form

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad .$$

[4.6.6 cont'd]

We have $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = t + 1$. Equation (1) then becomes

$$S_y = \int_0^4 2\pi(t + 1)\sqrt{1 + (t + 1)^2}\,dt \qquad (2)$$

If in (2) we let u = t + 1, we obtain

$$S = \int_1^5 2\pi u\sqrt{1 + u^2}\,du^{*}$$

$$= \frac{2\pi}{3}(1 + u^2)^{\frac{3}{2}}\Big|_1^5$$

$$= \frac{2\pi}{3}[26^{\frac{3}{2}} - 2^{\frac{3}{2}}]$$

$$= \frac{2\pi}{3}[26\sqrt{26} - 2\sqrt{2}]$$

Now, in many cases involving parametric equations, it is either impossible or else very difficult to eliminate the parameter so the method described above is important.

---

*Notice that $\int_0^4$ was expressed in terms of t. Recall that when we change variables in a definite integral, we must also make the appropriate changes in the limits of integration.

[4.6.6 cont'd]

However, the parameter is easily eliminated in this exercise. Specifically:

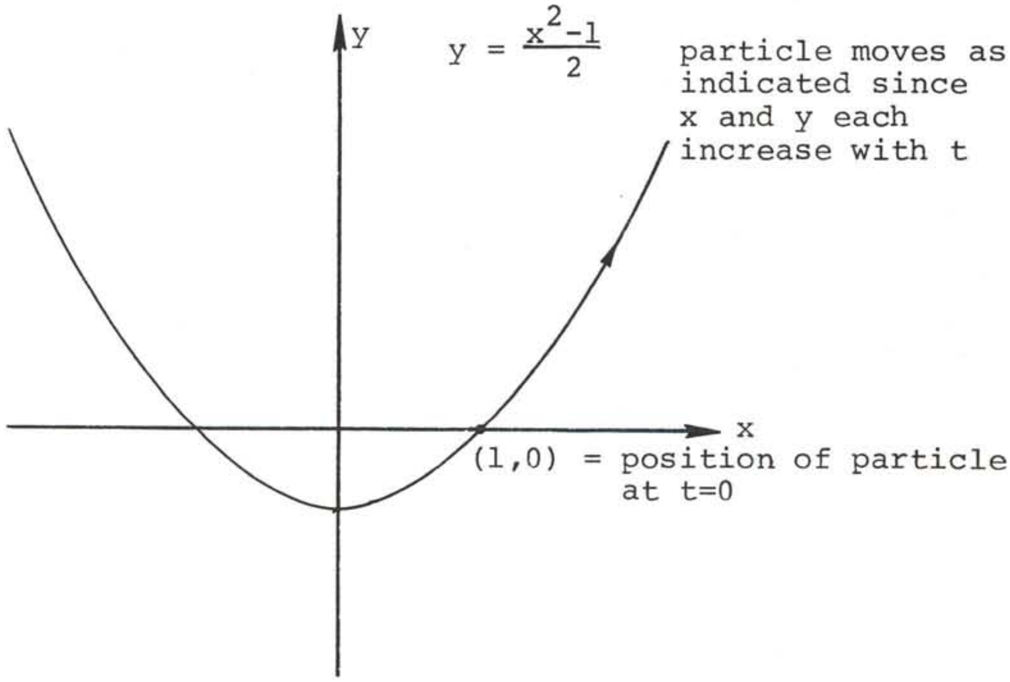
$$x = t + 1 \rightarrow t = x - 1 \rightarrow$$

$$y = \frac{(x-1)^2}{2} + (x-1) = \frac{(x-1)^2 + 2(x-1)}{2}$$

$$= \frac{(x-1)(x-1+2)}{2} = \frac{(x-1)(x+1)}{2}$$

$\therefore$ the path followed by the particle is the parabola

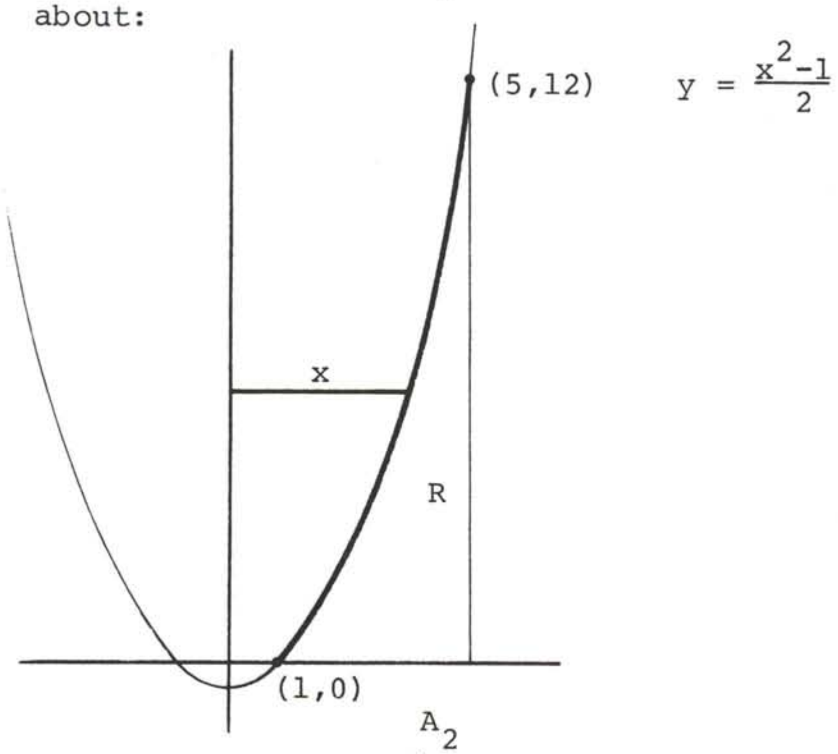
$y = \frac{x^2 - 1}{2}$. That is



Notice that the interval $0 \leqslant t \leqslant 4$ determines the section of the curve between (1,0) and (5,12)

[4.6.6 cont'd]

since $x = t + 1$, $y = \frac{1}{2}t^2 + t$. Thus we are talking about:



$$\text{Again } S = \int_{A_1}^{A_2} 2\pi x \, ds$$

Now, however, it is more convenient to express ds as

$\sqrt{1 + (\frac{dy}{dx})^2}\, dx$; $y = \frac{x^2 - 1}{2} \rightarrow \frac{dy}{dx} = x \rightarrow 1 + (\frac{dy}{dx})^2$

$= 1 + x^2$

$$\therefore S_y = \int_1^5 2\pi x \sqrt{1 + x^2}\, dx$$

$$= \frac{2\pi}{3}(1 + x^2)^{\frac{3}{2}} \Bigg|_1^5$$

[4.6.6 cont'd]

$$= \frac{2\pi}{3}\,(26\sqrt{26} - 2\sqrt{2})$$

which agrees with our previous result.

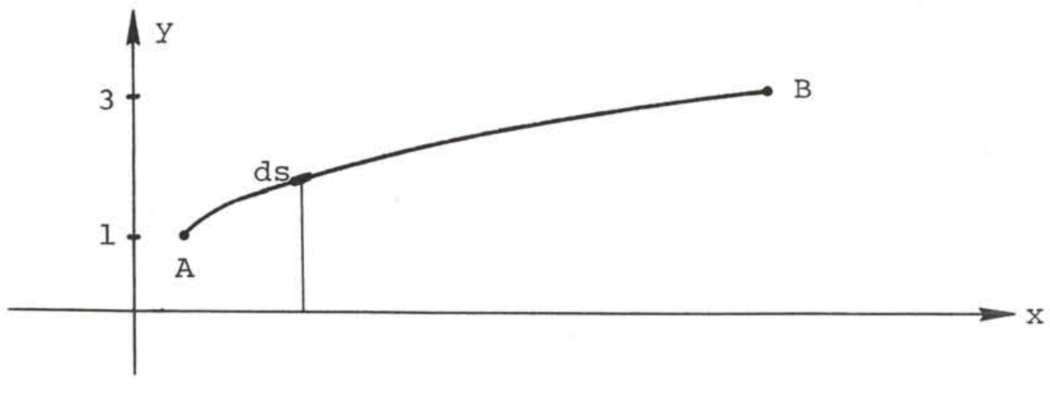
(Notice that the integral for surface area when R is revolved about the x-axis is a bit more difficult to evaluate; even though the theory is the same. Thus

$$S_x = 2\pi \int_1^5 y\sqrt{1 + x^2}\,dx$$

$$= \pi \int_1^5 (x^2 - 1)\sqrt{1 + x^2}\,dx$$

This is one reason why additional techniques for integration are so useful.)

<u>4.6.7</u>



[4.6.7 cont'd]

$$S_y = 2\pi \int x\, ds \;;\; ds = \left(y^2 + \frac{1}{4}y^{-2}\right)dy$$

$$= 2\pi \int_1^3 x\left(y^2 + \frac{1}{4}y^{-2}\right)dy\,,\quad x = \frac{y^3}{3} + \frac{1}{4y}$$

$$= 2\pi \int_1^3 \left(\frac{y^3}{3} + \frac{1}{4y}\right)\left(y^2 + \frac{1}{4y^2}\right)dy$$

$$= 2\pi \int_1^3 \left(\frac{1}{3}y^5 + \frac{1}{3}y + \frac{1}{16y^3}\right)dy$$

$$= 2\pi \int_1^3 \left(\frac{1}{3}y^5 + \frac{1}{3}y + \frac{1}{16}y^{-3}\right)dy$$

$$= 2\pi \left[\frac{1}{18}y^6 + \frac{1}{6}y^2 - \frac{1}{32}y^{-2}\right]_1^3$$

$$= 2\pi \left[\left(\frac{3^6}{18} + \frac{3}{2} - \frac{1}{32(9)}\right) - \left(\frac{1}{18} + \frac{1}{6} - \frac{1}{32}\right)\right]$$

$$= 2\pi \left[\frac{81}{2} + \frac{3}{2} - \frac{1}{288} - \left(\frac{16 + 48 - 9}{288}\right)\right]$$

$$= 2\pi \left[\frac{81}{2} + \frac{3}{2} - \frac{56}{288}\right] = 2\pi \left[\frac{81}{2} + \frac{3}{2} - \frac{7}{36}\right]$$

$$= 2\pi \left[\frac{1505}{36}\right]$$

$$= \frac{1505\pi}{18}$$

QUIZ

1. a. The key here is that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Hence, if $g(x) = \int_0^x \frac{du}{u^6 + 1}$ then $g'(x)$

$= \frac{1}{x^6 + 1}$. Therefore: $g'(\frac{1}{2}) = \frac{1}{(\frac{1}{2})^6 + 1}$

$= \frac{1}{\frac{1}{64} + 1} = \frac{64}{65}$.

b. With g as in (a), it follows that

$$g'(x) = \lim_{h\to 0}\left\{\frac{1}{h}\int_x^{x+h}\frac{du}{u^6 + 1}\right\}.$$

Namely:

$$\lim_{h\to 0}\left\{\frac{1}{h}\int_x^{x+h}\frac{du}{u^6 + 1}\right\} = \lim_{h\to 0}\left\{\frac{1}{h}\left(\int_0^{x+h}\frac{du}{u^6 + 1} - \int_0^x \frac{du}{u^6 + 1}\right)\right\}$$

$$= \lim_{h\to 0}\left\{\frac{1}{h}\Big(g(x + h) - g(x)\Big)\right\}$$

[1. cont'd]

$$= \lim_{h\to 0} \left\{ \frac{g(x+h) - g(x)}{h} \right\}$$

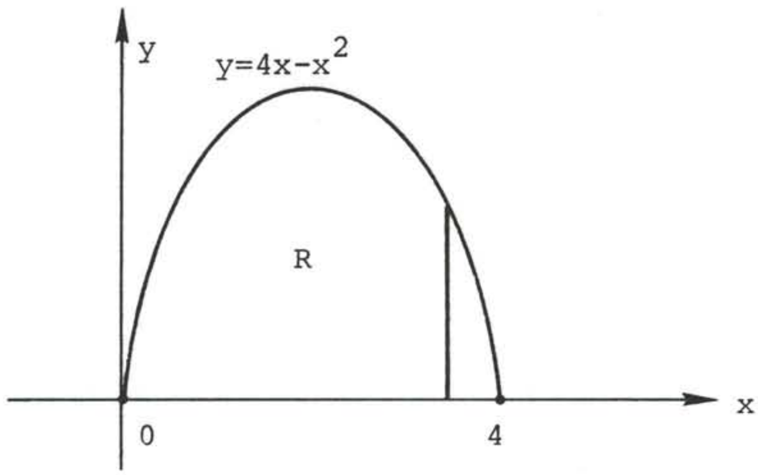
$$= g'(x)$$

Now, from (a), $g'(x) = \frac{1}{x^6 + 1}$

Hence:

$$\lim_{h\to 0} \left\{ \frac{1}{h} \int_x^{x+h} \frac{du}{u^6 + 1} \right\} = \frac{1}{x^6 + 1}$$

2.



a. $$A_R = \int_0^4 (4x - x^2)\, dx$$

$$= 2x^2 - \frac{1}{3} x^3 \Big|_0^4$$

[2. cont'd]

$$= 2(4)^2 - \frac{1}{3}(4)^3$$

$$= 32 - \frac{64}{3}$$

$$= \frac{32}{3}$$

[The concept used here was simply that $\int_a^b f(x)\,dx$ denotes the area of the region bounded above by $y = f(x)$, below by the x-axis, on the left by $x = a$ and on the right by $x = b$. Moreover, if f is continuous, as it is in our exercise, $\int_a^b f(x)\,dx = g(b) - g(a)$ where $g'(x) = f(x)$.]

b. Here we use the fact the $V_x = \pi \int_a^b [f(x)]^2 dx$

In this case:

$$V_x = \pi \int_0^4 (4x - x^2)^2\, dx$$

$$= \pi \int_0^4 (16x^2 - 8x^3 + x^4)\, dx\ ^*$$

---

*Do not make the mistake of identifying $\int (4x - x^2)^2\, dx$ with the form $\int u^2 du$. In other words $\int (4x - x^2)^2\, dx \neq \frac{1}{3}(4x - x^2)^3$. For this to be the correct answer we would have to have:

$$\int (4x - x^2)^2\, d(4x - x^2)$$

[2. cont'd]

$$\therefore V_x = \pi\left[\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5\right|_0^4$$

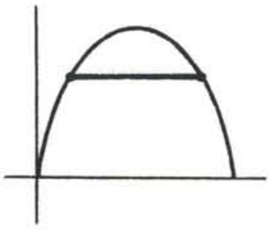
$$= \pi\left[\frac{16(4)^3}{3} - 2(4)^4 + \frac{1}{5}(4)^5\right]$$

$$= \pi(4^3)\left[\frac{16}{3} - 8 + \frac{16}{5}\right]$$

$$= \frac{64\pi}{15}[80 - 120 + 48]$$

$$= \frac{512\pi}{15}$$

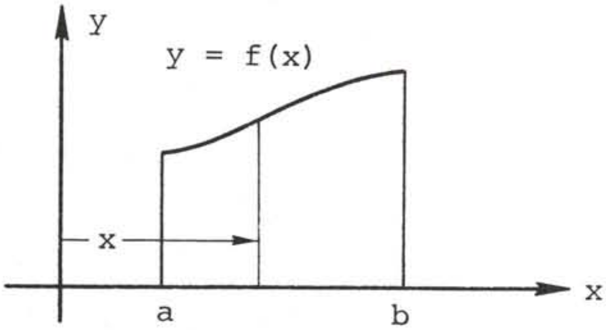
c. The key here is that it might be best to use the method of cylindrical shells. For to use the revolution-method of part (b) our "element of area" would be



and this would involve inverse function multi-valuedness. In the case of a quadratic equation, the "inversion" is not difficult, as we shall soon see. In more general cases, the problem is quite serious, perhaps, even hopeless.

Recall that the "recipe" for cylindrical shells is:

[2. cont'd]



$$V_y = \int_a^b 2\pi x f(x)\,dx$$

In our case, we have

$$V_y = \int_0^4 2\pi x(4x - x^2)\ dx$$

$$= 2\pi \int_0^4 (4x^2 - x^3)\ dx$$

$$= 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right|_0^4$$

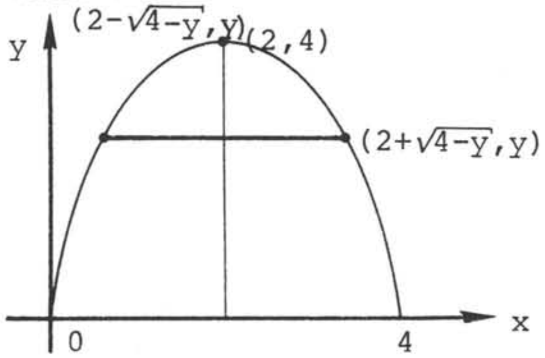
$$= 2\pi\left[\frac{4(4^3)}{3} - \frac{1}{4}(4)^4\right]$$

$$= 2\pi(4^4)[\tfrac{1}{3} - \tfrac{1}{4}] = 512\pi[\tfrac{1}{12}]$$

$$= \frac{128\pi}{3}$$

[2. cont'd]

Had we elected to use the technique in (b), we would have:



$$\begin{cases} y = 4x-x^2 \\ \therefore x^2 - 4x + y = 0 \\ \therefore x = \dfrac{4\pm\sqrt{16-4y}}{2} = 2\pm\sqrt{4y-y} \end{cases}$$

$$V_y = \pi \int_0^4 \left[(2 + \sqrt{4 - y})^2 - (2 - \sqrt{4 - y})^2\right] dy$$

$$= \pi \int_0^4 8\sqrt{4 - y}\, dy = 8\pi \left[-\frac{2}{3}(4 - y)^{\frac{3}{2}}\right]_0^4$$

$$= 8\pi\left[0 - \left\{-\frac{2}{3}(4)^{\frac{3}{2}}\right\}\right] = 8\pi\left(\frac{16}{3}\right)$$

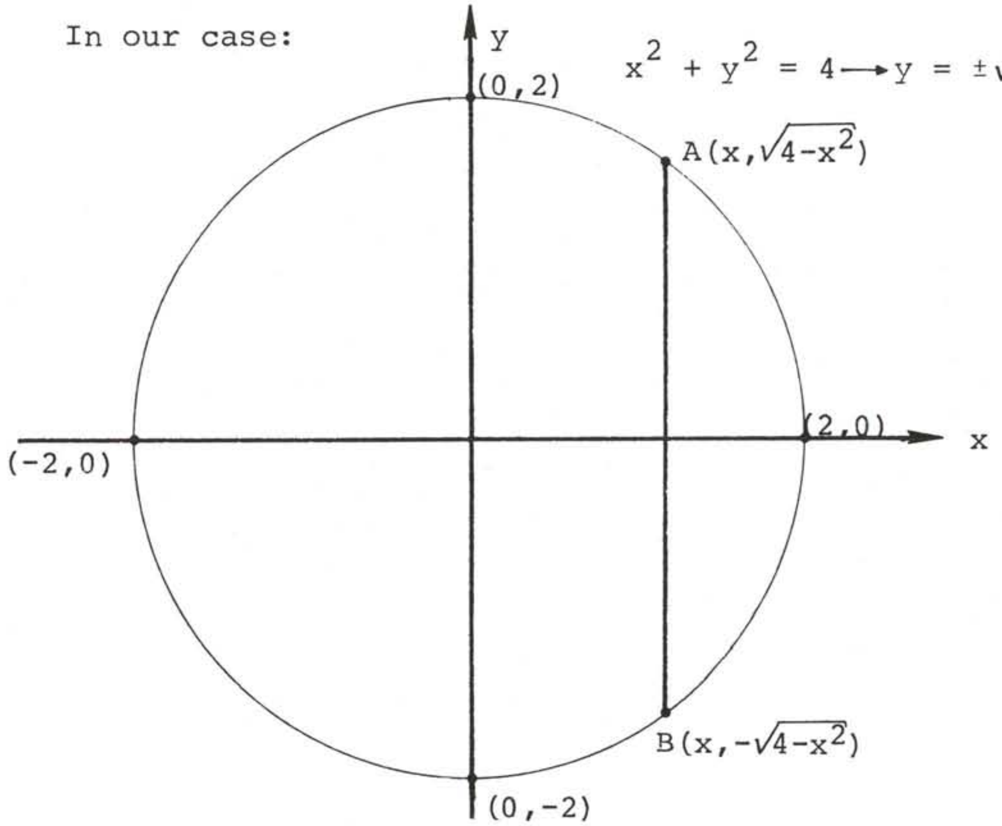
$$= \frac{128\pi}{3}$$

3. Here we wish to emphasize the general formula for area (lest we begin to feel that every section is either a circle, a "washer", or a cylindrical shell).

The general formula is $V = \int_a^b A(x)dx$ where $A(x)$ denotes the cross-sectional area of our solid as a function of x.

[3. cont'd]

In our case: $x^2 + y^2 = 4 \longrightarrow y = \pm\sqrt{4 - x^2}$



AB is the length of the side of the square

$\therefore A(x) = \overline{AB}^2 = (2\sqrt{4 - x^2})^2 = 4(4 - x^2)$.

$$\therefore V = \int_{-2}^{2} 4(4 - x^2)\,dx = \int_{-2}^{2} (16 - 4x^2)\,dx$$

$$= 16x - \frac{4}{3}x^3 \Big|_{-2}^{2} = 2\left[16x - \frac{4}{3}x^3\right|_{0}^{2}$$

$$= 2(32 - \frac{32}{3}) = \frac{128}{3}$$

4.

a. The length is $\int_{s_1}^{s_2} ds$ and in this case, since the equation is in the form $y = f(x)$, the most convenient form for ds is $\sqrt{1 + (\frac{dy}{dx})^2}\ dx$.

Therefore, the length of C is given by:

$$\int_0^4 \sqrt{1 + (\frac{dy}{dx})^2}\ dx \qquad (1)$$

Now, $y = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = x^{\frac{1}{2}} - \frac{1}{4} x^{-\frac{1}{2}}$$

$$(\frac{dy}{dx})^2 = x - \frac{1}{2} + \frac{1}{16} x^{-1}$$

$$1 + (\frac{dy}{dx})^2 = x + \frac{1}{2} + \frac{1}{16} x^{-1} = (x^{\frac{1}{2}} + \frac{1}{4} x^{-\frac{1}{2}})^2$$

$$\therefore \sqrt{1 + (\frac{dy}{dx})^2} = x^{\frac{1}{2}} + \frac{1}{4} x^{-\frac{1}{2}} \qquad (2)$$

Therefore, from (1), the length of C is:

[4. cont'd]

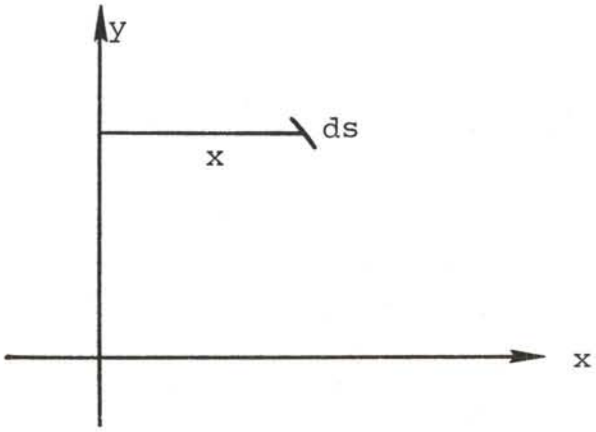
$$\int_0^4 \left(x^{\frac{1}{2}} + \frac{1}{4}x^{-\frac{1}{2}}\right) dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right|_0^4$$

$$= \frac{2}{3}(4)^{\frac{3}{2}} + \frac{1}{2}(4)^{\frac{1}{2}} = \frac{2}{3}(8) + 1$$

$$= \frac{19}{3}$$

b.



Our recipe is

$$S_y = \int_{s_1}^{s_2} 2\pi x \, ds$$

[4. cont'd]

Again, we write ds as $\sqrt{1 + (\frac{dy}{dx})^2}$ dx and use (2) to obtain:

$$S_y = \int_0^4 2\pi x(x^{\frac{1}{2}} + \frac{1}{4} x^{-\frac{1}{2}}) \, dx$$

$$= 2\pi \int_0^4 (x^{\frac{3}{2}} + \frac{1}{4} x^{\frac{1}{2}}) \, dx$$

$$= 2\pi \left[\frac{2}{5} x^{\frac{5}{2}} + \frac{1}{6} x^{\frac{3}{2}} \right|_0^4$$

$$= 2\pi \left[\frac{2}{5}(32) + \frac{1}{6}(8) \right]$$

$$= 2\pi \left[\frac{64}{5} + \frac{4}{3} \right] = \frac{2\pi}{15}[(64)(3) + 4(5)]$$

$$= \frac{424\pi}{15}$$

5. a. Here we are reviewing continuity. The point is that since sin x and x are both continuous, $\frac{\sin x}{x}$ will also be continuous except at x = 0. Now at x = 0, $\frac{\sin x}{x}$ is not defined. But we already know that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

[5. cont'd]

In other words, since $x \neq 0 \rightarrow f(x) = \frac{\sin x}{x}$,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad (1)$$

Moreover

$$f(0) = 1 \qquad (2)$$

Comparing (1) and (2) we have:

$$\lim_{x\to 0} f(x) = f(0),$$

hence, by definition of continuity, f is continuous at $x = 0$ (even though $\frac{\sin x}{x}$ isn't). Since $f(x) = \frac{\sin x}{x}$ is already known to be continuous when $x \neq 0$, the result follows.

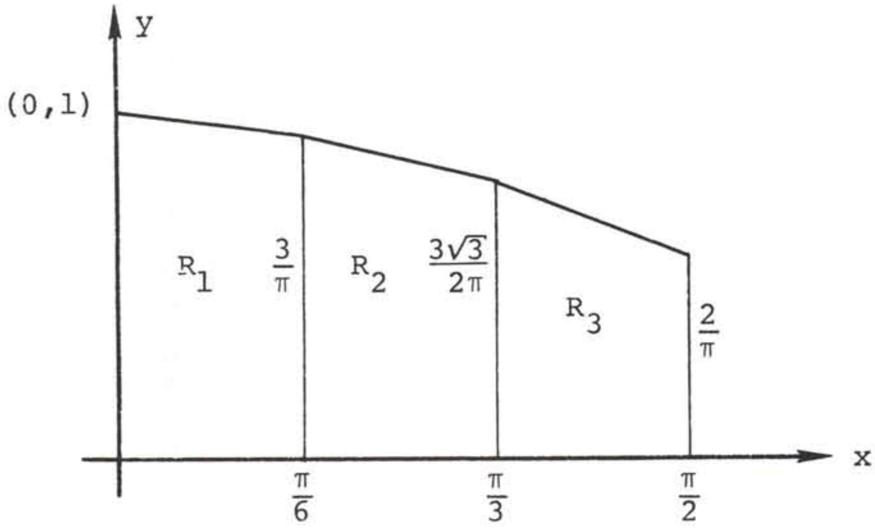
b. There is no need to sketch $y = f(x)$ accurately. The crucial points are that (1) f is continuous and (2) $f(x) \geqslant 0$ for all $x \in [0,\frac{\pi}{2}]$.

Dividing $[0,\frac{\pi}{2}]$ into 3 equal parts yields the partition $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$. Since $f(0) = 1$, $f(\frac{\pi}{6}) =$

$$\frac{\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\pi}{6}} = \frac{3}{\pi}, \quad f(\tfrac{\pi}{3}) = \frac{3\sqrt{3}}{2\pi}, \text{ and } f(\tfrac{\pi}{2}) = \frac{2}{\pi};$$

it follows that:

[5. cont'd]



$$A_{R_1} = \frac{1}{2}\left(1 + \frac{3}{\pi}\right)\left(\frac{\pi}{6}\right)$$

$$A_{R_2} = \frac{1}{2}\left(\frac{3}{\pi} + \frac{3\sqrt{3}}{2\pi}\right)\left(\frac{\pi}{6}\right)$$

$$A_{R_3} = \frac{1}{2}\left(\frac{3\sqrt{3}}{2\pi} + \frac{2}{\pi}\right)\left(\frac{\pi}{6}\right)$$

Therefore, our trapezoidal approximation with n = 3 yields

$$T_3 = A_{R_1} + A_{R_2} + A_{R_3}$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)\left[\left(1 + \frac{3}{\pi}\right) + \left(\frac{3}{\pi} + \frac{3\sqrt{3}}{2\pi}\right) + \left(\frac{3\sqrt{3}}{2\pi} + \frac{2}{\pi}\right)\right]$$

$$= \frac{\pi}{12}\left[1 + \frac{6}{\pi} + \frac{3\sqrt{3}}{\pi} + \frac{2}{\pi}\right]$$

[5. cont'd]

$$= \frac{\pi}{12} + \frac{6}{12} + \frac{3\sqrt{3}}{12} + \frac{2}{12}$$

$$= \frac{\pi + 8 + 3\sqrt{3}}{12} \qquad (\approx 1.37)$$