PROFESSOR: Hi. Today we start the part of our course that in the old days in the traditional treatment began the engineering form of calculus. In other words, what we have done is we have defined the derivative, the instantaneous change, as being a limit of an average rate of change. We spent a great deal of time proving certain theorems about limits.

Now what we're going to do is to analytically take the limit definition of a derivative, which we have already done qualitatively, apply our limit theorems to of this, and develop certain formulas for computing derivatives much more quickly from a computational point of view than having to resort to the original definition. In other words, what we're going to do is to go from a qualitative approach to the derivative to a more quantitative approach. And for this reason, I just call this lecture the 'Derivatives of Some Simple Functions' to create sort of a mood over here.

Now, the idea is this. Let's go back to our basic definition. The derivative of 'f of x' when 'x' is equal to 'x1', 'f prime of x1', on is by definition the limit as 'delta x' approaches 0, ''f of 'x1 plus delta x'' minus 'f of x1'' over 'delta x'. In other words, the average rate of change of 'f of x' with respect to 'x' as we move from 'x1' to some new position 'x1 + delta x', taking the limit as 'delta x' approaches 0.

Now, just to illustrate this, let's take, for example, 'f of x' equals 'x cubed'. In this case, if we now compute 'f of 'x1 plus delta x'' minus 'f of x1', recall that 'f' is the function that does what? Given any input, the output is the cube of the input, so 'f of 'x1 plus delta x' will be the cube of ''x1 plus delta x' minus 'f of x1''. That's minus 'x1 cubed'. We now use the binomial theorem. In other words, notice again how mathematics works.

You introduce a new definition, but then all of the old prerequisite mathematics comes back to be used as a tool over here. We use the binomial theorem. The 'x1 cubed' term here cancels with the 'x1 cubed' term here. And what we're left with is what? ''3x1 squared' times 'delta x'' plus '3x1 'delta x' squared'' plus 'delta x' cubed'. And to emphasize the next step that we're going to make, let's factor out a 'delta x'.

Now, going back to our basic definition, we must take this expression and divide it by 'delta x'. The key idea here is that in the last step, we're going to take the limit as 'delta x' approaches 0. The limit as 'delta x' approaches 0 means, in particular, that 'delta x' is not allowed to equal 0. And as long as 'delta x' is not 0, the 'delta x' in the denominator cancels the 'delta x' in the numerator to leave '3x1 squared' plus '3x1 'delta x'' plus 'delta x squared'.

And now finally to find what 'f prime of x1' is, we simply take this limit as 'delta x' approaches 0. And notice what you're doing over here when you let 'delta x' approach 0. We're going to be using the limit theorems. Namely, the limit of a sum is the sum of the limits. The limit of a product is the product of the limits. We can then take each of these limits term by term.

Notice that the first term, not depending on 'delta x', it's limit is just '3x1 squared'. Each of the remaining terms have a factor of 'delta x' in them. As 'delta x' goes to 0 then, each of the remaining terms go to 0, and we wind up with, by our basic definition, that 'f prime of x1' is '3x1 squared'. In particular, since 'x1' could have been any real number here, what we have shown is what? That if 'f of x' is equal to 'x cubed', then 'f prime of x' is '3x squared'. And the important point here is simply to observe that we've got this result by applying the basic definition to a specifically given function.

To generalize this result, let 'f of x' be 'x' to the 'n', where 'n' is any positive whole number. And I want to emphasize that because you're going to see as this proof goes on that I really use the fact then 'n' is a positive whole number, and I'll emphasize to you where I do this. I'm going to mimic the same thing I did in the special case in our first example when we simply chose 'n' to equal 3. Namely, I compute 'f of 'x1 plus delta x'' minus 'f of x1'. Since the output is the n-th power of the input, that's just 'x1 plus delta x' to the 'n' minus 'x1' to the 'n'.

Now, as long as 'n' is a positive whole number, we can expand it by the binomial theorem. The first term will be 'x sub 1' to the n-th. That will cancel this 'x sub 1' to the n-th. The next term will be 'n 'x sub 1'' to the 'n - 1' power times 'delta x'. And now I do something that's a little bit cheating, I guess. What I do next is I observe that every remaining term in the binomial theorem expansion here has a factor of at least 'delta x squared' in it. So rather than compute all of these terms individually, recognizing that I'm going to let 'delta x' eventually approach 0, I factor out a 'delta x squared' and say, lookit, the remaining terms have the form 'delta x squared' times some finite number. I don't know what it is, but it's some finite number. I'll put that-- well, let's just take a look and see what that means.

In particular now, if I take this expression and divide through by 'delta x', we're assuming again that 'delta x' is not 0 because we're going to take the limit as 'delta x' approaches 0. And remember our big harangue about that. When we approach the limit, we are never allowed to equal it. 'Delta x' never equals 0. Consequently, I can cancel a 'delta x' from the denominator with a 'delta x' from the numerator. Canceling a 'delta x' from the numerator knocks a 'delta x' altogether out of this term and leaves me with just a single factor of 'delta x' in this term.

What I now do is, referring to my basic definition, I now take the limit as 'delta x' approaches 0. As 'delta x' approaches 0, this term, not depending on 'delta x', remains 'nx' to the 'n - 1'. And here's the key step. The limit of a product is the product of the limits. This approaches 0. This is some number. And 0 times some number is 0. And therefore, all that's left here is 'nx1' to the 'n - 1'.

Again, since this was any 'x', what we have is what? If 'f of x' is 'x to the n', where 'n' is a positive whole number. And where do I use that fact? I use that fact to use the binomial theorem. Then the derivative is 'nx' to the 'n - 1'. In other words, if you want to memorize a shortcut now that we know the answer, observe that it seems to differentiate 'x' to the 'n'. All you do is bring the exponent down and replace it by one less.

But for heaven's sakes, don't look at that as being a proof. Rather, we gave the proof, then observed what the shortcut recipe was. Don't say isn't it easier just to bring the exponent down and replace it by one less instead of going through this whole long harangue over here? No, this is how we prove that result.

In other words, what we have now shown is that to differentiate 'x' to the 'n' for any positive integer 'n', the derivative is just 'nx' to the 'n - 1'. So far, we do not know if that result is true for any other numbers. We'll talk about that a little bit more later. Well, so much for that example. Let's look at another one.

Let's suppose that we have two functions 'f' and 'g', which are differentiable, say, at some number 'x1'. That means, among other things, that 'f prime of x1' and 'g prime of x1' exist. Now define a new function 'h' to be the sum of the given two functions.

By the way, we have to be very, very careful here. In general, 'f' and 'g' could have different domains. Notice that 'x' has to belong to both the domain of 'f' and the domain of 'g' for this to make sense. Consequently, I write down over here that 'x' belongs to the domain of 'f' intersect domain 'g'. In other words, it belongs to both sets, the domain of 'f' and the domain of 'g'. Because if 'x' didn't belong to at least one of these two, this expression wouldn't even make sense. In other words, 'x' must be a permissible input to both the 'f' and 'g' machines.

Now, what my claim is, is that in this particular case, 'h prime' exists. In other words, 'h' is also differentiable, and it's obtained by adding the sum of these two derivatives. In other words, this is the result that's commonly known as the derivative of a sum is the sum of the derivatives, which, of course, sounds like it should be right. But I'll comment on that in a moment also. what. I want to show is simply this, that 'h prime of x1' is simply 'f prime of x1' plus 'g prime of x1', OK?

And my main purpose is to highlight the basic definition. That's what I want to give you the drill on. Namely, how do we compute 'h prime of x1'? We must look at ''h of 'x1 plus delta x'' minus 'h of x1'' over 'delta x', then take the limit as 'delta x' approaches 0. That's the basic definition that will be used over and over and over. At any rate, sparing you all of the details here, let's simply observe what? That 'h of 'x1 plus delta x'', since 'h' is the sum of 'f' and 'g', is 'f of 'x1 plus delta x'' plus 'g of 'x1 plus delta x''. Consequently, 'h of 'x1 plus delta x'' minus 'h of x1' is 'f of 'x1 plus delta x'' plus 'g of 'x1 plus delta x''-- that's this part-- minus ''f of x1' plus 'g of x1''.

Now, the idea is I want to divide this by 'delta x'. I also have some knowledge of the differentiability of 'f' and 'g' separately, so I would like to combine or regroup these terms to highlight that fact for me. So what I do is I group these two terms together, these two terms together, and then divide through by 'delta x'. In other words, ''h of 'x1 plus delta x'' minus 'h of x'' over 'delta x' is what? ''f of 'x1 plus delta x'' minus 'f of x1'' over 'delta x' plus ''g of 'x1 plus delta x'' minus 'g of x1'' over 'delta x'.

Now, for my last step, I simply must take the limit as 'delta x' goes to 0. Well, just look at this bracketed expression. The limit of a sum is the sum of the limits. So to take the limit of this expression, I take the limit of these two terms separately and add them. By definition, the limit of the bracketed expressions as 'delta x' goes to 0-- by definition-- is 'f prime of x1'. And as 'delta x' goes to 0, this term here becomes 'g prime of x1'.

And by the way, for the last time, let me give you this word of caution again. Do not make the mistake of saying, gee, when 'delta x' goes to 0, my numerator is 0. Therefore, the fraction must be 0. No! When 'delta x' goes to 0, sure, if you replace 'delta x' by 0, the numerator is 0, but so is the denominator. In other words, this is our old story that the derivative becomes a study of 0/0 so if you let 'delta x' equals 0. In other words, the whole point is what? That as 'delta x' approaches 0, this by definition is 'f prime of x1'. This by definition is 'g prime of x1'. And therefore, we have proven as a theorem 'h prime of x1' is 'f prime of x1' plus 'g prime of x1', that the derivative of a sum is the sum of the derivatives.

Now, the main trouble so far is that every result that I've proven rigorously for you, you have probably guessed in advance that it was going to happen. You say who needs this rigorous stuff when we could have got the same result by intuition. And this is where the rift between the way the pure mathematician often teaches calculus and the way, say, the applied man teaches calculus often comes in.

The point is that the places that you can get into trouble often aren't stressed soon enough. So before we go any further, let me show you that you must be careful, that up until now, we have been very fortunate when we say things like the limit of a product is the product of the limits, fortunate in the sense that they worked out the way the wording seemed to indicate. Let me give you an example of what I mean by saying beware of self-evident.

I claim, for example, that the derivative of a product is not the product of the derivatives necessarily. And the best way to prove that to you is not to say take my word for it. This is one of the beauties of computational mathematics. You can always show by means of an example what's going on. For example, let 'f of x' be 'x + 1', let 'g of x' be 'x - 1'. Now, the derivative of 'x + 1' is simply 1. The derivative of 'x - 1' is also 1. Namely, we differentiate this as a sum. The derivative of 'x' is 1. The derivative of a constant is 0. In other words, 'f prime of x' is 1, 'g prime of x' is 1.

On the other hand, if we first multiply 'f' and 'g', 'x + 1' times 'x - 1' is 'x squared - 1', and the derivative of that is what? Well, to differentiate 'x squared', we just saw. Bring the exponent down. Replace it by one less. That's '2x'. The derivative of minus 1 is 0. In other words, if you multiply first and then take the derivative, you get '2x'. On the other hand, if you differentiate first and then take the product, you get 2.

Now, it is not true that '2x' is a synonym for 1 for all values of 'x'. In other words, if you differentiate first and then take the product, you get a different answer than if you multiply first and then differentiate. In other words, self-evident or not, the thing happens to be false.

In fact, let's see how the theory helps us in this case. See, what I've shown you here is that this result isn't true. Oh, I should say that this result is true, that the equality isn't true. What I haven't shown you is what is true. And again, to emphasize our basic definition, let's do that now. Let's show how we can use our basic definition to find the correct result.

What we do is we let 'h of x' equal 'f of x' times 'g of x'. And we're going to use the same structure as before. We are going to compute 'h of 'x1 plus delta x'', subtract 'h of x1' divide by 'delta x', and take the limit as 'delta x' approaches 0. We do that every time. All that changes is the property of the particular function that we're dealing with. In any event, if I do this, 'h of 'x1 plus delta x'' is obtained by replacing 'x' in here by 'x1 plus delta x'. And 'h of x1' is obtained by replacing 'x' by 'x1' over here, so I wind up with 'f of 'x1 plus delta x'' times 'g of 'x1 plus delta x'' minus 'f of x1' times 'g of x1'.

And now the problem is I would like somehow to be able to get terms involving 'f of 'x1 plus delta x'' minus 'f of x1', 'g of 'x1 plus delta x'' minus 'g of x1', and I use the same trick here that I used when I proved that the limit of a product is the product of the limits. Namely, I am going to add in and subtract the same term, and this will help me factor. In other words, what I'm going to do here is this. I write down this term. Now I say to myself I would like an 'f of x1' term to go with the 'f of 'x1 plus delta x''. So what I do is I write minus 'f of x1'. Then I observe that 'g of 'x1 plus delta x'' is a factor here, so I put that in over here. See, 'g of 'x1 plus delta x''.

See, in other words, I can now factor out a 'g of 'x1 plus delta x'' from these two terms and have the formula that I want left over. But, of course, this term doesn't belong here. I just added it myself, or I should say subtracted it, so now I put it in with the opposite sign, namely, plus 'f of x1' 'g of 'x1 plus delta x''. And now put this term back in here: minus 'f of x1' times 'g of x1'.

In other words, all I've done is rewritten this, but by adding and subtracting the same term, which does what? Dividing through by 'delta x', I factor out a 'g of 'x1 plus delta x' from these two terms. That leaves me with ''f of 'x1 plus delta x'' minus 'f of x1' over 'delta x'' because I'm dividing through by 'delta x'. Now, what I do is I factor out 'f of x1' from here and divide what's left through by 'delta x'-- that gives me 'f of x1'-- times the quantity ''g of 'x1 plus delta x'' minus 'g of x1'' over 'delta x'. I get this thing over here.

Now, my final step is what? I take the limit as 'delta x' approaches 0. Well, let's do the easy part first. As 'delta x' approaches 0, this becomes 'f prime of x1'. And as 'delta x' approaches 0, this becomes 'g prime of x1'. This, of course, remains 'f of x1'. And I guess you would argue intuitively that as 'delta x' approaches 0, 'x1 plus delta x' approaches 'x1'. Therefore, 'g of 'x1 plus delta x'' approaches 'g of x1'. That would give you this result over here.

I have put a little asterisk over here because I want to make a footnote about this. You know, even though you would have allowed me to say that this approaches 'g of x1' as 'delta x' approaches 0, it is not in general true that we can be this sloppy about this. I want to mention that later, but meanwhile, I don't want to obscure the result that I'm driving at. Namely, assuming that this is a proper step, what we have shown now is that to differentiate a product, you differentiate the first factor and multiply that by the second. Then add on to that the first factor times the derivative of the second.

Now, that may not be self-evident, but what we have shown is that self-evident or not, this result follows inescapably from our basic definitions and our previous theorems. And by the way, to finish off the example that we started where 'f of x' was 'x + 1' and 'g of x' was 'x - 1', notice that if we use this recipe, 'f prime of x1' is 1. 'g of x1' is 'x1 - 1'. 'f of x1' is 'x1 + 1'. 'g prime of x1' is 1. And if we now combine all of these terms, we get twice 'x1', and that's exactly what the derivative of the product should have been.

In other words, just to summarize this thing off, notice that when we were back over here, this was the result that we were supposed to get. In other words, to differentiate a product, our basic formula tells us that it's the first times the derivative of the second plus the second times the derivative of the first. And let me just go back to this parenthetical footnote that I wanted to mention for you.

It is not always that the limit of 'g of t' as 't' approaches 'a' is 'g of a'. Remember, our whole study of limits said you cannot replace 't' by 'a' or 'a' by 't' automatically. In fact what, type of examples did we see that this got us into trouble? For example, let 'g of t' be ''t squared - 1' over 't - 1'', OK? Then the limit of 'g of t' as 't' approaches 1, well, as long as 't' is not 1, 't - 1' is not 0. We can cancel the 't - 1' from the numerator and the denominator. That leaves us with 't + 1'. As 't' approaches 1, 't + 1' approaches 2.

In other words, the limit of 'g of t' as 't' approaches 1 is 2, but 'g of 1' doesn't even exist. 'g of 1' is 0/0. In other words, it's not true here that as 't' approaches 1, 'g of t' approaches 'g of 1'. In other words, you can say that when 'x1' approaches 'x2', 'f of x1' approaches is 'f of x2'. It doesn't have to be true. Why then could I use it in my particular example?

And by the way, the situation in which the limit of 'g of t' as 't' approaches 'a' is 'g of a' comes under the heading of a subject called continuity, and that is a later lecture in this particular block. We'll talk about it in more detail then. But for the time being, let me show you why we could use this in our present case.

Again, we use a trick. We want to show that as 'x1 plus delta x' gets close to 'x1', 'g of 'x1 plus delta x'' gets close to 'g of x1'. And to do that, that's the same as showing that this difference gets close to 0 as 'delta x' gets close to 0. What we do is utilize the fact, and this is very important. We utilize the fact that 'g' is differentiable. And what we do is-- and it doesn't look like a very significant step, but it is. What we do is we take this expression and both multiply it and divide it by 'delta x'. I don't know if you can see the method to my madness here. The whole thing I'm setting up here is that when I do this, the bracketed expression now looks like the thing that yields 'g prime of x1' when you take the limit. And now that's exactly what I'm going to do. I'm going to take the limit of this expression as 'delta x' approaches 0.

The limit of a product is the product of the limits. As 'delta x' is allowed to approach 0, this bracketed expression by definition becomes 'g prime of x1'. And obviously, this is just the limit of 'delta x' as 'delta x' approaches 0, which is 0. The fact that 'g' is differentiable means that this is a finite number, and any finite number times 0 is 0. In other words, the fact that 'g' was differentiable tells us that this limit is 0, and that's the same as saying that the limit 'g of 'x1 plus delta x'' as 'delta x' approaches 0 is, in fact, 'g of x1'. As self-evident as it seems, this is not a true result if the function 'g' is not differentiable, or it may not be a true result if 'g' is not differentiable.

Well, so much for that. There is one more recipe that's very important called the quotient rule. The one we just did was called the product rule, how do you differentiate the product of two functions. There is an analogous type of recipe for differentiating the quotient of two functions. And since the proof is very much the same as what I've already done, I leave the details to you. They're in the textbook. You can go through that in more detail if you wish.

But the result is this. If 'f' and 'g' are differentiable functions, the quotient is also differentiable. And the way you get the result-- and again, notice how nonintuitive this is. Get this in terms of logic if you're trying to memorize it logically. You take the denominator times the derivative of the numerator. And you subtract off the numerator times the derivative of the denominator. And then you divide the whole thing by the square of the denominator.

See, again, notice how overwhelming this course becomes if you try to memorize every result. Rather, do what? Memorize the basic definition and derive these results. And believe me, once you use these results long enough, you'll memorize them automatically just by repeated use. Well, at any rate, let me give you an example of using this.

Remember before we could only differentiate 'x' to the n if the exponent was positive? Suppose 'f of x' is 'x' to the 'minus n' where 'n' is now a positive integer. That means 'minus n' is negative. How would we differentiate this? And the idea here is we say lookit, if this had been a positive 'n', we could handle this. But 'x' to the 'minus n' by definition is 1 over 'x' to the 'n'. We know how to differentiate 'x' to the 'n' when 'n' is a positive number.

What do I have now? I have a quotient. So I use the quotient rule. 'f prime of x' is what? It's my denominator, which is 'x' to the 'n', times the derivative of my numerator. My numerator is a constant so it's the derivative of 0, minus the numerator-- that's minus 1-- times the derivative of the denominator. But for a positive integer 'n', we know that the derivative of 'x' to the 'n' is 'nx' to the 'n - 1', all over the square of the denominator. But the square of 'x' to the 'n' is 'x' to the '2n'.

And if I now simplify this, I have what? This is 0. This is 'minus n'. This is 'x' to the 'n - 1'. This comes upstairs as a 'minus 2n'. So I have 'minus x' to the 'n - 1 - 2n'. That's the same as what? 'Minus nx' to the 'minus 'n - 1'. And by the way, this is a rather interesting result now that we've proven it.

Suppose we said to a person, let's just bring down the exponential and replace it by one less. Well, if we brought down the exponent, that's 'minus n'. And if we replaced 'minus n' by one less, that would be 'minus 'n - 1'. And lo and behold, that's precisely the result that we got.

In other words, by using the quotient rule, for example, we can now show that the rule that says bring down the exponent and replace it by one less applies for all integers, not just the positive integers. In later lectures, we will show that it applies to fractional exponents as well as to any real number exponents as we go along. But I just wanted you to see the structure here. And in fact, I think in terms of what this lesson is all about, we have given enough examples.

I think we might just as well summarize here, and let it be, and let the rest come from the reading material and the learning exercises in general. And the summary is simply this. The basic definition of a derivative is 'f prime of x1' is the limit as 'delta x' approaches 0, ''f of 'x1 plus delta x'' minus 'f of x1'' over 'delta x'. That basic definition never changes. We always mean this when we write this. But what is important is that this basic definition can be manipulated to yield, and now I use quotation marks "convenient," because I don't know how convenient something has to be before it's really convenient, but convenient what? Recipes. Meaning what? Convenient ways to find the derivative.

You see, what's going to happen in the remainder of our course as we shall soon see in the next lecture, in fact, what we're going to start doing is using derivatives the same as they always were intended to be used. But now we are going to have much sharper computational techniques for finding these derivatives more rapidly than having to resort to the basic limit definition. Well, enough about that for now. Until next time, goodbye.

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