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PROFESSOR: Hi. Our lecture today actually has us backtrack a little bit to a concept which is actually more fundamental than that of differentiability. It's a topic which is called 'continuous functions'. In other words, our lecture today is concerned with a topic called Continuity. And actually, the topic of continuity had its roots way back at the beginning of our course when we first raised the question: does the limit of 'f of x' as 'x' approaches 'a' equal 'f of a'?

Remember, when we first started talking about the limit concept, our intuitive approach was to say, look-it, as 'x' gets arbitrarily close to 'a', 'f of x' gets arbitrarily close to 'f of a'. And we saw that this particular definition had a few loopholes in it, even though intuitively this is what we would have liked. And so what I want to do today is to see what happens if we can be sure that the limit of 'f of x' as 'x' approaches 'a' equals 'f of a'. What is implied if this happens to be a true statement?

And my first claim is that it is implicitly implied that 'f of a' must at least make sense. Otherwise, we wouldn't have written it over here. In other words, 'f' must be defined when 'x' equals 'a'. 'f of a' must be defined. Now to correlate that with material that we've had previously, let's go back to an example that we've discussed before, or at least an example which is close to something we've studied before. Let's define 'f of x' to be 'x squared minus 1' over 'x - 1'. Now you see as long as 'x' is not equal to 1, we can cancel 'x - 1' from both numerator and denominator, leaving this equal to 'x + 1'. And we then see what? That the limit of 'f of x' as 'x' approaches 1 is equal to 2.

On the other hand, notice that 'f of 1' turns out to be 0/0 in this case, which is undefined. In other words, in this particular case, the limit of 'f of x' as 'x' approaches 1 is not equal to 'f of 1' if only because 'f of 1' isn't even defined. And again, by way of a review, let's look at this thing pictorially.

Remember, 'x squared minus 1' over 'x - 1' is 'x + 1' except when 'x' equals 1. Consequently, to graph 'y' equals 'x squared minus 1' over 'x - 1', we simply graph 'y' equals 'x + 1' with the point corresponding to 'x' equals 1, meaning the point 1 comma 2 deleted. In other words, this

point deleted is the graph 'y equals x squared minus 1 over x - 1'. You see again there is no definition at 'x equals 1' because this point is missing. It would be the 0/0 form.

Now, to be sure, we can clean this up in the sense that we could define a new function 'g of x' to be 'x squared minus 1 over x - 1' as long as 'x' is not equal to 1 and define it to be 2 when 'x' equals 1, which, by the way, geometrically is just a fancy way of writing the equation of the line 'y equals x + 1'. In other words, all we're saying is that letting 'g of x' be 2 when 'x' equals 1 plugs the little hole to the periods over here. Now again, I hope that this seems fairly familiar from our discussion on limits, but for the time being, all I want us to see is that as soon as you write down that the limit of 'f of x' as 'x' approaches 'a' equals 'f of a', you at least imply that 'f of a' must be defined.

Let's see another property that's implied by our definition. I claim, as you might expect, that if the limit of 'f of x' as 'x' approaches 'a' is 'f of a', that means that 'f of x' is near 'f of a' when 'x' is near 'a'. In other words, my claim is that the curve 'y equals f of x' is unbroken in a neighborhood of 'x equals a'. And to see what I mean by that, let's look at this picture over here.

Suppose, for example, that our curve 'y equals f of x'-- let me darken this up so that we can see it a little better. Suppose that curve had a break at 'x equals a', OK? Now, let's say 'f of a' is over here. What I could do is now pick an interval surrounding 'f of a', if there was a break over here, such that this band never touched or included the top curve here. Now notice, from this point of view, that no matter how close 'x' gets to 'a', as long as 'x' is greater than 'a', 'f of x', which is up here on this dark curve, can never be within epsilon of 'f of a'. In other words, if there's a break in the curve I could always fix it up so that the limit of 'f of x' as 'x' approaches 'a' is not equal to 'f of a'.

At any rate, with these two properties as motivation, let us now give our basic definition. And the basic definition is simply this: a function 'f' is called continuous that 'x equals a' precisely if the property that we were discussing is present, namely, if the limit of 'f of x' as 'x' approaches 'a' is equal to 'f of a'.

We generalize this definition and say 'f' is called continuous on the entire interval 'I', or on the interval 'I', if the limit of 'f of x' as 'x' approaches 'a' is equal to 'f of a' for each 'a' in 'I'. In other words, notice that our first definition is a local property. Namely, we define continuous or continuity at a given value. And then if the function happens to be continuous at every point,

then we call the function itself continuous. What the thing means pictorially is that if the function is continuous at x equals a , it means that in terms of the graph of the function, that in a neighborhood of the point a comma f of a on the graph, in a neighborhood of that point, in other words, in a sufficiently small interval surrounding that point, the curve itself must be unbroken.

Now, you see, this may sound trivial. Remember, when we first started talking about limits, we had the feeling that limit of f of x as x approaches a should always equal f of a . And the reason for this was that instinctively we seem to always think of continuous functions. We think of what? Things changing in such a way the graph of the change is unbroken even though, of course, there are places where discontinuities occur. At any rate, what I'm saying is that many of the results that I now want to discuss with you, which are very important properties of continuous functions, may seem self-evident because we keep thinking about any function as being continuous. But I will try to emphasize the fact that these properties are not true if the function is not continuous.

I also would like to show a balance between geometric ideas and analytic ideas. In other words, let me start off by seeing what things seem to follow about continuous functions based primarily on the graph idea. For example, I claimed that continuous functions assume their maximum and minimum values on any closed interval. Now, this sounds like a big mouthful, and it also sounds kind of trivial. You say doesn't any function have a maximum and minimum value on a closed interval? And the answer is no.

For example, take the curve y equals 1 over $x - 1$ squared. Notice that when x is 1 , y is infinite, which means that if we graph this particular function, we find that the graph does something like this. In other words, it goes-- it jumps up here, comes back, et cetera. Now, what is the highest value or what is the highest point on our graph here? Well, notice that the graph is broken here. It's discontinuous when x equals 1 . Notice that the curve rises to infinity. In other words, the maximum value is undefined on this particular closed interval. You see there's a jump over here.

Of course, the fact that this is an infinite jump may make you feel uneasy if you don't feel comfortable with infinity. So let me paraphrase this just a little bit in terms of a finite jump. Let me write down the following function. Let's think of the function f of x equals x if x is less than 1 . Then as soon as x is at least as big as 1 , the function becomes $-x$. In other words, graphically, we have the line y equals x from 0 up to 1 , but not including 1 . And then

at 1, the function jumps down to minus 1 and becomes the line 'y' equals 'minus x'.

Let's look, for example, at the closed interval again from 0 to 2. What is the biggest value? What is the maximum that the function can have if 'x' is between 0 and 2? Well, notice that as 'x' gets closer and closer to 1, 'f of x' gets closer and closer to 1, But 'f of x' never equals exactly 1, because when 'x' is 1, the curve jumps down to here.

In other words, notice that as 'x' gets arbitrarily close to 1, 'f of x' increases and gets arbitrarily close to but never equals 1. In other words, notice that by picking 'x' to be less than 1, we can make 'f of x' as close to 1 as we want, but we can never make it exactly equal to 1 because of this jump which takes place over here. You see, this discontinuity causes us a little bit of difficulty.

By the way, we can use this example from another point of view. If we just look at the part where it says 'f of x' is equal to 'x' where 'x' is less than 1, that means that we'll just look at this part of the curve. Notice that this part of the curve is unbroken. And yet as 'x' approaches 1, 'f of x' approaches but never equals 1, which means that to get the maximum value in here, I would have to include the end point 'x' equals 1.

In other words, notice that even if the function is continuous, if the interval is open, it may not assume the maximum value. In other words, you may not be able to find the value of 'x' in that interval such as the function will be maximum at that particular value. Now again, this is making a mountain out of a mole hill in terms of your intuition because these are result which I'm sure you believe are true anyway. In other words, it seems to be true. But all I hope is that these little examples here show you what the importance of continuity is, because as we progress later in our work, these so called fine points are going to become crucial in many of our proofs.

And by the way, I think it's a truism to say that if you make fine points seem very, very important before you get to use them and don't emphasize them, people tend to think that you overestimated your case. So I prefer not to beat this to death and wait until such times in our course that we need these results before I emphasize them more. So these are results which are in the textbook, which I'll test you on in the exercises and the like, but just want to go through with you now so that you get an overall idea.

Another result that's rather clear geometrically is something called the intermediate value theorem. Suppose that 'f' is continuous on the closed interval from 'a' to 'b'. And without loss of

generality, let's suppose 'f of a' is less than 'f of b'. The same kind of an argument would hold if 'f of b' were less than 'f of a', but I just need some sort of an orientation.

Now I say let 'm' be any number such that m is what? Between 'f of a' and 'f of b'. In other words, 'm' is greater than 'f of a' but less than 'f of b'. You see, if 'f of b' were less than 'f of a', I would just reverse the inequality signs here. At any rate, what my claim is that we can find the number 'c' in the open interval from 'a' to 'b' such that 'f of c' equals 'm'. Again, this may sound kind of complicated. All it says in terms of a picture is this. If our curve goes from point 1 to point 2 where the height of 'p1' is 'f of a' and the height of 'p2' is 'f of b', then it must assume every intermediary height between 'f of a' and 'f of b' on that particular interval.

Again, you can think of that in terms of an automobile, if you want. If an automobile goes from a speed of 20 miles an hour to a speed of 30 miles per hour in what we think of as being continuous motion, then at some time in that interval, it must've had any particular speed you want to mention between 20 and 30 miles. In other words, we don't visualize a car going from 20 miles an hour to 30 miles an hour, say, without at one time having passed the speed of 27 miles per hour or some such thing. And geometrically, you see, this proof is very simple, namely, draw the line 'y' equals 'm'.

And now what you're saying is you want to get from this point to this point. And the idea is that the only way you can get from this point to this point without crossing this line is if someplace along the way, you jump over this line. In other words, geometrically, all you're saying is lookit, if I have to join 'p1' and 'p2' with an unbroken line, then I must cross the line 'y' equals 'm' someplace. By the way, notice I say that at least one place, the curve could have done something like this, for example. But the important point is what? There's at least one number 'c' such that 'f of c' equals 'm'.

Again, somebody might think in terms of an end run and say couldn't we have done something like this and not cross this line? That brings us back again to our concept of single-valuedness. In other words, notice, if we remove the restriction that 'f be a' single-valued function, in other words, if 'f' can double back, notice that if we think of the line 'y' equals 'm' as being endless, the point remains what? That you must someplace cross this line in going from 'p1' to 'p2'. But if we remove the single-valued restriction, then the point at which you cross this line would not have to be-- see, in this case, 'c' would not be in the open interval from 'a' to 'b' since 'f' is not single-valued.

And what I'm hoping this discussion does for us is gives us a good geometric feeling as to what is happening in terms of a continuous function. By the same token, remember our basic definition of continuity goes back to our concept of limit. And our concept of limit has been cemented down fairly firmly from an analytical point of view, and that means that we can also see these things theoretically as well as pictorially. And hopefully what we will do is combine pictorial and analytic aspects to the best possible advantage for solving problems.

But to show you what I mean by this, let's look at a few analytical properties of continuous functions. For example, suppose 'f' and 'g' are both continuous at 'x' equals 'a'. And suppose we define 'h' to be the sum of 'f' and 'g'. In other words, 'h of x' is 'f of x' plus 'g of x', OK? Let's compute the limit of 'h of x' as 'x' approaches 'a'. By definition, the limit of 'h of x' as 'x' approaches 'a' is the limit of the quantity 'f of x' plus 'g of x' as 'x' approaches 'a'.

Now, if we look at this particular expression, notice that because of our theorems on limits, we can do an awful lot with this without having to make any recourse to our picture at all. For example, notice that we can say right away that since the limit of a sum is the sum of the limits, this is the limit of 'f of x' as 'x' approaches 'a' plus the limit of 'g of x' as 'x' approaches 'a'. Secondly, since 'f' and 'g' are continuous at 'x' equals 'a', by definition of continuous that 'x' equals 'a', this says that the limit of 'f of x' as 'x' approaches 'a' is 'f of a'. The limit of 'g of x' as 'x' approaches 'a' is 'g of a'. Therefore, this expression is 'f of a' plus 'g of a'. But by the definition of 'h', this is just 'h of a'. And if we now look at this, we see what? That the limit of 'h of x' as 'x' approaches 'a' equals 'h of a'.

Notice that this result came about under the assumption that 'f' and 'g' were both continuous that 'x' equals 'a', and what have we proven over here? This is precisely the statement that 'h' is continuous at 'x' equals 'a'. In other words, what we've now proven analytically is that a sum of two continuous functions is a continuous function. By induction, we could prove this for the sum of more than two. We can in a similar way by using limit theorems prove things like the product of two continuous functions is continuous, et cetera.

We also were on the verge of a topic like this when we talked about differentiability sometime back. Namely, I claim that there is an interesting connection between differentiable and continuous, and that the connection is that any differentiable function is continuous. And by the way, the proofs utilizes that which we used in our first lecture on derivatives in this block. Namely, we take the expression 'f' of-- we want to show that if 'f prime of a' exists, that 'f' must be continuous at 'x' equals 'a'. That means we want to show that the limit of 'f of x' as 'x'

approaches 'a' is 'f of a', and that's the same as saying we want to show that this difference approaches 0 and the limit.

And the trick is we take 'f of x' minus 'f of a' and write it as "f of x' minus 'f of a' divided by 'x - a' times 'x - a', the idea being that we'll now take the limit as 'x' approaches 'a'. The limit of a product is the product of the limits. The limit of this as 'x' approaches 'a' is just going to be 'f prime of a'. That's our definition of derivative. And the limit of 'x - a' as 'x' approaches 'a' is just 0. In other words, putting these steps together, look what we get. The limit as 'x' approaches 'a', 'f of x' minus 'f of a', is the limit as 'x' approaches 'a' of this bracketed expression "f of x' minus 'f of a' over 'x - a" times the limit of 'x - a' as 'x' approaches 'a'.

By definition, 'f' being differentiable at 'a' means that this limit exists and is, in fact, 'f prime of a'. But any finite number times 0 is still 0. And therefore, it follows that the limit of 'f of x' as 'x' approaches 'a' is 'f of a'. There's a legitimate analytic proof that differentiability implies continuity.

Now, it's not necessarily true that continuity implies differentiability. And you see even though you can prove these things analytically, if you don't form some picture in your mind, you frequently will wind up memorizing results rather than having a feeling for them. What I mean by having pictures and proofs coexist is this.

Instead of saying to yourself, let's see, continuity is a weaker condition than differentiability, it's a strong-- which one is it? You say look-it. We identified continuity with unbroken. Remember, continuous means unbroken. What did differentiable mean? Differentiable meant that the curve had a tangent line, and that, in turn, meant that the curve was smooth. In other words, we may think of differentiable as the geometric analog of smoothness, continuity as the geometric analog of unbrokenness. And now I think it's very easy to see pictorially that a smooth curve must be unbroken, but an unbroken curve doesn't have to be smooth.

And in fact, a very trivial illustration of this is the graph 'y' equals the absolute value of 'x'. You see, well, what happens here? At the origin, notice that we have a sharp corner, but that the curve itself is unbroken there. You see, I can draw the curve without taking the pencil off the paper, the chalk off the board. That's what continuity means from an intuitive point of view. Yet I have a sharp corner. It's not smooth here. That's why I write smooth in quotation marks. Hopefully, the textbook, together with our exercises, will make this much clear to you. But notice again in terms of the picture how an unbroken curve doesn't have to be smooth, but a

smooth curve has to be unbroken.

Now, the point is we could say much more about continuous functions. I think that all we have to say, though, has already been said. It'll be reinforced in the text. We'll have exercises on this. But the important thing for now is to understand what we mean by continuous, how we will use it in the future, and at any rate, we'll be talking about that more later. And so until next time, goodbye.

NARRATOR:

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