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PROFESSOR: Hi. Our subject matter today concerns that phase of calculus known as the 'indefinite integral', or the 'antiderivative'. In terms of the concepts that we've talked about so far in our course, and keeping in mind that one concept with many applications is educationally more meaningful than many concepts each with one application, I prefer to call today's lesson the inverse derivative, or inverse differentiation. And again, bring into play a very well-known feature of our course, namely, our discussion of inverse functions.

Now, you see, the only problem that comes up in this context is a rather simple one and that is that all the time that we've been taking a derivative, we may not have realized that we were using the idea of a function in a rather different way. Namely, noticed that by taking a derivative, we had a rule which told us how, given a particular function, to assign to it a new function called its 'derivative'.

In other words, what we could have done is to have visualized a different function machine, which I will call the 'D-machine'. Hopefully, the 'D' will suggest differentiation, where the domain of my the machine-- in other words, the input of my 'D-machine' will be differentiable functions and the output, you see, the image of my 'D-machine' will be the derivative of that particular function.

So in other words then, if we want to use our typical notation, what we're saying is that the 'D-machine' does what? Given a differentiable function as its input, the output will be the derivative 'f prime'. Now again, we're used to talking about the derivative with respect to a given variable. Unless otherwise specified, the variable is 'x'. And so, perhaps with that in mind, maybe it would be more suggestive if I were to do something like this to indicate, you see, that the input is 'f of x' and the output is 'f prime of x'.

Now, you may recall also that long before we started dealing with functions of real variables, we discussed functions in general in terms of circle diagrams and the like. In this respect, notice that we can think of our 'D-machine' as operating on a particular domain, the domain being what? The set of all differentiable functions. And the range, or the image, will also be

what? All functions which are derivatives of differentiable functions.

Now, what kind of a function is 'D'? Don't lose track of the fact that things like one-to-oneness and ontoness are concepts that transcend dealing with numbers. They apply whenever we're dealing with any kind of a function. The idea is something like this. If 'x squared' were to go into the 'D-machine', the output would be '2x'. See? 'x squared' is mapped into '2x'. 'x cubed' is mapped into '3 x squared'. See, the derivative of 'x cubed' with respect to 'x' is '3 x squared'. 'x squared plus 7', also mapped into '2x'.

In other words, without going any further, notice that whatever 'D' is, 'D' is not 1 to 1. You see, two different functions can have the same derivative. In fact, many different functions can have the same derivative. For example, once we know that the derivative of 'x squared' is '2x', we certainly know that the derivative of 'x squared' plus any constant is also '2x'.

In other words then, we cannot make an inverse function machine as things stand here. And why is that? To have an inverse function, what you must be able to do is to reverse arrowheads and still have a function. Notice that, in a way, 'D inverse' would be a little bit of a tricky thing to talk about here. But let's pretend that we could anyway.

In other words, what would go wrong if we tried to build a 'D' inverse machine? Well, in terms of the example that we're talking about, if the input to the D inverse machine happened to be '2x', then there would be infinitely many different outputs that we're sure of. Namely what? Every function of the form 'x squared plus c' where 'c' is a constant.

And what do I mean, we can be sure of that many? You see, all I know going back to this diagram here is that every function of the form 'x squared plus c' will map into '2x' under the derivative function.

The other question that comes up is, how do you know that you can find a function whose derivative is '2x' that comes from a function which doesn't have the form 'x squared plus c'? How do you know there isn't some other function 'f of x' which has the property that 'f of x' maps into '2x'?

In other words, the derivative of 'f of x' is '2x' even though 'f of x' does not have the form 'x squared plus c'. And this is exactly where the mean value theorem comes in to help us over here. You see, what we're saying here is, suppose that 'D' of 'f of x' is '2x'. All we know is that when you run 'f of x' through the 'D-machine', you wind up with '2x'.

What does that mean? 'f prime of x' is '2x'. Well, we know the derivative of 'x squared' is also '2x'. Therefore, whatever 'f of x' is by the corollary to the mean value theorem that we've studied before, since 'f of x' and 'x squared' have identical derivatives, namely, they're both '2x', it means that they must differ by a constant. In other words, 'f of x' minus 'x squared' is a constant. But to say that 'f of x' minus 'x squared' is a constant is the same as saying that 'f of x' belongs to the family 'x squared plus c'.

In other words, 'f of x' equals 'x squared' plus a constant. What the mean value theorem tells us is not only does every function of the form 'x squared plus c' have its derivative equal to '2x' but that every function whose derivative is '2x' has the form 'x squared plus c'.

To generalize this, consider the following. Suppose we're given 'f of x', OK? And we now say, look, every time you differentiate a function, if you add on a constant, you don't change anything, meaning what? The derivative of a constant is 0. In other words, what we're saying is, if the derivative of 'f of x' is 'f prime of x', any function of the form 'f of x' plus 'c' will have the same derivative. And not only that, the only function whose derivative is 'f prime of x' must be one of these. That's what this 'e sub f' stands for. You see, what I'm really saying here is that certainly when you change the constant, you change the function. The point is, with respect to the thing called the derivative, you cannot tell the difference between two functions just by looking at their derivatives if they differ by a constant.

In other words, with respect to differentiation, two functions which differ by a constant are equivalent and that's why I invented this notation, 'e of f'. All I'm saying is that if we visualize now that the input of the 'D-machine' is not individual functions but whole classes of functions-- in other words, such that they differ only by a constant, then you see there is a one-to-one correspondence between the input and the output. Now that's a subtle point. It's a point which I'm sure most of you will grasp. But more importantly, the important thing is simply that once we've seen one function with a given derivative, in a manner of speaking, we've seen them all. In other words, we only have enough leeway as to fool around with an arbitrary constant.

So let's generalize again. What do we mean by 'D inverse' of 'f of x'? We mean what? The set of all function 'g of x' whose derivative with respect to 'x' is the given 'f of x'. See, that's exactly what the inverse function means. And by the way, notice that this tells me my set implicitly. Let me put that in parentheses over here. Namely, suppose somebody says, does 'g of x' belong to this particular set called 'D inverse'? All I have to do is differentiate the given 'g' and see if I

get 'f'. If I do, 'G' belongs to the set. If I don't, it doesn't belong to the set.

But as we've so often stressed about inverse functions, notice that the test to see whether something belongs to 'D inverse', it's sufficient to know how to differentiate. By the way, to summarize what we did before, to list this thing explicitly, notice that what we're saying is, to find the set of all functions whose derivative is little 'f of x', all we have to do is what? Find one function, capital 'F', whose derivative is little 'f'. And then our set, explicitly, is what? The set of all functions of the form capital 'F of x' plus 'c' where 'c' is an arbitrary constant.

Now, I'm sure this concept is not difficult for you to grasp. For those of you who have been through calculus before, as our course is intended to be for this type of person too, you see, you may not be familiar with the notation, the 'D inverse' notation, which I want to stress here. But the concept, I hope, is clear in its own right. Now of course, you see, the problem that comes up is that it's much easier to differentiate a function than it is to be given a function and then have to try to find what you have to differentiate to get it. Without meaning it as facetiously as it may sound, I use this in on many occasions, I prefer to say is much easier, you see, to scramble an egg than to unscramble one. Told what to differentiate, that's easy enough to do. Given the derivative, that may not be so easy.

Let's look in terms of an example. But this is, again, a very nice teachers trick. To get the right answer, you start with the answer and then work to the problem. I'll start with the function 'h of x' equals 'x' times the 'square root of 'x squared plus 1''. Or written more conveniently in exponential notation, 'x' times ''x squared plus 1' to the 1/2'. Let me differentiate that. Remember, this is a product. The derivative of a product is what? It's the first factor which is 'x' times the derivative of the second. That means what? I bring the 1/2 down to a power 1 less, times the derivative of what's inside with respect to 'x'. See, the some old chain rule again. That's '2x'.

Then what? Plus the second factor, 'x squared plus 1' to the 1/2 times the derivative of the first with respect to 'x', which is 1. At any rate, simplifying, bringing the minus 1/2 power into the denominator, then putting everything over a common denominator, I wind up with the fact that 'h prime of x' is '2 x squared plus 1' over the 'square root of 'x squared plus 1''. I hope I haven't made a careless error here. But again, one of the beauties of the new mathematics is that it's the method that's important, OK?

Now, at any rate, what do I know? Starting with 'h of x' equaling 'x' times the 'square root of 'x

squared plus 1'', I now know that its derivative is '2 x squared plus 1' over the 'square root of 'x squared plus 1''. Let's write that in terms of our 'D inverse' notation. Namely, what we're saying is that 'x' times the 'square root of 'x squared plus 1'' has the property that its derivative is '2 x squared plus 1' over the 'square root of 'x squared plus 1''. Consequently, every function which has the property that its derivative is '2 x squared plus 1' over the 'square root of 'x squared plus 1'' must come from this family. You see? Everything of this form we'll have its derivative equal to this.

And secondly, any other function cannot have its derivative equal to this by our corollary to the mean value theorem. Now you see, what I'm saying is-- and here's the beauty of what we mean by inverse operations and the like. It's conceivable that you might not have been sophisticated enough at this stage in the game to have been able to deduce this had we been given this.

Notice that my cute trick was I started with this, found out what the derivative was, and then just inverted the emphasis. A change in emphasis again. However, notice the following. Suppose you weren't able to find this. And somebody said to you, I wonder if 'x' times the 'square root of 'x squared plus 1'' is a function whose derivative is equal to this.

And all I'm saying, without going through the work again because I've already done that, is to simply observe that even if you did not know this explicit representation, by definition, 'D inverse' of '2 x squared plus 1' over the 'square root of 'x squared plus 1'' is simply what? The set of all functions 'g of x' such that 'g prime of x' is equal to '2 x squared plus 1' over the 'square root of 'x squared plus 1''. In other words, given any function at all, I could differentiate it. If the derivative came out to be this, then I have a solution. It belongs to the solution set. Otherwise, it doesn't. But again, notice that's to solve any 'D inverse' problem, it's sufficient to understand a corresponding derivative property.

In fact, maybe now is a good time to show how we get certain recipes for 'D inverse' type things. Let me just write down a typical one. You see, 'D inverse of 'x to the n'' is 'x to the 'n + 1'' over 'n + 1' plus a constant. And of course, observe that as soon as you see something like this, you have to beware of 'n' equals negative 1. Otherwise we have a 0 denominator.

Now, a person says, this doesn't look familiar to me. Again, keep in mind what D inverse means. Essentially, to say this is just a switch in emphasis from saying what? That if you run the family of functions 'x to the 'n + 1'' over 'n + 1' plus a constant through your 'D-machine',

you get 'x to the n'. Or more familiarly, what? The derivative of any member in this family is 'x to the n' provided that 'n' is not equal to minus 1.

Now again, this may look a little bit abstract to you. So to avoid this problem, let's just do a concrete illustration. Let's pick a particular value of 'n' and work with this thing. Let's suppose we're told to determine 'D inverse' of 'x to the 7th'. What does that mean? What it really means is, let's find a function whose derivative is 'x to the 7th'. And why do I say let's find a function? Because once I find a function, all I have to do is tack on arbitrary constants and the family that I get that way is the unique family of functions which have this particular derivative.

So I play the detective game. I know from differential calculus that if I differentiate 'x' to the 8th power, I'll wind up with the exponent 7, at least. In other words, the derivative of 'x' to the 8th power. 'D of 'x to the 8th'' is '8 'x to the 7th''. Well, what answer did I want to get? I wanted to get 'x to the 7th', not '8 'x to the 7th''. So I fudge this thing a little bit. I say, evidently what I should have done was to have started with 1/8 as much. In other words, multiplying equals by equals, I multiply both sides of this equation by 1/8 and I wind up with '1/8 D 'x to the 8th'' equals 'x to the 7th'.

And now comes a very crucial step. And let me write that down because I think it's something that we should pay very close attention to. And that is that the derivative has the property that if you want to differentiate a constant times a function, you can take the constant out and differentiate just the function. This is a very crucial point because you see-- and by the way, notice that in general, not all functions have this property. For example, if you're squaring something, if you double the number that's being squared, the output is 4 times as much because twice something squared is 4 times as much.

In other words, in general, you do not say that if you double the input, you're going to double the output. Not every function has that property. But the function called 'D', the derivative, does have this particular property. And you see, with that in mind, I can bring this 1/8 inside. This is the key step, that 1/8 times the derivative of 'x to the 8th' is a derivative of ''1/8' x to the 8th'. That's this key step over here.

And now, you see, putting all this together, I find what? That the derivative of ''1/8' x to the 8th' is 'x to the 7th'. And therefore, since I found one function whose derivative is 'x to the 7th', I have, in a sense, found them all, namely, ''1/8' x to the 8th' plus 'c'. In other words, that's what I call the equivalent class of ''1/8' x to the 8th'. All the functions that differ from ''1/8' x to the

8th' by a constant.

And again, to emphasize this very important point, let me again mention, beware of non-constant factors. Let me give you a for instance. Let's suppose I take almost the same problem. And that almost as a big almost. Let's suppose I say, let me find all functions whose derivative, say, is 'x squared plus 1' to the 7th power. In other words, still something to the 7th power.

So I argue something like, well, since whenever I differentiate I reduce the exponent by 1, to wind up with a 7th power, maybe I should have started with an 8th power. So I say, OK, that's what I'll do. I'll start with an 8th power. So I say, OK, what is the derivative of 'x squared plus 1' to the 8th power? Now notice I know how to differentiate, hopefully. And again, let me make this point very strongly. There is no sense studying inverse functions if we don't know the original function itself. Because the whole purpose of the inverse function, or the whole strategy behind it, is to reduce it to the original function, namely, to switch the emphasis.

So at any rate, I differentiate 'x squared plus 1' to the 8th power. I get '8 'x squared plus 1' to the 7th'. But now, by the chain rule, I must remember that this part was only the derivative with respect to 'x squared plus 1'. The correction factor is what? The derivative of what's inside with respect to 'x'. That's '2x'. And so I wind up with that if I differentiate ''x squared plus 1' to the 8th', I get '16x' times ''x squared plus 1' to the 7th'.

Now, how much did I want to get? I wanted to get just ''x squared plus 1' to the 7th'. That put me off by a factor of '16x'. Now I say, OK, I'll fix that up. Namely, I'll divide both sides by '16x', assuming, of course, that 'x' is not 0. And this, by the way, is perfectly valid. I can now go from here to here and say, look, '1 over 16x' times the derivative of ''x squared plus 1' to the 8th' with respect to 'x' is ''x squared plus 1' to the 7th'.

However, notice that I cannot take this factor and bring it inside here. And again, as I so often have said, also, of course I can bring it inside here. I just did. What I mean is, I don't get the right answer. And what's the best proof that I don't get the right answer? Very, very simple. Take this function, differentiate it, and see if you get ''x squared plus 1' to the 7th'. You won't, unless you differentiate incorrectly. Don't be like the person who just differentiates, brings the 8 down, replaces this by 1 less, multiplies by a derivative of what's inside, and cancels everything out.

Notice that the expression inside the brackets that I've just circled is a quotient. And the

derivative of a quotient is obtained in a very special way. The denominator times the derivative of the numerator minus the numerator times the derivative of the denominator over the square of the denominator. And all I'm saying is, you won't get ''x squared plus 1' to the 7th' power if you do that. Again, notice that you do not have to know the right answer in order to see what answer is wrong, OK? So this would be a wrong answer.

By the way, this would also be a wrong answer because we've already seen that the derivative of ''1/8' x squared plus 1' to the 8th power is what? You bring the 8 down, which kills off the 1/8. You replace this to a power of one less, which gives you ''x squared plus 1' to the 7th'. But the correction factor here is you must also multiply by a derivative of what's inside with respect to 'x'. And that's to '2x'.

In other words, you do not get ''x squared plus 1' to the 7th'. What you do get is what? ''x squared plus 1' to the 7th' times '2x'. And the question that may now come up is, how come this worked when you were raising 'x' to the 8th power but it didn't work when you were raising 'x squared plus 1' to the 8th power?

Again, as always in these cases, the answer is immediately available in terms of derivatives. We can talk, in fact, about the inverse chain rule. That when we really talked about 'D'-- remember, we mentioned this at the very beginning of the lecture, that the variable inside the parentheses was the one with respect to which you were differentiating. In other words, what we saw was that if you wanted to get something to the 7th power, what you had to differentiate was 1/8 that same something to the 8th power if you were differentiating with respect to that same variable. In other words, what would have been ''x squared plus 1' to the 7th' would have been what? If you would differentiating not with respect to 'x' but with respect to 'x squared plus 1'.

See, what we really wanted when we wrote this down was the derivative with respect to 'x'. And even though this notation may look a little bit strange to you, observe that once you get used to the notation, this is just another way of talking about the chain rule. Namely, to find the derivative of this with respect to 'x', you first take the derivative with respect to 'x squared plus 1' and then multiply that by the derivative of 'x squared plus 1' with respect to 'x'. And if we do that, you see, we get the answer that we've talked about before.

In other words, what we could say is that the function that you have to differentiate to get ''x squared plus 1' to the 7th' times '2x' is ''1/8' x squared plus 1' to the 8th' plus a constant. And

how do I know that? Well, the way I know that is simply what? In terms of inverse functions, I started with "'1/8' x squared plus 1' to the 8th', differentiated it and found out I got "x squared plus 1' to the 7th' times '2x'. And so this became the recipe.

And again, a rather interesting aside, if you look at this, and look at this, it would appear at first glance that the top one should be a more difficult problem than the bottom one. The reason being that the input seems more simple in the bottom one. Yet, the interesting point is that the '2x', which seems to make this thing more complicated, is precisely the factor you need by the chain rule to make this thing work. Because when you differentiate 'x squared plus 1' to the 8th power, you're going to get a factor of '2x' by the chain rule.

Now again, the main aim of the lectures is not to take the place of the computational drill supplied in our exercises and in the text but to give you sort of an insight. And they'll be plenty of drill on the mechanics of this in our exercises on this section. Let me just at least continue on with the concept of our recipes. For example, here's another one. And this one says what? That if you run the sum of two functions through the 'D inverse' machine, the output is the same as if you sent the functions through separately and then added them up, OK?

I'll talk about that in more detail later. Again, all I want to see is that this is the analogous result of, again, a beautiful property of the derivative. And that is that the derivative of a sum is the sum of the derivatives. And how does that work over here? Again, a very interesting property throughout advanced calculus, linear algebra and the like. These properties are very, very special. And we'll have occasion, as the course goes on, to talk about them more. For the time being, rather than have you get lost in the maze of details, let me work a specific illustration.

Let's suppose I would like to find the family of functions whose derivative is 'x to the 5th' plus 'x cubed', OK? Now, what I'm saying is this. By my previous result, I certainly know how to find the function whose derivative is 'x to the 5th', namely, "'1/6' x to the 6th'. I also know how to find the function whose derivative is 'x cubed', namely, '1/4' x to the 4th'. Putting these two steps together and say equals added to equals are equal, I can conclude that 'D of "1/6' x to the sixth" plus 'D of "1/4' x to the fourth" is 'x to the 5th' plus 'x cubed'.

Now, the key step is that since the derivative of a sum is the sum of the derivatives, I can say that the sum of these two derivatives is the derivative of the sum of the two functions. Namely, that this is what? 'D of "1/6' x to the 6th" plus "1/4' x to the 4th', a very important power and property of the derivative. Therefore, have I found one function whose derivative is 'x to the

5th' plus 'x cubed'? The answer is yes. ''1/6' x to the 6th' plus ''1/4' x to the 4th'.

Therefore, what is the family of all functions whose derivative is 'x 5th' plus 'x cubed'? And again, as before, the answer is, take your one solution that you found, tack on an arbitrary constant, and I suppose, technically speaking, I should put the braces in here to indicate that my solution is an infinite set, all belonging to one family called an equivalent set of functions because they have the same derivative.

Well, let's continue on and do a little bit more harder stuff. Remember, we talked about implicit differentiation. Well, is there an analogue to implicit 'D inverses'? You see, notice that in every problem so far, when I wrote 'D inverse', I was essentially telling you explicitly what came out of the 'D-machine'. Now, suppose I twist the emphasis a little bit. Suppose I tell you, look, I run a certain function through the 'D-machine'. In other words, I form its derivative. What the output is, is the square of the reciprocal of the function. In other words, if 'g of x' comes in, '1 over 'g of x'' squared comes out.

And now the question is, what is the function 'g of x'? And again, notice that if we don't know what the right 'g of x', is you can certainly test a given 'g of x' to see whether it's right or not. Namely, what could you do? You differentiate 'g of x', see what you get, and if what you get isn't 1 over the square of 'g of x', you've got the wrong answer. But the question that comes up is, given this type of a problem, how do we handle it? See?

In other words, where does the inverse idea come in here? The implicit relationship that 'g of x' is determined by this particular property. And again, notice how we use properties of derivatives. We're given that 'g prime of x' is a synonym, identity, for '1 over 'g squared of x'. We can cross-multiply and we get 'g squared of x' times 'g prime of x' is identically one.

Now, if you're clever about this-- and remember, notice this very, very importantly. For example, ordinary division is the inverse of ordinary multiplication. Notice that to be really cute in division, you have to be pretty cute in multiplication. Since all you're doing is changing the emphasis, notice that to handle hard problems in antiderivatives, you have to be able to handle tough derivative problems.

What I'm driving at is, you look at something like this and begin to wonder, do you know a function whose derivative is this? See? The idea is, if you're familiar with your chain rule, what is the derivative of 'g cubed of x'? To differentiate 'g cubed', what do you do? You bring the 3 down to a power 1 less times the derivative of 'g of x' with respect to 'x'. That's your chain rule.

That's 'g prime of x'.

In other words, if you're clever enough to see this, what you say here is, OK, now I multiply both sides by 3. The left hand side is just the derivative of 'g cubed of x'. The right hand side is the derivative off '3x'. Therefore, whatever g of x is, its cube has the same derivative with respect to 'x' as '3x'. And we've already learned that if two functions have identical derivatives, they differ by, at most, a constant. Consequently, 'g cubed of x' must equal '3x' plus some constant. In other words, 'g of x' is the cube root of '3x plus c'.

Now, time is running short in terms of other things that I want to teach you in today's lesson. Let me leave this, then, just for you to check. Simply differentiate the cube root of '3x plus c'. And make sure that when you get that derivative, it does turn out to be '1 over 'g of x' squared'. As I say, it's a straightforward demonstration. I leave the details to you. But the point that's really important is that whenever you do get an answer-- the hard part is to get the answer. Whenever you do get the answer, you can check by means of just taking a derivative.

Now, with all of this talk about 'D inverse' in mind, let me now go back to the more traditional notation, the notation that you'll find in most textbooks, the notation that, as I say, if you've had calculus before, most likely, you're more familiar with. And that is the following, that when we write 'D inverse' of 'f of x', the average textbook writes a symbol like this. It's called the 'integral' of 'f of x'. I will have later lectures to bemoan this choice of notation from a different point of view. But for the time being, all we're saying is, instead of writing 'D inverse'-- again, what's in the name? Just use this particular notation. That when you see this particular thing, perhaps read this as what? That this particular symbol is a code to tell you to find all functions whose derivative is 'f of x'.

And by the way, to summarize the results that we've obtained so far, let me just rewrite some of these basic results in terms of the more traditional notation. When you write this, this is called the 'indefinite integral' and what we're saying is the indefinite integral of ''x to the n' dx' is 'x to the 'n + 1'' over 'n + 1' plus a constant when it is not equal to minus 1. The integral of a sum is the sum of the integrals. And the integral of a constant times a function is a constant times the integral.

All this says is what? That this is a consequent of the fact that the derivative of the sum is the sum of the derivatives. This is a consequent of the fact that the derivative of a constant times a function is the constant times the derivative of the function. And again, all this is, same thing

we were talking about before only with the 'D inverse' notation replaced by the more common indefinite integral.

Now, we come to one more problem which will finish us up for the day, once we get through talking about it. And that's this. If all this means is find the functions whose derivative is 'f of x', why write this notation here? Why couldn't we have just written, for example, the so-called integral sign with a little 'x' underneath? You see, we've talked about misleading notations before. You see, in terms of our differential notation, when you see ''f of x' dx', you have every right to think of a differential.

Now, in all fairness, the chances are that this notation would not have been invented if there weren't some connection between derivatives and differentials. So let me mention this point. Going back as if we were starting the lecture all over again, could I have invented a different machine, which I'll call the 'script D- machine'? In other words, I don't want to call it the same 'D-machine' as before because now it's going to have a different set of outputs. You see, now the input will still be differentiable functions. But the output, instead of being the derivative of the function, will be the differential of the function. For example, before, we said what? If 'x squared' goes in, the output would be '2x'. With the script 'D-machine', the output would be '2x dx'.

Now, even though these machines are different because one machine gives you a differential as an output and the other gives you a derivative as an output, notice that they are equivalent. Namely, knowing the differential, we can pin down the function the same as when we knew the derivative. Now, the question that comes up is, why should we use this particular type of notation?

And to use the examples that are most prevalent in the text and also in our notes, let me give you the same problem that we've done before, but now from a different point of view. Suppose we're still given the problem 'g prime of x' equals '1 over 'g of x' squared'. We say, OK, let 'y' equal 'g of x'. When we do that, this problem now translates into what? 'dy dx' equals '1 over 'y squared''. Now, if we allow ourselves to use the differential notation, which we justified in previous lectures, this says what? Cross-multiplying ''y squared' dy' equals 'dx'. Notice, by the way, that 'y' is implicitly a differentiable function of 'x'.

So what we're saying over here is, look, here are two functions which have the same differential, therefore, if we integrate them, they differ by a constant. Well, to mimic what we

were doing before, let's just say-- let's see, this, I think, I have a little bit twisted here. If ''y squared' dy' equals '3x', '3 'y squared' dy' equals '3dx'. If the differentials are equal, then the functions differ by a constant. But 'y cubed' is the function whose differential is '3 'y squared' dy'. '3x' is the function whose differential is '3dx'. And we wind up with 'y cubed' equals '3x + c', or, 'y' equals the 'cube root of '3x + c''.

By the way, let me just pull this board down so that we can make a little bit of a comparison here. You see, notice that in using this differential notation, which allowed us to use some nice algebraic devices and didn't seem quite as difficult for us to recognize certain things, I hope you notice that there seems to be a correspondence between the steps that we had here and the steps that took place over here. In other words, having done at the so-called rigorous way, notice that differentials give us a very nice, intuitively-simpler technique for solving certain types of problems when we have implicit relationships between 'y' and 'x'.

And so, just to illustrate this in terms of one more problem, let's take one that's fairly geometric. Let's suppose we're given the following problem. 'dy dx' is 'minus x' over 'y' and we know that when 'x' equals 3, 'y' equals 4. Using the language of differentials, what we would do over here is we would cross-multiply. Whatever you want to do. we recognize that, to get '2y', you must differentiate 'y squared'. This is a step you don't really need. I don't care how you want to do this. All I'm saying is that from this step, I can get to here.

Recognizing that y squared has '2y dy' as its differential and that ''minus 'x squared'' has 'minus 2x dx' as its differential, I know that 'y squared' is equal to ''minus 'x squared'' plus a constant. I transpose. I get that 'x squared' plus 'y squared' is a constant. Knowing that when 'x' equals 3, 'y' equals 4, I can determine that the constant must be 25 when 'x' equals 3 and 'y' is 4. But since it's a constant, if it's 25 when 'x' equals 3 and 'y' equals 4, it's 25 everyplace. And again, if you want a geometric interpretation of this, the circle centered at the origin with radius equal to 5 is the only curve in the whole world whose derivative at any given point is the negative of the x-coordinate over the y-coordinate and passes through the point (3 , 4).

And if you want to see that geometrically, let me just take a second here. You see, notice that any point on the circle (x , y), notice that the tangent line to the curve, which has slope 'dy dx', is what? Perpendicular to the radius, again. The radius has slope 'y' over 'x'. And to be perpendicular-- if two lines are perpendicular, their slopes are negative reciprocals. This is this particular problem. And now, all I wanted to show you is that if you're nervous about differentials and you don't like to use them, notice that the same problem could have been

stated without differentials. Namely, we are thinking of a function which we'll call 'g of x' such that the derivative of 'g of x' is 'minus x' over 'g of x'.

In other words, if the input of the 'D-machine' is 'g', the output is 'minus x' over 'g'. And we also know that when 'x' equals 3, the output is 4. So to 'g' of 3 equals 4. And without going through the great details here, let's just notice that we could cross-multiply the same as before. We could multiply both sides by 2, the same as before. We could recognize that the left-hand side was the derivative of ''g of x' squared', that the right-hand side is the derivative of 'minus 'x squared''. Therefore, since they have the same identical derivatives, they must differ by a constant. OK.

We actually have plus or minus here, but notice the fact that when the input is 3, the output is 4 means that we're on the positive branch of the curve, et cetera. I say et cetera not because these points aren't important but because every point that comes up now has already been discussed under the heading of differential calculus. In other words, that the inverse of differentiation can be handled very, very neatly just by knowing differentiation with a switch in emphasis.

Why do we want to know this particular topic? Well, because in many cases, to look at it geometrically, in the past, we were given the curve. We wanted to find out what the slope was. In many physical applications, we are, in a sense, told what the slope is and have to figure out what the curve is. Hence the motivation. Well, at any rate, there'll be more about this in the text and in our exercises. So until next time, good bye.