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# Calculus Revisited Part 1

A Self-Study Course



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## Study Guide

Block I  
Sets, Functions,  
and Limits

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Center for Advanced  
Engineering Study

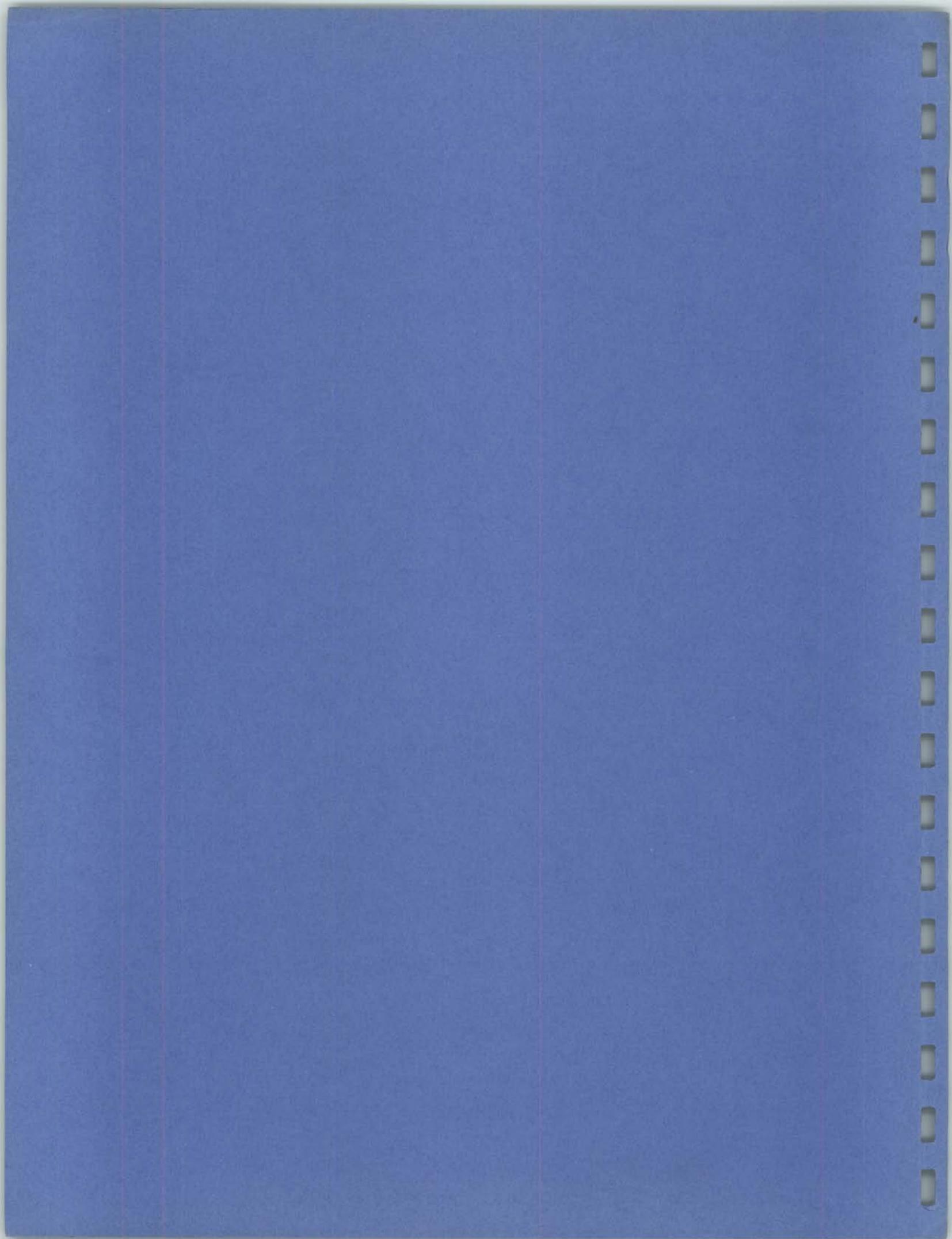
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Institute of Technology

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PART 1  
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STUDY GUIDE  
Block I  
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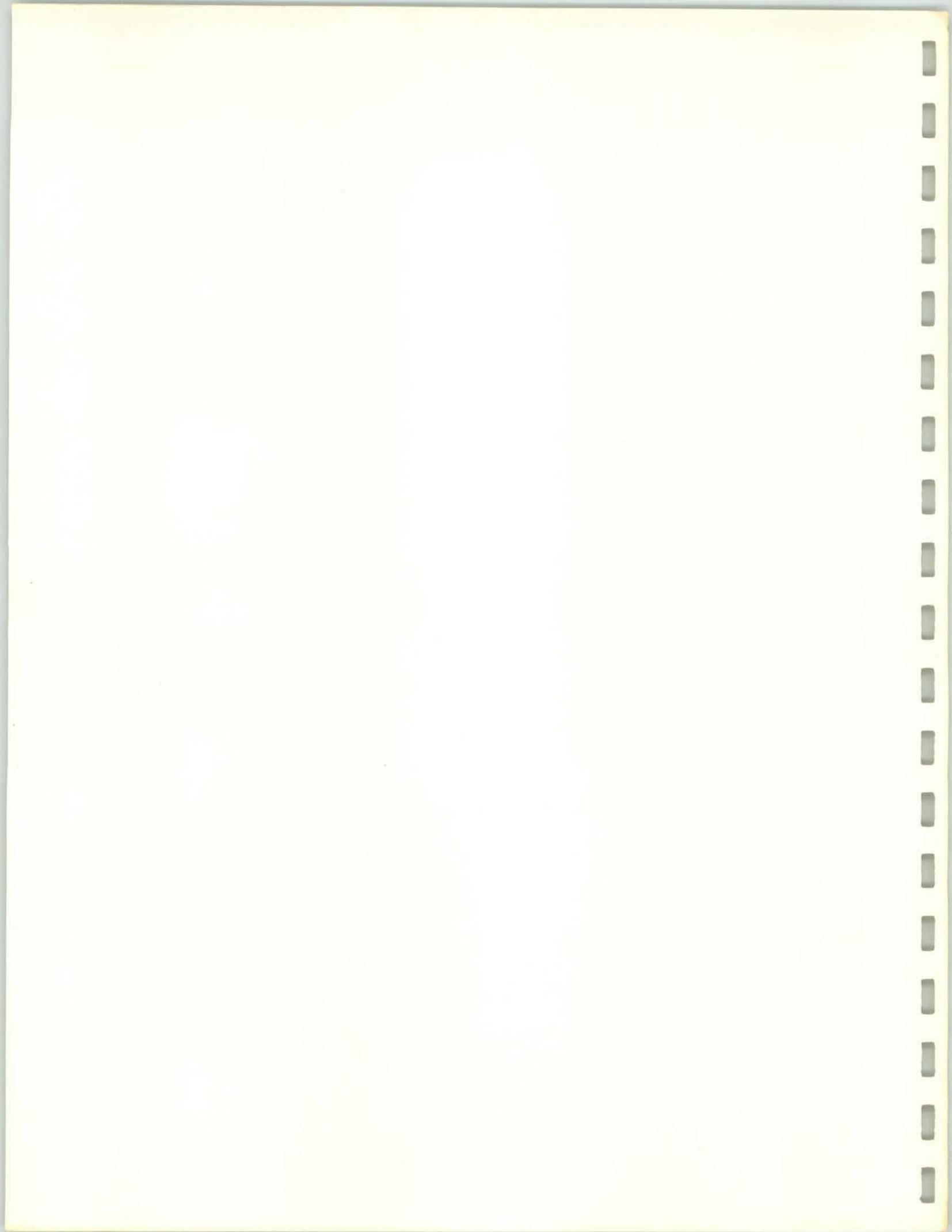
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## STUDY GUIDE: Calculus of a Single Variable

### INTRODUCTION

This self-study course consists of several elements which supplement one another. As in most courses, the central building block is the textbook. The remaining parts augment the text. First, there are the lectures. These are designed to give an overview of the material covered in the text and to supply motivation and insight in those areas where the oral word is more helpful than the written word.

Because the lectures are on film (or tape) it is assumed that you will be able to view a lecture more than once. You may use the lecture as an introductory overview and then review the unit by watching the lecture again when the rest of the assignment for that unit has been completed.

Yet, the fact remains that most students will not, for one reason or another, watch the lecture as often as might be advisable. For this reason, photographs of the blackboards, exactly as they appeared at the end of the lecture, have been made and reproduced as "Lecture Notes." Consequently, as you proceed through an assignment, there is always a rather convenient reminder and summary of the lecture. In fact, it might very well happen that once you have seen the lecture and done the assignment, the photographs of the blackboards will be sufficient to supply you with an "instant replay" whenever needed.

After the lecture, there are times when additional material is needed to bridge the gap between where the lecture ends and the text begins. Certain topics in the text are particularly difficult and as a result the student requires additional points of view or even a rehash of what's in the text. Frequently, a choice of approaches to a difficult topic is the best psychological boost to the student. Certain important topics are sometimes presented in the text in order to solve a specific problem, but it turns out that these topics have applications far beyond the

## STUDY GUIDE: Calculus of a Single Variable - Introduction

specific problem in question; consequently, the student would benefit from a more detailed explanation. Finally, there are certain topics that form a "twilight zone" for the student. Roughly speaking, these are the topics that the college professor assumes the student learned in high school and the high school teacher thinks he will learn in college. In such cases a student may need more explanation than is offered in the text. For these reasons, the course includes a volume entitled "Supplementary Notes."

While good lectures and well-written text material are vital in the learning of any course, it seems that certain ideas are best transmitted through their application in various problems. For this reason, we have designated certain problems as LEARNING EXERCISES labelled with an (L). These exercises serve as a springboard for emphasizing important ideas and, accordingly, their solutions are presented in great detail.

Since these exercises lead to points that may not be stressed elsewhere in the package, it is important that you do each learning exercise and that you study the solution, EVEN IF YOU CAN DO THE PROBLEM.

If you have done a learning exercise correctly and have read its solution, you may, if you wish, omit all exercises prior to the next learning exercise. Notice that after each learning exercise there is at least one other exercise which is somewhat similar to the learning exercise. These exercises are supplied for you to get extra experience in the event you had difficulty with a learning exercise but feel that you would like another "chance" after having read the solution. The solutions to these additional exercises are more concise than the solutions to the learning exercises.

Finally, for those who desire even more experience, the text-book offers a very large assortment of exercises to practice with,

## STUDY GUIDE: Calculus of a Single Variable - Introduction

and the answers to almost all of the exercises are given in the answer section at the end of the textbook.

Since this is a refresher course, there is always the chance that you are particularly well versed in certain aspects of the course. If you prefer, you may omit the corresponding blocks in this package. To help you make this decision, we have included a block pretest before each block. If you can do ALL of the problems in the pretest, you may omit the block. Otherwise, it is much wiser to study the block.

You will notice that the solutions to the block pretest problems are rather sketchy. Do not be alarmed or frustrated. We have chosen each problem to be important enough so that each is presented somewhere in the block as what we call a LEARNING EXERCISE. Each learning exercise will be solved in a very detailed manner (with the solution appearing separately from the problem) in order that you may learn as much as possible from the problem.

We sketch the solutions only to the extent we think is necessary for you to be able to decide whether you can do the problem, i.e., to help you decide whether any error you made was merely a careless computational mistake or a more serious conceptual error.

As important as the pretest is the post-test or what is more colloquially known as the quiz. Somehow or other, there is no substitute for a comprehensive test to see what the student has retained. For this reason there is a "final examination" at the end of each block. The correct answers together with rather detailed solutions are supplied so that the student can better analyze his difficulties.

In summary, then, our typical format for a lesson unit is:

1. See a lecture.
2. Read some supplementary notes.
3. Read a portion of the text.
4. Do the exercises.

## STUDY GUIDE: Calculus of a Single Variable - Introduction

When assigned, these four steps almost always occur in the given order, but there are some assignments in which (1) and/or (2) are omitted, and there are a few places where the supplementary notes form the only reading assignment, especially in the treatment of topics not covered in the text.

Finally, I would like to acknowledge the very able assistance I have received from several people in the preparation of this self-study course. First and foremost, I am deeply indebted to John T. Fitch, the manager of our self-study development program. He discussed and helped me plan the lectures and written material -- unit by unit. He made suggestions, offered improvements, and, in many cases, put himself in the role of the student to help me "tone down" certain topics to the extent that they became (hopefully) more understandable to the student. In addition to all this, he was a friend and colleague, and this went a long way towards making a very difficult undertaking more palatable for me.

I would also like to thank Harold S. Mickley, the first director of CAES, whose idea it was to make "Calculus Revisited" available as a self-study course. Most of this present course reflects his ideas as to what constitutes a meaningful continuing education, calculus course. He, too, during his stay at the Center was a constant source of inspiration to me, and, in a certain sense, this course belongs more to him than to anyone else.

If you think that having to read all this material is difficult, imagine what it would have been like to have had to type the entire manuscript. Yet this job was accomplished, in an efficient and good-natured manner, by our able staff of secretaries - in addition to maintaining all the other responsibilities of their office. In particular, I am grateful to Miss Elise Pelletier who worked on the manuscript from its very inception and to Mrs. Richard Borken for their help in the preparation of the manuscript.

STUDY GUIDE: Calculus of a Single Variable - Introduction

I hope that your studying this course will be as rewarding and enjoyable as preparing the course has been to me. Good luck.

Cambridge, Massachusetts  
May 1970

Herbert I. Gross

STUDY GUIDE: Calculus of a Single Variable

TABLE OF CONTENTS

Introduction

Block I - Sets, Functions, and Limits

Content

Pretest

Unit 1. Sets

Unit 2. Arithmetic of Sets

Unit 3. Analytic Geometry

Unit 4. Functions

Unit 5. The Derivative as a Limit

Unit 6. A More Rigorous Approach to Limits I

Unit 7. A More Rigorous Approach to Limits II

Unit 8. Mathematical Induction

Quiz

Solutions

Pretest

Unit 1. Sets

Unit 2. Arithmetic of Sets

Unit 3. Analytic Geometry

Unit 4. Functions

Unit 5. The Derivative as a Limit

Unit 6. A More Rigorous Approach to Limits I

Unit 7. A More Rigorous Approach to Limits II

Unit 8. Mathematical Induction

Quiz

STUDY GUIDE: Calculus of a Single Variable

BLOCK I: SETS, FUNCTIONS, AND LIMITS

Content

- Unit 1. Sets
- Unit 2. Arithmetic of Sets
- Unit 3. Analytic Geometry
- Unit 4. Functions
- Unit 5. The Derivative as a Limit
- Unit 6. A More Rigorous Approach to Limits I
- Unit 7. A More Rigorous Approach to Limits II
- Unit 8. Mathematical Induction

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits

PRETEST

1. Given the equation  $x^2 - 4x + 3 = 0$ ,
  - (a) Write the solution set using set-builder notation
  - (b) Write the solution set in roster notation
  
2. (a) Show that  $(A \cap B) \cup C$  and  $A \cap (B \cup C)$  need not be the same set.
  - (b) What is the most general case in which  $(A \cap B) \cup C$  and  $A \cap (B \cup C)$  are the same?
  - (c) What is wrong with the expression  $A \cap B \cup C$ ?
  
3. Let  $A = \{1, 2, 3\}$ 
  - (a) How many elements belong to the set  $\{f \mid f: A \rightarrow A\}$ ?
  - (b) How many of these elements are 1-1?
  - (c) If  $f(1) = 2$ ,  $f(2) = 3$ , and  $f(3) = 1$ , describe  $f^{-1}: A \rightarrow A$ .
  
4. Define  $f$  by

$$f(x) = \begin{cases} 2 - x, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Describe

- (a)  $f(-x)$
- (b)  $-f(x)$
- (c)  $f(x+3)$
- (d)  $f(2x + 3)$

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Pretest

5. Let  $f$  and  $g$  be defined by:

$$f(x) = \frac{x^2 - 9}{x - 3}$$

$$g(x) = x + 3$$

(a) How do  $f$  and  $g$  differ?

(b) Does  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} (x + 3)$ ?



STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions and Limits.

UNIT 1: Sets

1. View: Lecture 0.000
2. Read: Supplementary Notes, Chapter I, "An Introduction  
the Theory of Sets."
3. Exercises:
  - 1.1.1 (L) Let A be the set whose elements are a, b, c, and d.
    - (a) Write A in roster form.
    - (b) How many subsets does A have?
    - (c) How many of these subsets have exactly two elements?
    - (d) Use the roster method to list each of the subsets of  
A which have exactly two elements.
    - (e) If a coin is tossed four times (and we assume that  
the coin is not "loaded") is there a fifty-fifty  
chance that we will get two heads and two tails?  
How is this problem related to parts a and c above?
  - 1.1.2 Let A be a set which has six members.
    - (a) How many subsets does A have?
    - (b) How many of these subsets have exactly three elements?
    - (c) How many of these subsets have exactly two elements?
    - (d) How many of these subsets have exactly four elements?
    - (e) How are c and d related?
    - (f) If a "fair" coin is tossed six times what is the  
probability that we will obtain three heads and three  
tails?
  - 1.1.3 (L) Consider the equation
$$x^4 = 1$$
    - (a) Write the solution set for this equation in set-builder  
notation.

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions and Limits - Unit I: Sets

[1.1.3 (L) cont'd]

- (b) Write the solution set in roster form if:
  - (i) The universe of discourse is the set of complex numbers.
  - (ii) The universe of discourse is the set of real numbers.
  - (iii) The universe of discourse is the set of positive real numbers.
  - (iv) The universe of discourse is the set of even integers.

1.1.4 Consider the equation  $(2x - 3)(x + 1)(x + 7) = 0$ ,  
and let  $I$  denote the universe of discourse.

- (a) Write the solution set of the given equation in set-builder notation.
- (b) Write the solution set in roster form, if:
  - (i)  $I$  is the set of rational numbers.
  - (ii)  $I$  is the set of real numbers.
  - (iii)  $I$  is the set of integers.
  - (iv)  $I$  is the set of positive integers.

UNIT 2: Arithmetic of Sets

1. Read: Supplementary Notes, Chapter II, "The Arithmetic of Sets"

2. Exercises:

1.2.1 (L)

- (a) Show that  $(A \cap B) \cup C$  and  $A \cap (B \cup C)$  need not be equal sets.
- (b) What is the most general case in which  $(A \cap B) \cup C$  and  $A \cap (B \cup C)$  will be equal?
- (c) Why is it ambiguous to write  $A \cap B \cup C$ ?
- (d) Is it ambiguous to write  $A \cap B \cap C$ ?

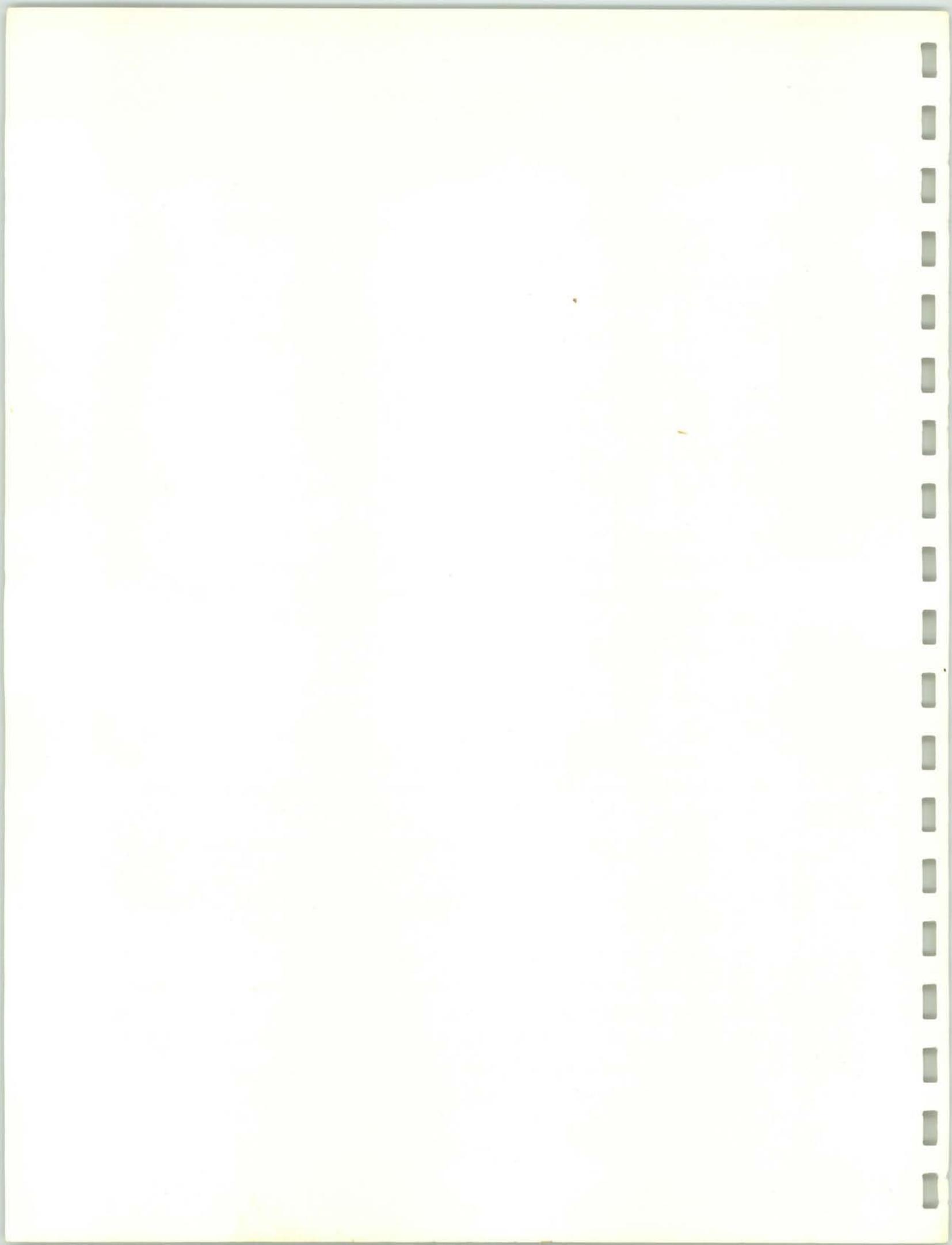
1.2.2

- (a) Show that  $(A \cup B) \cap C$  and  $A \cup (B \cap C)$  need not be equal. Under what conditions will they be equal?
- (b) Show that  $(A \cup B) \cap C$  and  $(A \cap C) \cup (B \cap C)$  are equal sets.

1.2.3 (L)

- (a) Show that  $(A \cup B)' = A' \cap B'$ . (This result is often referred to as DeMorgan's Rule.)
- (b) Show that it is possible that  $X \cup B = X \cup C$  but that  $B \neq C$ .
- (c) If  $X \cup B = X \cup C$  and if  $X \cap B = X \cap C$  show that  $B = C$ .

1.2.4 A school offers three foreign languages which we shall call A, B, and C. Each of the 280 students takes at least one of these three languages. The following data is available: (1) 20 students take all three languages, (2) 30 students take both A and B, (3) 50 students take both A and C, (4) 60 students take both B and C, (5) 120 students take A, and (6) 160 students take C. How many students take only B as their language?



UNIT 3: Analytic Geometry

1. View: Lecture 1.010
2. Read: Thomas 1.1 through 1.5
3. Exercises:
  - 1.3.1 (L) The x- and y- intercepts of a line L are respectively a and b. Show that AN equation of L is
$$(x/a) + (y/b) = 1, \text{ if } ab \neq 0$$
  - 1.3.2 The perpendicular distance ON from the origin to line L is p, and ON makes an angle  $\theta$  with the positive x-axis. Show that AN equation of L is given by
$$x\cos\theta + y\sin\theta = p$$
  - 1.3.3 (L) Find the coordinates of the point P(x,y) which is so located that the line  $L_1$ , which passes through P and origin, has slope equal to 2, and the line  $L_2$ , which passes through the point P and the point A(-1,0), has slope equal to 1.
  - 1.3.4 (L)
    - a. Find the equation of the line L through A(-2,2) and perpendicular to the line L':  $2x + y = 4$ .
    - b. Find the point B at which the lines L and L' of a intersect.
    - c. Find the perpendicular distance from the point A(-2,2) to the line L' whose equation is  $2x + y = 4$ .

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

- 1.3.5 Find the perpendicular distance from the point  $B(4,1)$  to the line  $3x - y = 5$ .
- 1.3.6 (L) In terms of the constants  $m$ ,  $b'$ , and  $b''$ , find the perpendicular distance between the parallel lines whose equations are  $y = mx + b'$  and  $y = mx + b''$ .
- 1.3.7 (L) If  $a$  and  $b$  are any two real numbers we say that  $a$  is less than  $b$  and write  $a < b$  if (and only if)  $b - a$  is positive (that is,  $b = a + h$  where  $h$  is positive). If  $a < b$  we also say that  $b$  is greater than  $a$  and write  $b > a$ . Prove that:
- (a) If  $a < b$  then  $a + c < b + c$  and  $a - c < b - c$ .
  - (b) If  $a < b$  and  $c < d$  then  $a + c < b + d$ . Is it also true in this case that  $a - c < b - d$ ? Explain.
- 1.3.8 Prove the following properties of inequalities:
- (a) If  $a < b$  and  $c$  is positive then  $ac < bc$ . What happens if we remove the restriction that  $c$  be positive?
  - (b) If  $a < b$  and  $a$  and  $b$  are either both positive or else both negative (another way of writing this is to say that  $ab > 0$ ) then  $1/b < 1/a$ .

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits

UNIT 4: Functions

1. View: Lectures 1.020 and 1.025
2. Read: Supplementary Notes, Chapter III, "An Introduction to Functions"
3. Read: Thomas 1.6 and 1.7
4. Exercises:
  - 1.4.1 (L) Let  $A = \{1, 2, 3\}$ 
    - (a) How many elements belong to the set  $\{f: A \rightarrow A\}$ ?
    - (b) How many of these elements are 1-1?
    - (c) If  $f(1) = 2$ ,  $f(2) = 3$  and  $f(3) = 1$ , describe  $f^{-1}: A \rightarrow A$ .

- 1.4.2 (L) Define  $f$  on the set of real numbers by:

$$f(x) = \begin{cases} 2 - x, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Graph  $y = f(x)$
- (b) Describe each of the following functions and sketch the graph of each:

- |                 |             |                |
|-----------------|-------------|----------------|
| (1) $f(-x)$     | (2) $-f(x)$ | (3) $f(x + 3)$ |
| (4) $f(x) + 3$  | (5) $f(2x)$ | (6) $2f(x)$    |
| (7) $f(2x + 3)$ |             |                |

- 1.4.3 (L) Find the minimum and maximum values of  $f(x, y)$  if  $f(x, y) = 3x + 4y$ , and the domain  $S$  of  $f$  is defined by the simultaneous inequalities

$$x + y \geq 2, \quad y \leq 3x + 2, \quad 5x \leq 10 - y$$

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

1.4.4 (L)

- (a) Let  $C$  and  $F$  denote, respectively, corresponding centigrade and Fahrenheit temperature readings. Given that the  $F$  versus  $C$  graph is a straight line, find its equation from the data that  $C = 0$  when  $F = 32$  and  $C = 100$  when  $F = 212$  (that is,  $(0, 32)$  and  $(100, 212)$  are points on the graph).
- (b) Is there a temperature at which  $C = F$ ? If so, what is this temperature?

1.4.5 (L)

- (a) Prove that  $|a| < |b|$  if and only if  $a^2 < b^2$ .
- (b) Prove that  $|a + b| \leq |a| + |b|$ .
- (c) Prove that  $|a - b| \geq \left| |a| - |b| \right|$ .

- 1.4.6 (L) Describe the domain of the variable  $x$  without the use of the "absolute value" symbol if it is known that  $2 < |x - 3| \leq 4$ .

- 1.4.7 (L) Solve the following equation for  $x$  in terms of  $y$ . On the basis of the assumption that  $x$  and  $y$  are real variables, discuss the possible sets of values of  $x$  and of  $y$ , the domain and the range of the functions being defined by the formula

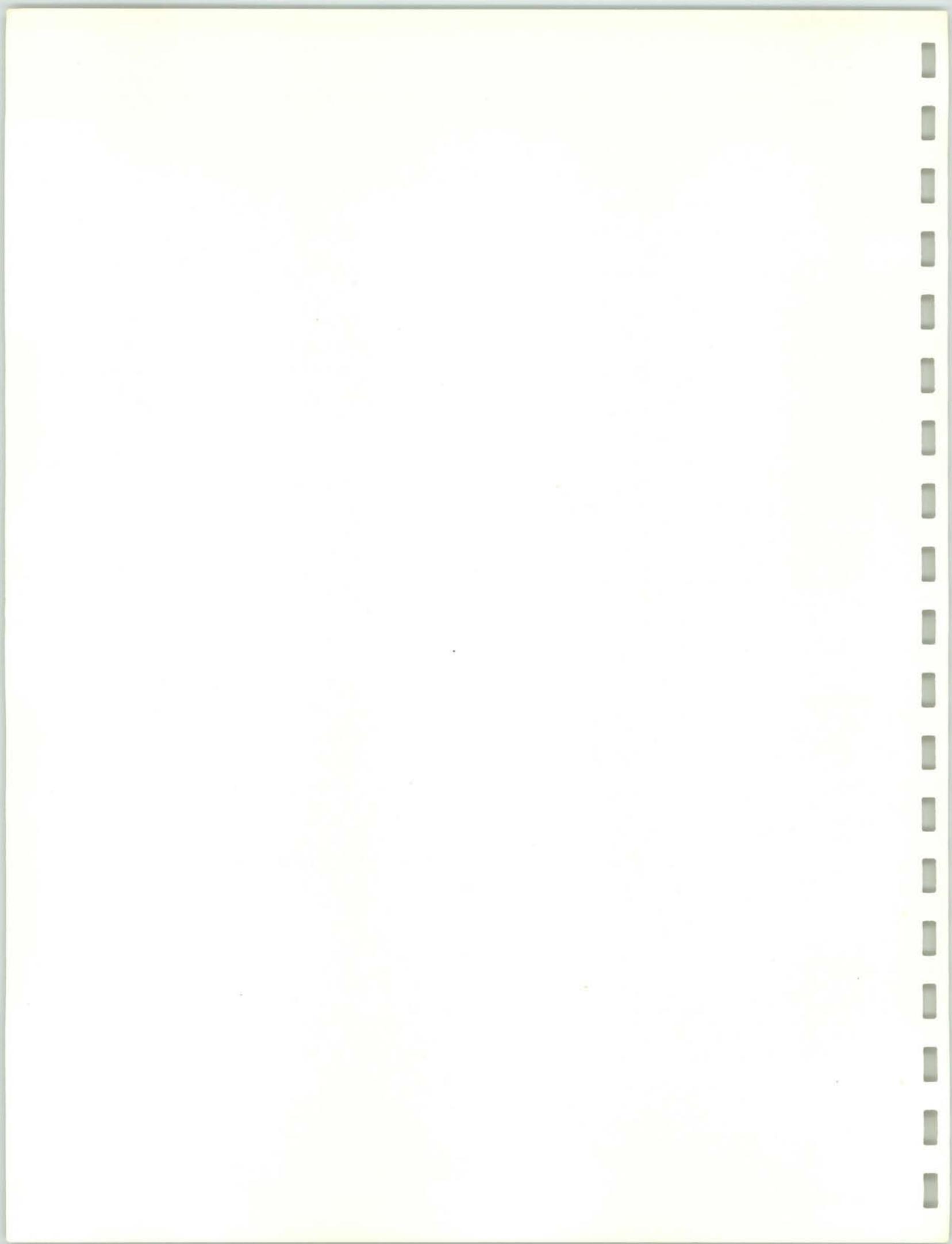
$$y = \sqrt{\frac{x}{x+1}} \quad (1)$$

which here gives  $y$  in terms of  $x$ .

- 1.4.8 (L) Separate the equation  $x^2 + xy + y^2 = 3$  into two equations, each of which determines  $y$  as a (single-valued) function of  $x$ .

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

1.4.9(L) Given  $f(x) = 2x - 7$ , show that  $f^{-1}(x)$  is not  
the same as  $\frac{1}{f(x)}$ .



STUDY GUIDE: Calculus of a Single Variable - Block I: Sets, Functions, and Limits

UNIT 5: The Derivative as a Limit

1. View: Lecture 1.030
2. Read: Supplementary Notes, Chapter IV, Sections A, B, C, and D.
3. Read: Thomas 1.8 through 1.11
4. Exercises: (Note: All derivatives should be computed as limits. Do not use other formulas for differentiation, since, in particular, they haven't been developed yet in this course.)

1.5.1 (L) Find  $f'(x)$  if  $f(x) = x^3$ .

1.5.2 (L) Find  $f'(x)$  if  $f(x) = \sqrt{3x}$ .

1.5.3 Find  $f'(x)$  if  $f(x) = \frac{1}{\sqrt{2x}}$

1.5.4 (L) Find the slope of the curve  $y = x^3$  at any point  $(x_1, y_1)$  on the curve.

1.5.5 Find the slope of the curve  $y = \sqrt{3x}$  at  $(12, 6)$ .

1.5.6 (L) At what point does the line which is tangent to the curve  $y = x^2 - 2$  at  $(2, 2)$  intersect the x-axis?

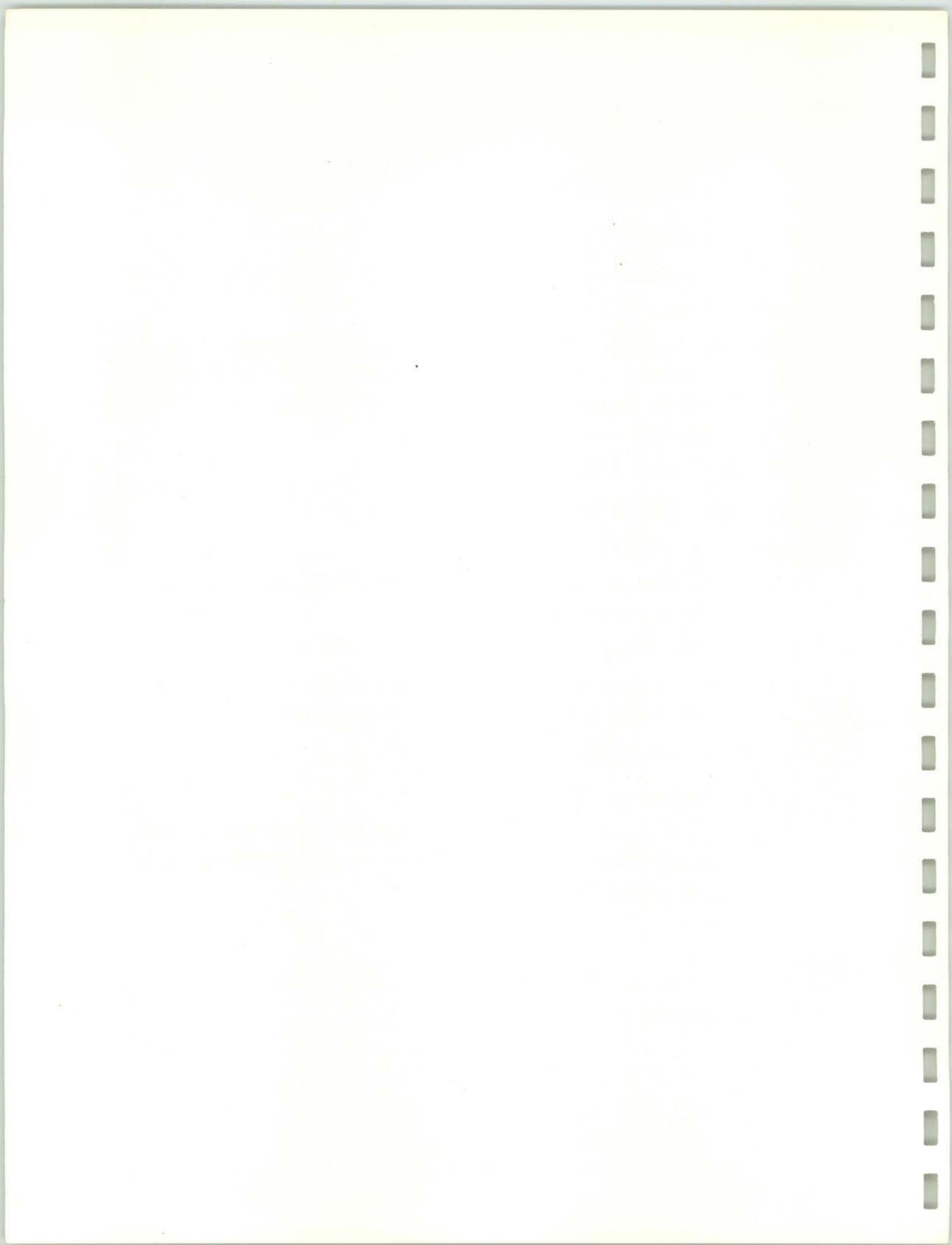
1.5.7 (L) Determine  $\frac{dA}{dr}$  if  $A = \pi r^2$ .

1.5.8 Determine  $\frac{dV}{dr}$  if  $V = \frac{4}{3} \pi r^3$ .

1.5.9 (L) A particle is projected vertically upward in such a way that after  $t$  seconds it attains a height of  $h$  feet, where  $h$  is given by

$$h = 128t - 16t^2.$$

Find the maximum height which is attained by the particle.



STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits

UNIT 6: A More Rigorous Approach to Limits I

1. View: Lecture 1.040

2. Read: Supplementary Notes, Chapter IV, Section E

3. Read: Thomas 2.1 and 2.2

4. Exercises:

1.6.1 (L) Suppose  $f$  is defined on  $[a,b]$  and that  $\lim_{x \rightarrow c} f(x) = L$  for some  $c$  in  $(a,b)$ . Choose any two numbers  $m$  and  $M$ , subject only to the condition that  $m < L < M$ . Show that there exists a number  $\delta$  such that  $m < f(x) < M$  provided that  $0 < |x - c| < \delta$ .

1.6.2 (L) Suppose  $\lim_{x \rightarrow c} f(x) = 0$  and that  $g(x)$  is bounded (that is, there exists a number  $K > 0$  such that  $|g(x)| < K$ ). Define  $h$  by  $h(x) = f(x)g(x)$ . Prove that  $\lim_{x \rightarrow c} h(x) = 0$ .

1.6.3 (L) Given  $\epsilon > 0$ , determine  $\delta$  (in terms of  $\epsilon$ ) such that  $|t^2 + t - 12| < \epsilon$  if  $0 < |t - 3| < \delta$ .

1.6.4 Given  $\epsilon > 0$ , find  $\delta$  so that  $|x^2 - 5x - 6| < \epsilon$  if  $0 < |x - 6| < \delta$ .

1.6.5 (L) Define  $f$  by  $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$   
Discuss  $\lim_{x \rightarrow 0} f(x)$ .



STUDY GUIDE: Calculus of a Single Variable - Block I: Sets, Functions, and Limits

UNIT 7: A More Rigorous Approach to Limits II

1. Read: Thomas 2.3 and 2.4. Omit all references to trigonometric functions, in particular, Example 3 on p. 52, and skip section 2.5 entirely. We shall return to this material in a different context later in the course.

2. Exercises:

1.7.1 (L)

(a) Suppose that  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} g(x) = L_2$ . Define  $h$  by  $h(x) = f(x)g(x)$ . Prove that  $\lim_{x \rightarrow c} h(x) = L_1 L_2$ .

(b) If  $\lim_{x \rightarrow c} f(x) = L$  show that  $\lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{L}$ ,  $L \geq 0$ , provided that  $\lim_{x \rightarrow c} f(x)$  exists. Generalize this result to cover the case  $\lim_{x \rightarrow c} \sqrt[n]{f(x)}$  where  $n$  is any positive integer.

1.7.2 (L) Prove that:

(a)  $\lim_{t \rightarrow 3} (t^2 + t) = 12$

(b)  $\lim_{x \rightarrow 6} (x^2 - 5x) = 6$

1.7.3 Given that  $\lim_{x \rightarrow 1} x = 1$ , compute  $\lim_{x \rightarrow 1} [(x + 1)^5 (x^2 + x + 2)]$

1.7.4 (L) Suppose that  $\lim_{x \rightarrow c} f(x) = L$  and that  $L \neq 0$ . Define  $g$  by  $g(x) = \frac{1}{f(x)}$ . Prove that  $\lim_{x \rightarrow c} g(x) = \frac{1}{L}$ .

1.7.5 (L) Show that  $\lim_{x \rightarrow \infty} f(x)$  has the same meaning as  $\lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right)$ .

1.7.6 (L) Compute  $\lim_{x \rightarrow \infty} \left[ \frac{3x^2 - 7x + 1}{4x^2 + 5x - 7} \right]$ .

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

1.7.7 (L) Compute  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$

1.7.8 (L) Compute  $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$ .

1.7.9 (L) Suppose that  $\lim_{x \rightarrow c} f(x)$  and  $\lim_{x \rightarrow c} g(x)$  exist and that  $f(x) < g(x)$  for all  $x$ . Prove that  $\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$ .

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits

UNIT 8: Mathematical Induction

1. View: Lecture 1.060
2. Read: Supplementary Notes, Chapter V, "Mathematical Induction."
3. Exercises:

1.8.1 Prove by induction that  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$  .

1.8.2 Prove by induction that  $1^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$  .

1.8.3 Prove by induction that  $1 + 3 + \dots + (2n-1) = n^2$  .

1.8.4 Prove by induction that:

$$|a_1 + \dots + a_n| \leq |a_1| + \dots + |a_n| .$$



STUDY GUIDE: Calculus of a Single Variable - Block I: Sets, Functions, and Limits

QUIZ

1. (a) Use circle diagrams to prove that  $(A \cup B)' = A' \cap B'$ .  
(b) Apply mathematical induction to part (a) to deduce that  $(A_1 \cup A_2 \cup \dots \cup A_n)' = A_1' \cap A_2' \cap \dots \cap A_n'$ .  
(c) Use circle diagrams to show that

$$[A \cap (B \cup C)]' = A' \cup (B' \cap C')$$

2. (a) Let  $A = \{1, 2, 3, 4\}$ . Define  $f: A \rightarrow A$  by  $f(1) = 2, f(2) = 4, f(3) = 1, \text{ and } f(4) = 3$ . Describe  $f^{-1}$ .  
(b) Let  $R$  denote the set of real numbers and define  $f: R \rightarrow R$  by

$$f(x) = \frac{3x - 4}{5}$$

Describe  $f^{-1}$ .

3. The curve  $C$  is described by the equation  $y = x^3 - 2x^2$ . Let  $P(x_1, y_1)$  and  $Q(x_1 + \Delta x, y_1 + \Delta y)$  denote any two points on  $C$ .  
(a) Find the slope of  $PQ$  in terms of  $x_1$  and  $\Delta x$ .  
(b) Find the equation of the line tangent to  $C$  at  $P(x_1, y_1)$ .  
(c) A line is drawn tangent to  $C$  at the point  $(3, 9)$ . At what point does this line intersect the  $x$ -axis?

4. Let  $f$  be defined by  $f(x) = 2x - 3$ , where  $0 \leq x \leq 4$ . Sketch each of the following curves:

- (a)  $y = f(x)$       (b)  $y = f(x + 5)$       (c)  $y = f(x) + 5$   
(d)  $y = f(4x)$       (e)  $y = 4f(x)$       (f)  $y = |f(x)|$

STUDY GUIDE: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Quiz

5. We know that  $\lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x-1}} = \infty$ .

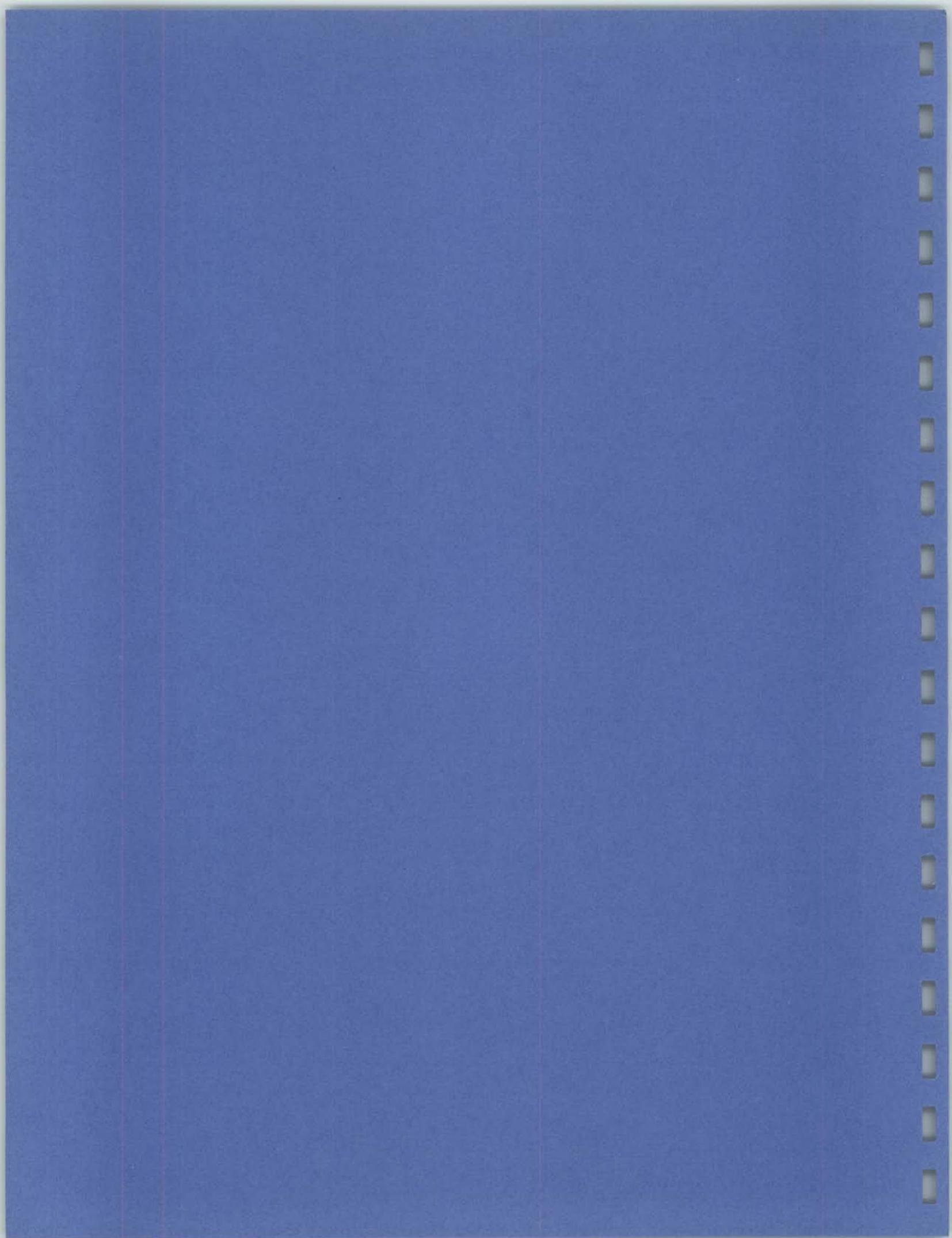
(a) For a given  $M > 0$  determine how close  $x$  must be to 1 if

$$\frac{1}{\sqrt{x-1}} > M.$$

(b) In particular, describe the size of  $x$  if  $\frac{1}{\sqrt{x-1}} > 1000$ .

Calculus of a Single Variable

SOLUTIONS



SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits

PRETEST

1. (a)  $\{x: x^2 - 4x + 3 = 0\}$  (b)  $\{1,3\}$
2. (a) and (b) They are unequal unless all C's are A's  
(c) It is ambiguous since  $(A \cap B) \cup C$  need not equal  
 $A \cap (B \cup C)$
3. (a) 27 (b) 6 (c)  $f^{-1}(1) = 3, f^{-1}(2) = 1, f^{-1}(3) = 2$
4. (a)  $f(-x) = \begin{cases} 2+x, & \text{if } -2 \leq x \leq 0 \\ 0, & \text{otherwise} \end{cases}$  (b)  $-f(x) = \begin{cases} x-2, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$   
(c)  $f(x+3) = \begin{cases} -x-1, & \text{if } -3 \leq x \leq -1 \\ 0, & \text{otherwise} \end{cases}$   
(d)  $f(2x+3) = \begin{cases} -2x-1, & \text{if } -\frac{3}{2} \leq x \leq -\frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$
5. (a)  $g(3) = 6$  but  $f(3)$  is undefined; otherwise  $f(x) = g(x)$   
(b) Yes



UNIT 1: Sets

1.1.1(L)

(a) Recalling the convention that when we list a set we enclose it in braces, we have

$$A = \{a, b, c, d\}$$

(b) A has 16 subsets. To see this, let us observe that each element of A either belongs to a chosen subset or it doesn't; moreover, the choice of whether one element belongs to the subset in no way affects the choice of whether a second element belongs to the subset. Thus, there are  $2 \times 2 \times 2 \times 2 = 2^4 = 16$  subsets\* of A.

Another way of seeing this result might be to think in terms of a code wherein 0 means that the element does not belong to the subset and 1 means that it does. So, as far as whether a belongs to a particular subset we have the two choices

$$\begin{array}{l} \underline{a} \\ 0 \text{ (a doesn't belong)} \\ 1 \text{ (a does belong)} \end{array} \qquad (1)$$

Moreover either of the above two choices holds, whether or not b belongs to the subset. That is

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\*If how we arrived at  $2 \times 2 \times 2 \times 2$  is not clear, continue for a more complete explanation.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 1: Sets

[1.1.1(L) cont'd]

<u>b</u>	<u>a</u>	
0	0	(neither a nor b belong)
0	1	(a belongs but b doesn't)
1	0	(b belongs but a doesn't)
1	1	(both a and b belong)

(2)

Again, either of these four choices may occur, whether or not c belongs to the given subset. Thus

<u>c</u>	<u>b</u>	<u>a</u>	
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

(3)

Finally, we observe that each of the 8 possibilities that exist in (3) exists regardless of whether d belongs to the given subset. This leads to

<u>d</u>	<u>c</u>	<u>b</u>	<u>a</u>	
0	0	0	0	→ This subset corresponds to $\emptyset$
0	0	0	1	
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	

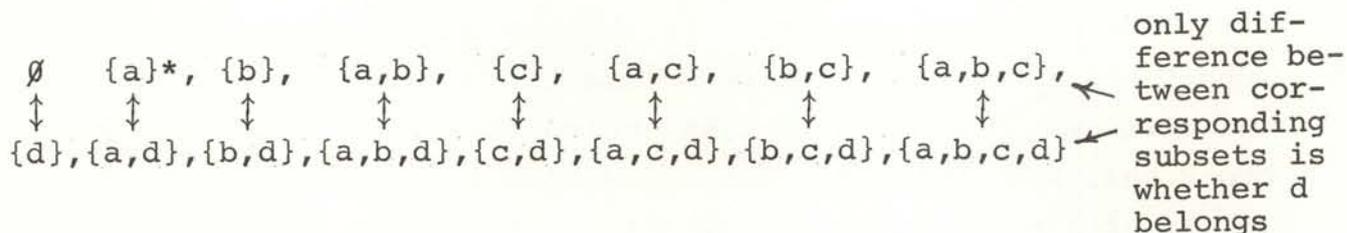
(4)

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 1: Sets

[1.1.1 (L) cont'd]

0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	(4)
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	→ This subset corresponds to A, itself

Moreover, if we now read the rows of (4) from top-to-bottom and use our code we see that our sixteen subsets are:



Notice also that there are other ways of reconstructing the results shown in (4). For one thing, if we read each row of (4) as a binary number, then our rows name the sixteen numbers, 0 through 15 (in our usual decimal notation). For another thing, we could use the following "branch" diagram:

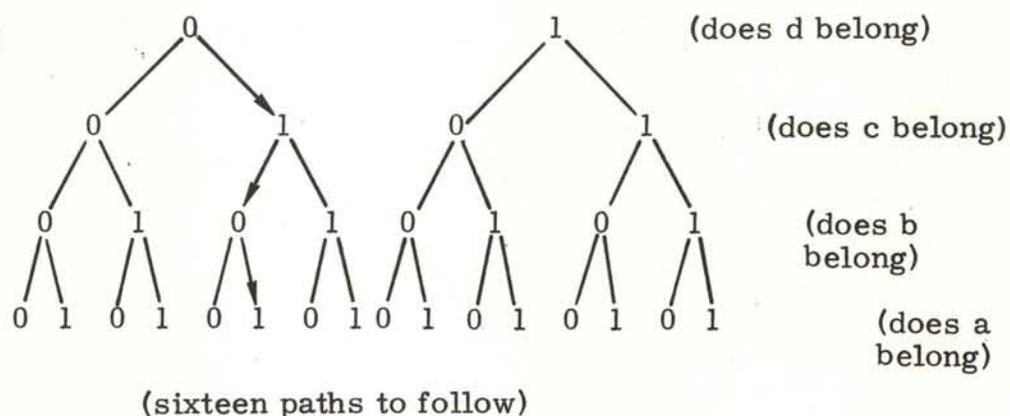
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\*Again, notice that we write {a} rather than a. Aside from our agreement upon symbolism, there is a conceptual difference as well. For example, if  $S$  denotes the solution set of  $x - 1 = 0$ , there is a difference between saying  $1 \in S$  and  $S = \{1\}$ . Among other things,  $1 \in S$  does not imply that  $S = \{1\}$ . That is,  $1 \in \{x: x^2 - 4x + 3 = 0\}$  but  $\{x: x^2 - 4x + 3 = 0\} = \{1, 3\}$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 1: Sets

[1.1.1 (L) cont'd]

(path shown by  
 arrows is read as:  
 d doesn't belong  
 c does belong  
 b doesn't belong  
 a does belong)



Notice also that (1) implies that there are two subsets of a set which has one element; (2) implies that there are four subsets of a set which has two elements; and (3) implies that there are eight subsets of a set which has three elements. Our other analysis indicates that every time we add a member to a set we double the number of subsets (since every subset of the original set is a subset of the augmented set whether we add the additional element to it, or not. In other words, then, we would expect that if a set has  $n$  elements it has  $2^n$  subsets.

(c) To answer this question we can return to our chart (4) and count the rows which have exactly two 1's. In this way, we find that there are six such rows, and the six subsets are listed explicitly as

$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\},$  and  $\{c,d\}.$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 1: Sets

[1.1.1 (L) cont'd]

Of course we might not want to have to list all the possibilities just to get the right number. For example, as we saw before if  $A$  has  $n$  elements our chart will have  $2^n$  rows and  $2^n$  gets large very rapidly. (For instance,  $2^{20}$  exceeds one million as can be checked by direct computation; hence if  $A$  has twenty elements it has more than one million subsets. Thus the chart for listing the subsets of  $A$  would have to have more than one million rows.)

In short, this may be one of those times when we would like a clever way of counting that does not involve the usual type of explicit enumeration. To this end, recall the notation  $C(n,r)$ , meaning the number of ways in which  $r$  elements can be chosen from a set of  $n$  elements, without regard to order. We emphasize that order is not important since the sets  $\{a,b\}$  and  $\{b,a\}$  are equal. That is, a set does not depend on the order in which we list its members. Recall that  $C(n,r)$  is given by:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

In our present example, we have  $n = 4$  and  $r = 2$ . That is, there are  $C(4,2)$  subsets of  $A$  which have two elements; and  $C(4,2) =$

$$\frac{4!}{2!2!} = \frac{24}{4} = 6.$$

Without falling back on a mechanical-type of formula, we see that one of our elements could have been any one of the four, while the second could have been any one of the remaining three. Thus, we could have picked a pair of elements in  $4 \times 3 = 12$  different ways. Of these twelve ways, six would be a different listing of the other six, such as  $\{a,b\}$  and  $\{b,a\}$ . Thus, the answer of 12 takes order into account. Without regard to order, the answer is 6.

[1.1.1 (L) cont'd]

(d) We have solved (d) in our solution of (c). Again, we wish to emphasize the difference between being able to compute a particular number of things, as opposed to having to list them all.

(e) Here we wish to emphasize how the same abstract mathematical situation can apply to different physical situations. The assumption that we do not have a "loaded" coin (such a coin is often referred to as a "fair" coin) means that the event of a head turning up on a given toss is as likely as that of a tail turning up. At any rate, we can now think of A as being the set of four coins (where we assume that tossing one fair coin four times is the same problem as tossing four fair coins once). If we now think of 0 as denoting heads and 1 as denoting tails, our chart (4) can now be interpreted to show that there are 16 different ways in which the coins may fall, of which 6 yield exactly two heads and two tails and the other 10 don't. Since the events are EQUALLY LIKELY, we say that the odds are 10 to 6 (5 to 3) AGAINST THE EVENT. We also say that the probability of the event is  $6/16$ , or  $3/8$ , where the probability is defined to be the ratio of the number of successes to the total number of possible outcomes, PROVIDED THAT ALL OUTCOMES ARE EQUALLY LIKELY.

If the events are not equally likely, we must be a bit more cautious and introduce appropriate weighting factors. For example, when we toss four coins there are five\* mutually-exclusive

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\*There are, of course, many other possibilities. For example, we can get the two mutually-exclusive possibilities that (1) we get four heads, and (2) we don't. This does not mean that the probability that we get four heads is  $1/2$ . Rather, four heads occurs in only one way, while not getting four heads happens 15 ways. Thus, (2) should have a "weighting" factor of 15.

SOLUTIONS: Calculus of a Single Variable - Block 1: Sets,  
Functions, and Limits - Unit 1: Sets

[1.1.1 (L) cont'd]

possibilities, one of which must happen. Namely: four heads and no tails, three heads and one tail, two heads and two tails, one head and three tails, or no heads and four tails. Only one of these five includes two heads and two tails; yet, we do not say that the probability of getting two heads and two tails is  $1/5$ , SINCE THE OUTCOMES ARE NOT EQUALLY LIKELY. More explicitly,

<u>outcome</u>	<u>number of ways</u>	(according to chart (4))
4H,0T	1	
3H,1T	4	
2H,2T	6	
1H,3T	4	
0H,4T	<u>1</u>	
	16	

Notice that 2H,2T is more likely than any other single possibility, but the total of all other possibilities exceeds the number of ways in which 2H,2T may occur.

1.1.2

- (a) We have  $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$  subsets.
- (b) There are  $C(6,3)$  subsets which have exactly three members. More explicitly  $C(6,3) = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{6} = 20$ .
- (c) There are  $C(6,2) = \frac{6 \times 5}{2} = 15$  such subsets.
- (d) There are  $C(6,4) = 15$  such subsets.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 1: Sets

[1.1.2 cont'd]

(e) The answers to both are the same. The reason for this is that each time we explicitly choose two elements, we implicitly choose four other elements. For example, if  $A = \{a,b,c,d,e,f\}$  and we choose the subset  $\{b,e\}$  then we implicitly determine the set of remaining elements  $\{a,c,d,f\}$ . (In modern language, there is a one-to-one correspondence between the set of subsets of  $A$  which have exactly two elements and the set of subsets of  $A$  which have exactly four elements.)

(f) There are 64 possible (equally-likely) outcomes [see (a)] of which 20 involve exactly three heads (or tails) [see (b)]. Thus the probability of obtaining exactly three heads and three tails is  $\frac{20}{64} = \frac{5}{16}$ . (That is, the odds against this happening are 11 to 5.)

1.1.3 (L)

(a) Regardless of the universe of discourse, the set-builder notation has us write

$$S = \{x : x^4 = 1\}.$$

(b) If the universe of discourse is allowed to be the set of complex numbers, then our solution set would be given in roster form by  $\{1, -1, i, -i\}$  where, as usual,  $i$  denotes  $\sqrt{-1}$ . That is,  $i^2 = -1$ .

Of these four numbers, all are complex numbers; 1 and -1 are the only ones which are called real numbers, 1 would be the only real, positive numbers, and none of the four are even integers. Thus we would have

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 1: Sets

[1.1.3 (L) cont'd]

- (i)  $\{1, -1, i, -i\}$
- (ii)  $\{1, -1\}$
- (iii)  $\{1\}$
- (iv)  $\emptyset$

Note especially well that the answer to (iv) is written as  $\emptyset$ . We do not write  $\{\emptyset\}$ . That is, if  $S$  is the solution set, we are saying that  $S = \emptyset$  (obviously, we can't do any more than that if we wish to list the elements of a set which has no members) and that it is false to write  $S = \{\emptyset\}$ .

This is more than a whim. The sets  $\emptyset$  and  $\{\emptyset\}$  are conceptually very different. Among other things,  $\emptyset$  has no elements while  $\{\emptyset\}$  has one element, namely  $\emptyset$ .

As a second illustration, let  $S$  denote the set of all sets which have no members. Certainly there is at least one set with this property, namely  $\emptyset$ . Since  $S$  has at least one member, it cannot be the empty set. That is,  $S \neq \emptyset$ . On the other hand, only  $\emptyset$  has no elements, hence  $S = \{\emptyset\}$ . Hence,  $\{\emptyset\} \neq \emptyset$ .

If we desire still other reasons as to why  $\{\emptyset\}$  and  $\emptyset$  are different, recall that for any set  $A$ ,  $A \notin A$ . In particular, since  $\emptyset$  is a set, we must have that  $\emptyset \notin \emptyset$ . On the other hand,  $\emptyset \in \{\emptyset\}$ .

As a final interpretation, recall that we may view a subset of a set as being whatever part of the set we wish to choose. Our extreme choices range from taking nothing (the empty set) to taking everything (the universe of discourse in this case, the set itself.) In this sense, then, it is easy to see why both the set itself and the empty set are considered subsets of any set (another reason would be that our nice result that a set of  $n$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 1: Sets

[1.1.3 (L) cont'd]

elements has  $2^n$  subsets would have to be replaced by  $2^n - 2$  subsets if we excluded the set itself and the empty set from being subsets). If we took all of the elements we elected to choose, the empty set would correspond to our taking an empty bag. Clearly, we can recognize the difference between an empty bag and a bag which has another empty bag inside it.

As a final word, observe that the relation between  $\emptyset$  and  $\{\emptyset\}$  is a more subtle form of the relation discussed earlier involving the difference between  $b$  and  $\{b\}$ .

1.1.4

(a)  $\{x: (2x - 3)(x + 1)(x + 7) = 0\} = S$

(b) The roots of the equation are  $3/2$ ,  $-1$ , and  $-7$ . Of these, all are real and all are rational; only  $-1$  and  $-7$  are integers; and none are positive integers. Hence,

(i)  $S = \{3/2, -1, -7\}$

(ii)  $S = \{3/2, -1, -7\}$

(iii)  $S = \{-1, -7\}$

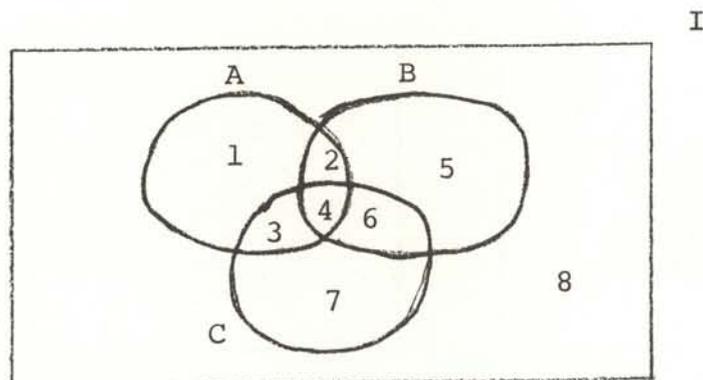
(iv)  $S = \emptyset$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits

UNIT 2: Arithmetic of Sets

1.2.1(L)

You will recall in the supplementary notes that we shaded and/or hatched appropriate regions of a Venn (circle) diagram to represent sets. For various reasons, it is sometimes difficult (for example, we must shade the regions very carefully lest we later misinterpret what region is actually shaded; such care is very time consuming). An equivalent technique is to number (or otherwise code) the mutually-exclusive regions in the circle diagram. By way of illustration, we might indicate the Venn diagram for the three sets A, B, and C by writing:



In this way, we would read A as  $A = \{1, 2, 3, 4\}$  (where it is important to note that we really mean this as an abbreviation for  $A = 1 \cup 2 \cup 3 \cup 4$ . Notice also that our regions are numbered so that  $1 \cap 2 = 1 \cap 3 = 1 \cap 4 = 2 \cap 3 = 2 \cap 4 = 3 \cap 4 = \emptyset$ ). In a similar way,  $B = \{2, 4, 5, 6\}$  and  $C = \{3, 4, 6, 7\}$ .

With this new notation in mind, we solve the exercise as follows:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 2: Arithmetic of Sets

[1.2.1(L) cont'd]

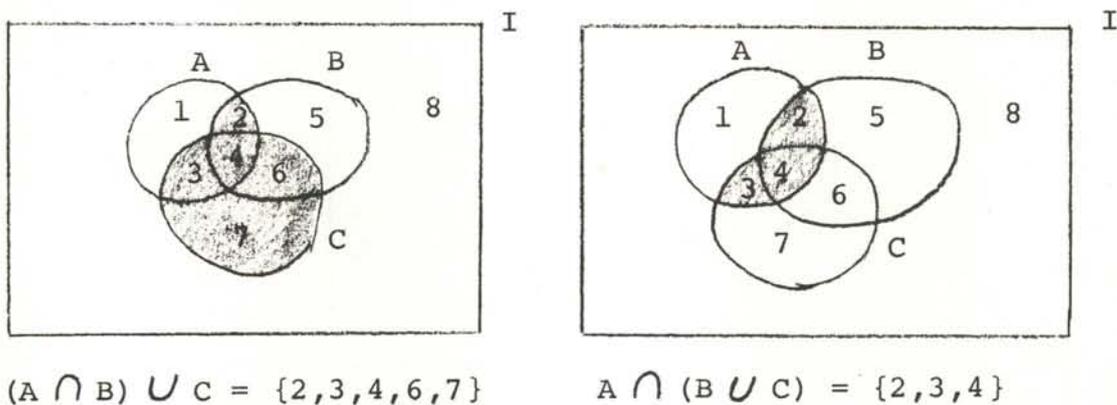
(a)  $A \cap B = \{2, 4\}$  (and a quick look at the above diagram shows that this result agrees with what we would have obtained by appropriately shading the proper region).

$$\text{Thus } (A \cap B) \cup C = \{2, 4\} \cup \{3, 4, 6, 7\} = \{2, 3, 4, 6, 7\} \quad (1)$$

$$\text{Similarly } B \cup C = \{2, 4, 5, 6\} \cup \{3, 4, 6, 7\} = \{2, 3, 4, 5, 6, 7\}$$

$$\begin{aligned} \text{Hence } A \cap (B \cup C) &= \{1, 2, 3, 4\} \cap \{2, 3, 4, 5, 6, 7\} \\ &= \{2, 3, 4\} \end{aligned} \quad (2)$$

A comparison of (1) and (2) shows that the two sets are not the same. In fact, we can now translate (1) and (2) very neatly into the appropriate circle diagrams. Namely:



The point is that while we could shade the regions directly, our method seems to be a bit more objective. Our method, more importantly, lends itself to certain types of analysis, as we shall see in the next part.

(b) From (a) we have that  $(A \cap B) \cup C = \{2, 3, 4, 6, 7\}$  while  $A \cap (B \cup C) = \{2, 3, 4\}$ . Since  $\{2, 3, 4\} \subset \{2, 3, 4, 6, 7\}$  we have that for any choice of A, B, and C:

$$[A \cap (B \cup C)] \subset [(A \cap B) \cup C] \quad (3)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 2: Arithmetic of Sets

[1.2.1(L) cont'd]

Moreover, we see at once that it is regions 6 and 7 that prevent us from asserting equality in (3). Since our regions are mutually-exclusive the only way we can offset regions 6 and 7 is to make sure that they are empty. In other words, if  $6 = 7 = \emptyset$  then:

$$(A \cap B) \cup C = \{2,3,4,6,7\} = \{2,3,4,\emptyset,\emptyset\} = \{2,3,4\} (= A \cap (B \cup C))$$

$$\begin{aligned} \text{(In terms of unions, } (A \cap B) \cup C &= 2 \cup 3 \cup 4 \cup 6 \cup 7 \\ &= 2 \cup 3 \cup 4 \cup \emptyset \cup \emptyset \\ &= 2 \cup 3 \cup 4) \end{aligned}$$

Thus we get equality if and only if both 6 and 7 are empty. This, in turn, implies that  $6 \cup 7 = \emptyset$ . If we now look at the original circle diagram we see that  $6 \cup 7$  represents those C's which are not A's. Therefore, the most general condition that guarantees equality is All C's are A's (that is,  $C \subset A$ ).

Again, another value of our coding system is that we do not have to be able to recognize from the picture that  $6 \cup 7$  means  $C \cap A'$  (C's which are non-A's). Rather we know that  $A = \{1,2,3,4\}$  and  $C = \{3,4,6,7\}$ . Thus as soon as we know that 6 and 7 are both empty, we have that  $C = \{3,4,\emptyset,\emptyset\} = \{3,4\}$  and  $A = \{1,2,3,4\}$ . Clearly  $\{3,4\} \subset \{1,2,3,4\}$ , and the assertion follows.

(c) If we look at  $A \cap B \cup C$  it seems that there are two different ways of "pronouncing" it. One is  $(A \cap B) \cup C$  and the other is  $A \cap (B \cup C)$ . But in both (a) and (b) we showed that  $(A \cap B) \cup C$  and  $A \cap (B \cup C)$  were, in general, different sets. Thus, with the parentheses missing  $A \cap B \cup C$  can be interpreted as two different sets.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 2: Arithmetic of Sets

[1.2.1(L) cont'd]

This is somewhat similar to what happens in ordinary algebra. For example  $a \times b + c$  is ambiguous (unless we add parentheses or some other convention) since it can be read as  $(a \times b) + c$  or  $a \times (b + c)$ . In general these two numbers are different. For example,  $(2 \times 3) + 4 = 10$  while  $2 \times (3 + 4) = 14$ . Thus  $2 \times 3 + 4$  can mean either 10 or 14 depending on our voice inflection in reading the expression. (Notice, for example, that if  $c = 0$  then  $a \times (b + c)$  and  $(a \times b) + c$  are equal. We agree, however, that equality means for all choices of  $a$ ,  $b$ , and  $c$ , not just for some special cases. Indeed, in (b) we were showing a special case in which the two sets were equal even though in (a) we showed that they were not always equal.)

Obviously, it is bothersome to have mathematical operations which depend on "voice inflection". In a mathematical structure, we prefer those operations (if we can get them) which do not depend on this (such operations, i.e. those which do not depend on voice inflection are said to have the ASSOCIATIVE property). Ordinary addition (as well as multiplication) are associative since  $a + b + c$  gives the same answer for all  $a$ ,  $b$ , and  $c$  regardless of whether we read it as  $(a + b) + c$  or as  $a + (b + c)$ . This brings us to (d).

Namely:

(d)  $A \cap B \cap C$  can be read as  $(A \cap B) \cap C$  or as  $A \cap (B \cap C)$ .  
From our diagram, we have  $A \cap B = \{2,4\}$  and  $B \cap C = \{4,6\}$ .

Hence  $(A \cap B) \cap C = \{2,4\} \cap \{3,4,6,7\} = \{4\}$

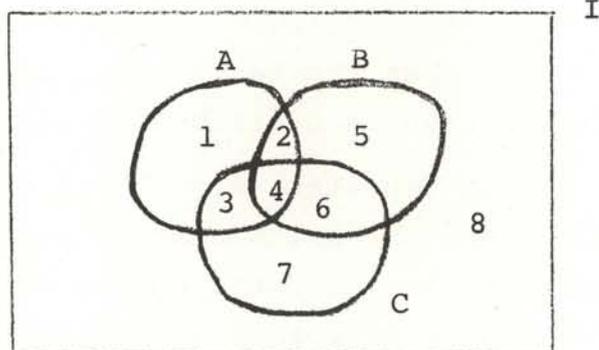
and  $A \cap (B \cap C) = \{1,2,3,4\} \cap \{4,6\} = \{4\}$

Thus  $(A \cap B) \cap C = A \cap (B \cap C) (= \{4\})$  and we may omit the parentheses in  $A \cap B \cap C$  because we get the same answer no matter where the parentheses are. That is,  $A \cap B \cap C$  is not ambiguous. In other words, we may say that the operation known as intersection is associative.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 2: Arithmetic of Sets

1.2.2

Let us use the coding system:



Then:  $A \cup B = \{1, 2, 3, 4, 5, 6\}$

$B \cap C = \{4, 6\}$

$A \cap C = \{3, 4\}$

(a)  $(A \cup B) \cap C = \{3, 4, 6\}$

$A \cup (B \cap C) = \{1, 2, 3, 4, 6\}$

Thus  $[(A \cup B) \cap C] \subset [A \cup (B \cap C)]$ . Moreover equality holds if and only if  $1 = 2 = \emptyset$ . This in turn means that:

All A's are C's

(b)  $(A \cup B) \cap C = \underline{\{3, 4, 6\}}$

$(A \cap C) \cup (B \cap C) = \{3, 4\} \cup \{4, 6\} = \underline{\{3, 4, 6\}}$

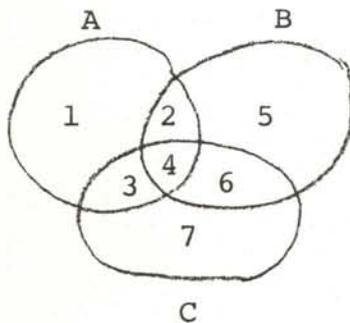
Therefore the two sets are equal, since both equal  $\{3, 4, 6\}$ .

It is a widely accepted custom to write intersection as multiplication and union as addition. Thus one finds  $A + B$  used to denote  $A \cup B$  and  $AB$  to denote  $A \cap B$ . With this notation in mind (b) would look like:  $(A + B)C = AC + BC$ . This greatly resembles a well-known result of ordinary arithmetic.

[1.2.2 cont'd]

By the way, in the arithmetic expression  $a(b + c) = ab + ac$ , we say that multiplication is DISTRIBUTIVE OVER ADDITION since the multiplier is "distributed" over the terms being added. In a similar way, one says that intersection is distributive over union.

Now, however, we see an important structural difference between the arithmetic of numbers and the arithmetic of sets. Namely, in ordinary arithmetic addition is NOT distributive over multiplication. That is, it is not true that  $a + (b \times c) = (a + b) \times (a + c)$ . However, as the following circle-diagram shows, the corresponding statement for sets is true, namely union is distributive over intersection.



$$\begin{array}{ll}
 B \cap C = \{4, 6\} & A \cup B = \{1, 2, 3, 4, 5, 6\} \\
 A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{4, 6\} & A \cup C = \{1, 2, 3, 4, 6, 7\} \\
 = \{1, 2, 3, 4, 6\} & (A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 6\}
 \end{array}$$

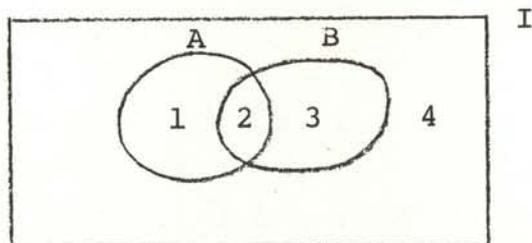
$$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\text{That is } A + (BC) = (A + B) (A + C)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 2: Arithmetic of Sets

1.2.3(L)

(a) We now use a coding device for two circles. Say:



Then:  $A \cup B = \{1, 2, 3\}$

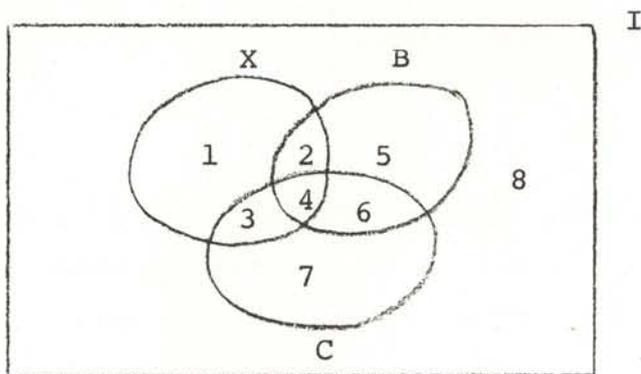
Therefore:  $(A \cup B)' = \{1, 2, 3\}' = \{4\}$  (1)

On the other hand,  $A' = \{3, 4\}$  and  $B' = \{1, 4\}$ .

Hence  $A' \cap B' = \{3, 4\} \cap \{1, 4\} = \{4\}$  (2)

The result now follows from comparing (1) and (2). (Notice that it might seem more "natural" to have  $(A \cup B)' = A' \cup B'$ , but natural or not, it would be incorrect. In fact  $A' \cup B' = \{1, 3, 4\}$  and this equals  $(A \cup B)'$  if and only if  $1 = 3 = \emptyset$ ; but this says, if and only if  $A = B$ . That is, if  $A \neq B$  then  $(A \cup B)'$  cannot equal  $A' \cup B'$ .)

(b) Referring to the given diagram:



[1.2.3(L) cont'd]

$$X \cup B = \{1,2,3,4,5,6\}$$

$$X \cup C = \{1,2,3,4,6,7\}$$

Thus  $X \cup B = X \cup C$  if and only if  $5 = 7 = \emptyset$ . So, all that is required for  $X \cup B = X \cup C$  is that all B's be either X's or C's (that is, 5 denotes  $B \cap X' \cap C'$ ) and that all C's be either X's or B's. Certainly this can happen without having  $B = C$ . (For example, one possibility would be that B and C were different subsets of X. In this event both  $X \cup B$  and  $X \cup C$  would equal X but  $B \neq C$ .)

(c) From (b) we know that  $X \cup B = X \cup C$  implies that both 5 and 7 are empty. Thus we already know that  $B = \{2,4,6\}$  and  $C = \{3,4,6\}$ . We next have that:  $X \cap B = \{2,4\}$  and  $X \cap C = \{3,4\}$ . Hence  $X \cap B = X \cap C$  implies that 2 and 3 are empty. Coupling this with our previous knowledge that 5 and 7 are empty, we see that  $B = \{4,6\}$  and  $C = \{4,6\}$ . Hence  $B = C$ . (This is known as the "cancellation" law for sets. That is unlike the analog for numbers if either  $X \cup B = X \cup C$  or  $X \cap B = X \cap C$  we cannot be sure that  $B = C$  - in other words, we cannot "cancel" X from both sides of the equation. However if both conditions apply simultaneously, we can.)

#### 1.2.4

This problem gives us a nice application of circle diagrams and also serves as an illustration of how the diagram may actually be easier to handle in some situations than the more analytic "recipes".

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 2: Arithmetic of Sets

[1.2.4 cont'd]

Stated analytically, this problem asks us to find  $N(A' \cap B \cap C')$ .  
[Note that  $A' \cap B \cap C'$  is another way of saying the set of all B's  
which are neither A's nor C's], given that:

$$\begin{aligned}N(A \cup B \cup C) &= 280 \\N(A \cap B \cap C) &= 20 \\N(A \cap B) &= 30 \\N(A \cap C) &= 50 \\N(B \cap C) &= 60 \\N(A) &= 120 \\N(C) &= 160\end{aligned}$$

The recipe stated in the supplementary notes is fine if  
all we want to do is determine  $N(B)$ . Namely:

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - [N(A \cap B) + N(A \cap C) + N(B \cap C)] + N(A \cap B \cap C)$$

or:

$$280 = 120 + N(B) + 160 - 30 - 50 - 60 + 20$$

or:

$$N(B) = 120$$

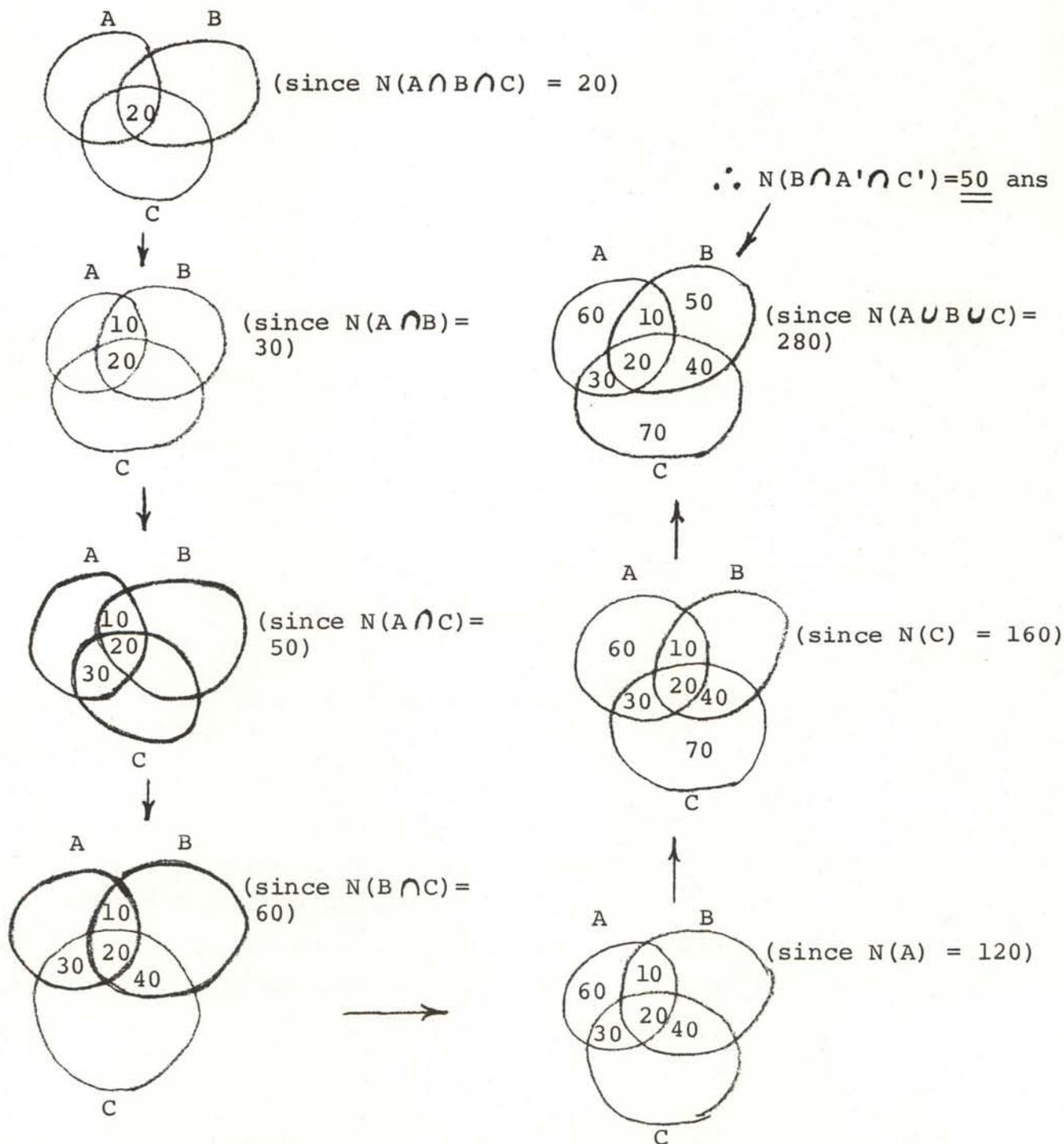
But we want only part of B and analytically this means that we  
must find some way of expressing  $A' \cap B \cap C'$  in terms of the given  
sets.

In a more rigorous treatment of sets we would have to come  
to grips with such problems, since we often would have several  
sets rather than just three. As we have seen in the previous  
exercise, the circle diagrams (and equivalently, the charts) get  
out of hand quite rapidly when we deal with many sets.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 2: Arithmetic of Sets

[1.2.4 cont'd]

However, with just three sets, we can utilize the given information to arrive at:



UNIT 3: Analytic Geometry

1.3.1(L)

A given line (or, for that matter, any curve) can be represented by many different equations. For example, the equation of any line not parallel to the  $y$ -axis can be written in the form  $y = mx + b$ , where  $m$  is the slope of the line and  $b$  is its  $y$ -intercept. Such a form is particularly convenient if we want to write the equation of the line in a way which emphasizes its slope and  $y$ -intercept. In this exercise, we are being asked to find a convenient equation for the line if we wish to emphasize its  $x$ - and  $y$ -intercepts.

One way of tackling this problem is to use the previous form  $y = mx + b$ . Since the line passes through  $(0, b)$  and  $(a, 0)$ , its slope is given by  $(b - 0)/(0 - a)$ , or  $-b/a$ . Thus our equation becomes:

$$y = (-b/a)x + b$$

Clearing of fractions, we obtain

$$ay = -bx + ab; \text{ or } bx + ay = ab$$

Dividing both sides by  $ab$  (which we can do since  $ab \neq 0$ ) we obtain the desired result:

$$(x/a) + (y/b) = 1$$

---

\*The fact that in this exercise  $a$  and  $b$  are numbers means that our line intersects both the  $x$ - and  $y$ -axis. Hence our line is not parallel to the  $y$ -axis. Therefore  $y = mx + b$  is an acceptable form for the equation of  $L$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.1(L) cont'd]

Of course there was nothing crucial about the equation  $y = mx + b$  except for its convenience. We could have fallen back to the basic equation:

$$\frac{y - y_0}{x - x_0} = m$$

where  $(x_0, y_0)$  is  $(0, b)$  or  $(a, 0)$ , and  $m = \frac{-b}{a}$ .

At any rate, it is hoped that our presentation emphasizes that the line remains the same - only its equation changes. For example,  $\frac{x}{2} + \frac{y}{3} = 1$  and  $y = -\frac{3}{2}x + 3$  are different (but equivalent) equations for the same line. One form emphasizes that  $(2, 0)$  and  $(0, 3)$  are points on the line, while the other emphasizes that  $(0, 3)$  is on the line and  $-\frac{3}{2}$  is its slope.

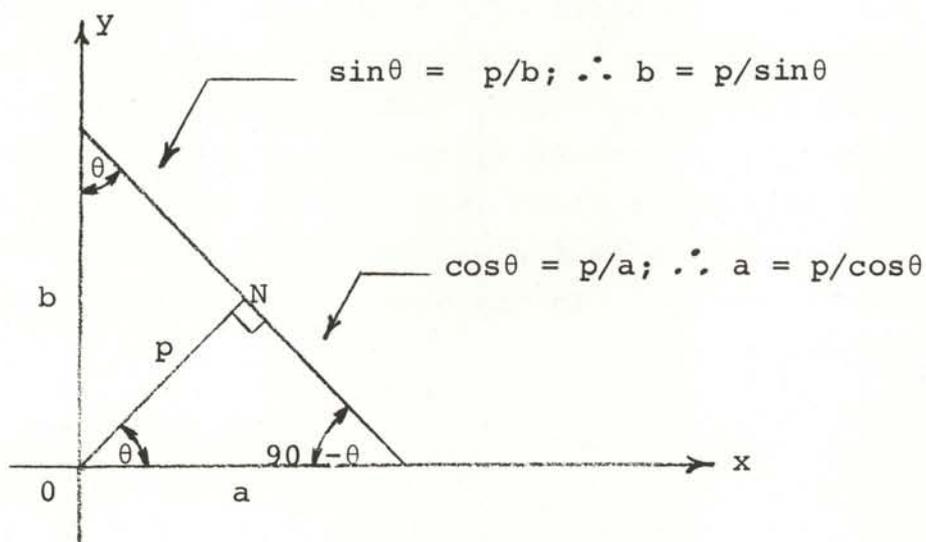
Which of the two forms is better depends on the particular problem involved. Perhaps there are situations in which neither form is advantageous. For example, it is possible that we have a "radar-type" situation, in which we know the perpendicular distance of the line from the origin and we also know the angle this perpendicular makes with the positive x-axis. Letting  $\theta$  denote the angle and  $p$  the distance, it might be advisable to express the equation of the line in terms of  $p$  and  $\theta$ . (This is true in general in mathematics. When there are alternative ways of expressing a relationship, we try to choose the way which utilizes the measurements we have available. As a rather trivial example, consider the case of a freely-falling body near the surface of the earth, starting from rest. The

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.1(L) cont'd]

typical elementary physics text explains the three recipes:  
 $v = gt$ ;  $s = 1/2gt^2$ ; and  $v^2 = 2gs$ . The three recipes are equivalent, but the one we use in a given problem depends on what we want to determine and what is known. Thus if  $t$  is known and we wish to find  $v$ , we would use  $v = gt$ , etc.)

1.3.2



Observing that  $a = \frac{p}{\cos \theta}$  and  $b = \frac{p}{\sin \theta}$ , we can invoke the result of Exercise 1.3.1(L) to obtain:

$$\frac{x}{p/\cos \theta} + \frac{y}{p/\sin \theta} = 1 \quad \text{or;}$$

$$\frac{x \cos \theta}{p} + \frac{y \sin \theta}{p} = 1 \quad \text{or;}$$

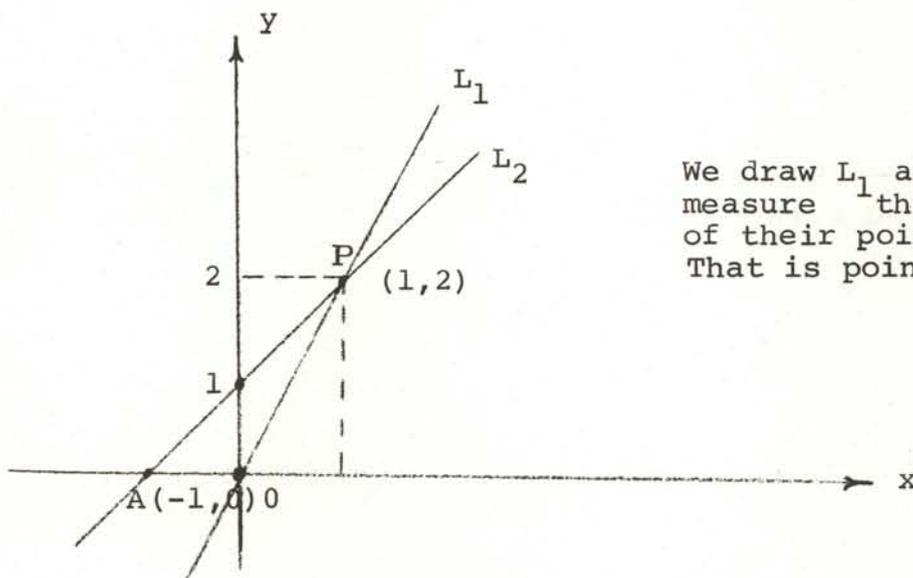
$$x \cos \theta + y \sin \theta = p$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

1.3.3(L)

(a) Here again we can solve the problem graphically - but we don't have to. (By the way, once again let us point out that we are not belittling the concept of drawing to scale. Rather, (1) it is unnecessary to take the time to do so and (2) if we were dealing with more variables - for example had we been dealing with planes rather than lines - it would be difficult to draw things to "true" scale (not to mention the fact that we can't even draw the graph of a linear equation in more than three variables).

Had we wished to solve this problem graphically, the key construction would be to observe that if we know a point  $Q(x_1, y_1)$  on the line whose slope is  $m$ , then we can locate a second point on the line merely by taking  $(x_1 + 1, y_1 + m)$ , since slope  $m$  means  $y$  changes by  $m$  units when  $x$  changes by one unit. [Whether the line is rising or falling is taken into account algebraically by  $y_1 + m$ . That is, if  $m$  is negative we are in effect subtracting the magnitude of  $m$  from  $y_1$ .] In any event, we obtain a graphical solution as follows:



We draw  $L_1$  and  $L_2$  and measure the coordinates of their point of intersection. That is point  $P(x, y)$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.3(L) cont'd]

Of course, in terms of our lecture concerning the concept of points-versus-dots, we are faced with the general problem (although it is rather trivial in this particular example) that we can, at best, only approximate the coordinates of the point of intersection, since all of our constructed lines have a certain amount of thickness. That is, with the usual line, how shall we distinguish between, say, (1,2) and (1.00002,1.99998)?

The advantage of the analytic solution is that when we write an algebraic expression, such as  $x = 1$ , we have the equivalent of a point rather than a dot. We only run into trouble when we locate  $x = 1$  on the number line, even though there are many times when the accuracy of a diagram is sufficient for the problem and the diagram is much easier to arrive at than the more analytic result.

To find the point P analytically, we know that it belongs to both lines  $L_1$  and  $L_2$ . The slope of  $L_2$  is the same as the slope of the line segment AP, since we have already seen that the slope of a line is WELL-DEFINED, in the sense that we get the same slope no matter what two distinct points we choose on the line. Since  $(-1,0)$  is on  $L_2$  the slope of AP is given by  $[y - 0]/[x - (-1)]$ ; or  $y/(x + 1)$ . (Notice again how the algebraic signs take care of themselves, without our having to draw a picture.) On the other hand, we are given that the slope of this line is also +1. Since a line has but one slope, it follows that:

$$\frac{y}{x + 1} = 1 \quad (1)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.3(L) cont'd]

We are told that the slope of OP equals 2. Since (0,0) is on OP, the slope of OP is also given by  $(y-0)/(x-0) = y/x$ . This tells us that:

$$\frac{y}{x} = 2 \quad (2)$$

Where the lines intersect, their x- and y- coordinates are the same. Here we solve by using y.

(1) and (2) can be written in the form:

$$\begin{cases} y = x + 1 \\ y = 2x \end{cases} \quad (3)$$

Solving the pair of simultaneous equations in (3), we find rather quickly that  $x = 1$  and  $y = 2$ .

A few passing remarks may now be in order.

(a) The fact that the slope of  $L_1$  is 2 means that

$$\frac{\Delta y}{\Delta x} = 2 \quad (4)$$

If we compare (4) with (2), it seems that the  $\Delta$ 's "cancelled" just as in ordinary cancellation of common factors. Observe however that  $\Delta$  is not a number - it is an OPERATOR, which, when operating on a number, prefaces it by "the change in...". We must never cancel  $\Delta$  or confuse it with a number. The only reason that (2) and (4) looked so much alike was that, if our point of origin happens to be (0,0), our point of termination (x,y) also happens to be labeled ( $\Delta x$ ,  $\Delta y$ ). Although we won't prove it here, the only time  $\Delta y/\Delta x$  and  $y/x$  are identical is when we have a straight line passing through the origin (0,0).

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.3(L) cont'd)

(b) The wording of this problem refers to "a point P". Actually P is a UNIQUE point, since it lies on the intersection of two non-parallel lines (we know that  $L_1$  and  $L_2$  are not parallel since they have different slopes); and two non-parallel lines in the plane intersect at ONE and ONLY one point. In this way, notice how understanding the geometrical interpretations involved help us to VISUALIZE what must be happening analytically.

In terms of Analytic Geometry, observe that what we did was to relate an algebraic expression,  $y = x + 1$ , with a picture - the line passing through  $(-1,0)$ , with slope equal to  $+1$ .

In the language of sets:

$$\{(x,y): y = x + 1\}$$

can be viewed either as a set of ordered pairs of numbers or as a set of points in the plane. As a set of points, the relation determines the line  $L_2$ .

1.3.4(L)

This three-part exercise breaks down the problem of finding the perpendicular distance from a point to a line into three consecutive steps. In essence we are saying that, knowing the slope of a given line, we also know that its negative reciprocal must be the slope of the line  $L$  through the given point  $A$  which is perpendicular to the line  $L'$ . If we then solve the two equations of the lines  $L$  and  $L'$  simultaneously, the solution labels the point on  $L'$ ,  $B$ , which is the foot of the perpendicular from  $A$  to  $L'$ . The length of this segment,  $\overline{AB}$ , is then clearly the perpendicular distance of the point  $A$  from the line  $L'$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.4(L) cont'd]

This is precisely the mechanism behind the rather abstract proof of the formula for finding the perpendicular distance from a point to a line.

At any rate, returning to the specifics of this problem, if we rewrite  $L'$  as  $y = -2x + 4$ , we see at once that its slope is  $-2$  (why?). Hence, since  $L$  is perpendicular to  $L'$  its slope must be  $1/2$ .

Then since  $(-2, 2)$  is a point on  $L$ , we have that a point  $(x, y)$  is on  $L$  if and only if  $\frac{y - 2}{x - (-2)} = 1/2$ . Upon proper simplification of this equation, we see that an equation for the line  $L$  is given by  $2y - 4 = x + 2$ , which in turn may be written as  $x = 2y - 6$ ; and this is the answer to part (a) (although, of course, there are other forms that the correct answer may assume).

For part (b), we have that  $(x, y)$  is on  $L'$  if and only if  $2x + y = 4$ ; and  $(x, y)$  is on  $L$  if and only if  $x = 2y - 6$ . Observing that  $B$  the point of intersection of  $L$  and  $L'$  is the point which belongs simultaneously to both lines, we see that the point  $B(x, y)$  which we seek must be determined as the solution to the pair of simultaneous equations:

$$\left. \begin{array}{l} 2x + y = 4 \\ x = 2y - 6 \end{array} \right\}$$

Solving this set of simultaneous equations we find that the point of intersection of  $L$  and  $L'$  is given by  $(2/5, 16/5)$ .

As for part (c) we observe that for the line segment  $AB$ ,  $\Delta x = -2 - 2/5$  and  $\Delta y = 2 - 16/5$ . Since the length of  $AB$  is precisely  $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ , we see that the required distance is given by  $\sqrt{(12/5)^2 + (6/5)^2}$  or  $\sqrt{144/25 + 36/25} = \sqrt{180/25} = 6\sqrt{5}/5$ , which is the correct answer.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.4(L) cont'd]

As a final check that we obtained the same answer as that which we would have obtained from the formula

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

(where  $(x_1, y_1)$  denotes the given point and  $Ax + By + C = 0$  is the equation of the given line), we need only let  $x_1 = -2$  and  $y_1 = 2$ . Then we could rewrite the equation of  $L'$  in the form  $2x + y - 4 = 0$ ; whence  $A = 2$ ,  $B = 1$  and  $C = -4$ . Plugging these values into (1) we see that we obtain the correct value for  $d$ .

In closing this problem, it might be valuable to you to return to the proof given for the distance between a point and a line and to see if the abstract proof seems more "real" now that you have dabbled with an example in which "concrete" numbers are used. Quite in general many abstract mathematical situations become more understandable if we first "play around" with examples which use specific numbers.

1.3.5

The recipe  $(x_1, y_1) = (4, 1)$  and  $Ax + By + C = 0$  in this case means  $3x - y - 5 = 0$   $\therefore x_1 = 4, y_1 = 1, A = 3, B = -1, C = -5$ .

---

\*We shall study the significance of absolute values in the next unit. For now it is sufficient to know that for any number  $x$   $|x|$  (called the absolute value of  $x$ ) means the magnitude (size) of  $x$ . In the present context, observe that distance is by definition non-negative while  $Ax_1 + By_1 + C$  might be negative. Writing  $|Ax_1 + By_1 + C|$  stresses the fact that we are dealing with magnitudes.

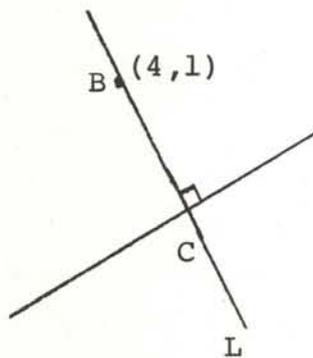
SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 3: Analytic Geometry

[1.3.5 cont'd]

Hence by formula:

$$\bar{d} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3(4) - 1(1) - 5|}{\sqrt{9 + 1}} = \frac{6}{\sqrt{10}} = \frac{6\sqrt{10}}{10} = \underline{\underline{\frac{3}{5}\sqrt{10}}}$$

To do this more systematically, in the spirit of Exercise 1.3.4(L), we have



$$L' \therefore 3x - y = 5$$

$$\text{or } y = 3x - 5$$

$$\therefore m_{L'} = 3 \therefore m_L = -\frac{1}{3}$$

equation of L:  $\frac{y - 1}{x - 4} = -\frac{1}{3}$  or  $x + 3y = 7$

Since C belongs to both L and L' we find that it is determined by:

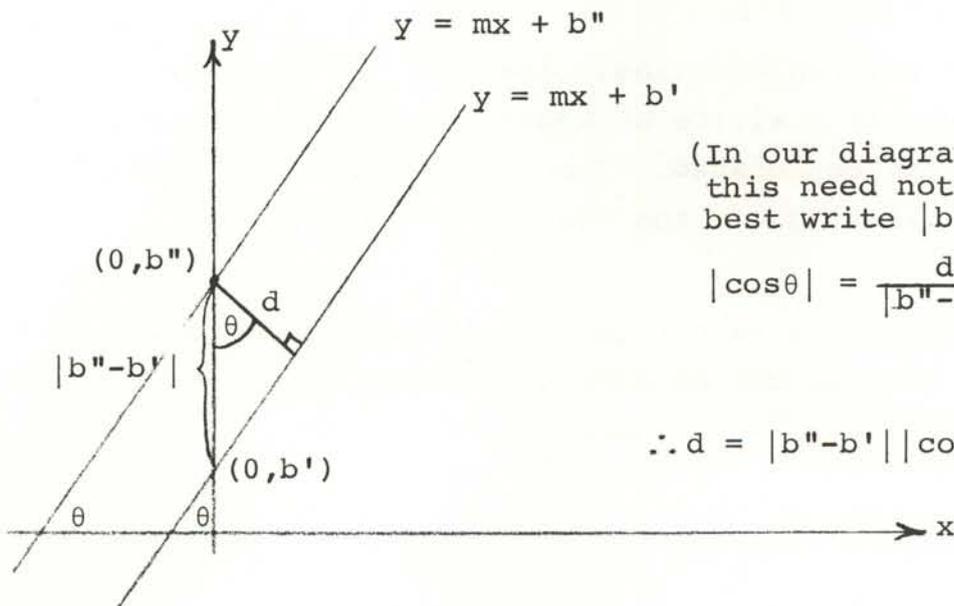
$$\left. \begin{array}{l} x + 3y = 7 \\ 3x - y = 5 \end{array} \right\} \therefore \begin{cases} x = \frac{11}{5} \\ y = \frac{8}{5} \end{cases} \therefore C = \left(\frac{11}{5}, \frac{8}{5}\right)$$

$$\therefore \overline{BC} = \sqrt{\left(4 - \frac{11}{5}\right)^2 + \left(1 - \frac{8}{5}\right)^2} = \sqrt{\frac{81}{25} + \frac{9}{25}} = \sqrt{\frac{90}{25}} = \underline{\underline{\frac{3}{5}\sqrt{10}}}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 3: Analytic Geometry

1.3.6(L)

It should be clear that the two given lines are indeed parallel (since they have equal slopes). The major place for misinterpretation in a problem such as this is to oversimplify what is happening. For example, it is possible that you might be tempted to subtract one equation from the other and conclude that the distance between the lines was  $|b' - b''|$ . (Again use absolute values since we are looking for a magnitude. In other words, one of the pair of numbers  $b' - b''$  or  $b'' - b'$  is non-negative; and that is the one we choose). However, if we do this we have found the VERTICAL distance (which is not necessarily the perpendicular distance. In fact unless  $m = 0$  which occurs when the lines are horizontal, the perpendicular and the vertical distances will be different). In terms of a diagram:



(In our diagram  $b'' > b'$ . Since this need not be the case, we had best write  $|b'' - b'|$ )

$$|\cos\theta| = \frac{d}{|b'' - b'|}$$

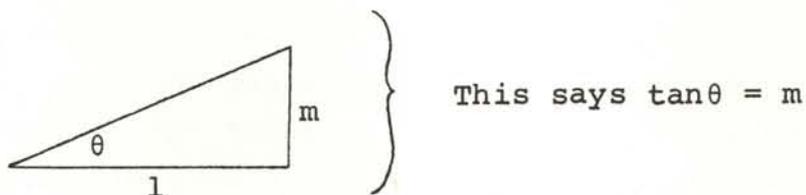
Here we write  $|\cos\theta|$  since the lines could have a negative slope - in which case  $\theta > 90$ , and  $\therefore \cos\theta < 0$ .

$$\therefore d = |b'' - b'| |\cos\theta|$$

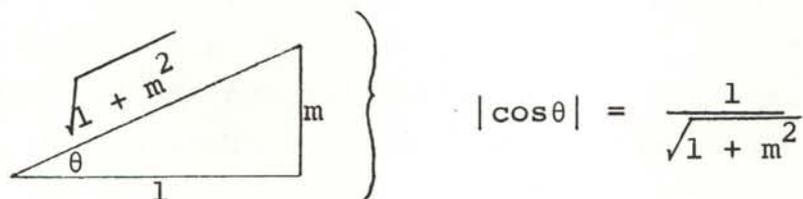
Since  $m = \tan\theta$ , a convenient diagram for determining  $|\cos\theta|$  (or for that matter, any other trigonometric function of  $\theta$ ) is to draw the "reference triangle"

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 3: Analytic Geometry

[1.3.6(L) cont'd]



Then by the Pythagorean Theorem,



(The problem with our reference triangle is: (1)  $m$  might be negative, (2) whether  $m$  is positive or negative there are two quadrants in which  $\theta$  can be located. Such considerations "merely" affect the algebraic signs; hence the use of absolute values solves our problem.)

That is, if  $\tan\theta = m$ , we cannot be sure that  $\tan\theta$  and  $\cos\theta$  have the same sign. What we can be sure of is that  $\cos\theta = \pm \frac{1}{\sqrt{1+m^2}}$ . However this insures that  $|\cos\theta| = \frac{1}{\sqrt{1+m^2}}$ .)

$$\therefore d = \frac{|b'' - b'|}{\sqrt{1+m^2}} \quad (1)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.6(L) cont'd]

Although it wouldn't be considered a proof in any way, notice how (1) reaffirms our claim that vertical and perpendicular are the same only when the lines are horizontal. Namely only if  $m = 0$  is our correction factor equal to 1.

An interesting aside is that this result gives us an alternative method for finding the distance from a point to a line: We find the equation of the line which passes through the given point parallel to the given line and then use the result of this exercise to find the perpendicular distance between the two lines.

For example, suppose the line is given by  $ax + by + c = 0$  and the point is  $(x_1, y_1)$ . From the fact that  $ax + by + c = 0$  it follows that:

$$y = (-a/b)x + (-c/b)$$

Thus the slope of our line is given by  $m = -a/b$  and its y-intercept is given by  $b' = -c/b$ .

Now to find the equation of the line which passes through  $(x_1, y_1)$  parallel to  $ax + by + c = 0$ , we need only solve the slope equation that  $(y_1 - b'')/(x_1 - 0) = -a/b$ , where  $b''$  is the y-intercept of this line. Solving this equation for  $b''$ , we obtain:

$$b'' = y_1 + ax_1/b$$

Combining this with the fact that  $b' = -c/b$ , we obtain

$$|b'' - b'| = |y_1 + ax_1/b + c/b| = \left|\frac{1}{b}\right| |ax_1 + by_1 + c|. \text{ Moreover}$$

$$\sqrt{1 + m^2} = \left|\frac{1}{b}\right| \sqrt{a^2 + b^2}; \text{ whence it follows that}$$

$$d = |ax_1 + by_1 + c|/\sqrt{a^2 + b^2} \text{ (what happens if } b = 0?).$$

1.3.7(L)

From one point of view the study of inequalities may be viewed as being a bit premature at this stage of our course. From another point of view, however, the fact is that in the study of calculus (or for that matter in most numerical mathematics situations) we are often more interested in inequalities than in equalities. For example, in this course we shall often be interested in investigating cases in which the difference between two numbers must be LESS THAN a certain amount.

If we apply this notion to the idea of straight lines, let us observe that we are often as interested in finding all points (or ordered pairs of numbers),  $(x,y)$  for which, say,  $y < 3x - 5$  as we are in finding those points for which  $y = 3x - 5$ .

For these and a multitude of other reasons, we wish to introduce inequalities as quickly as possible.

In this exercise we shall also try to emphasize our theme of analytic geometry by juxtapositioning the algebraic and geometric techniques for solving the problem.

With this in mind we proceed with the solution of the exercise.

(a) We are given that  $a < b$  and we want to show that  $a + c < b + c$ . According to the definition,  $a + c < b + c$  if and only if  $(b + c) - (a + c)$  is positive. But  $(b + c) - (a + c)$  is equal to  $b - a$ ; and SINCE WE ARE GIVEN THAT  $a < b$ , the definition tells us that  $b - a$  is positive. Therefore in this case  $(b + c) - (a + c)$  is positive; whence the result follows. An almost word-for-word treatment can be used to show that  $a - c < b - c$ .

In terms of geometry  $a < b$  merely means that  $a$  is to the left of  $b$  on the number line. If  $a$  is to the left of  $b$  and we move the same amount ( $c$ ) from both points  $a$  and  $b$ , the point we arrive at having left point  $a$  will be to the left of the point we arrived at after leaving  $b$ . ( $-c$  merely means that we moved in the direction opposite that of  $c$ .)

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.7(L) cont'd]

We are not advocating one method over the other. Rather both are important. The picture is easier for us to interpret, and the analytic approach captures the spirit of logic wherein we only accept those results which follow inescapably from our stated assumptions. Moreover, the logical (analytic) approach still makes sense even in those cases for which there is no convenient picture.

(b) To show that  $a + c < b + d$  we must show that  $(b + d) - (a + c)$  is positive. This may be rewritten as  $(b - a) + (d - c)$ . Since we are given that  $a < b$  and  $c < d$  it follows by our definition that both  $b - a$  and  $d - c$  are positive, then since the sum of two positive numbers is positive, it follows that  $(b - a) + (d - c)$  is positive - and this is precisely what we had to show to prove the given result.

Again in terms of lengths this means that if a "is shorter than b" and c is shorter than d; then the combined length of a and c is shorter than the combined length of b and d.

As for the second part of this problem, a counter-example (that is, merely one case in which the result is false) is the fact that  $4 < 6$  and  $1 < 5$  yet it is not true that  $4 - 1 < 6 - 5$ , in other words, when we subtract, the fact is that the more we take away the less we have and this causes problems. By the way, observe that the assertion in the problem CAN be true SOMETIMES (for example  $4 < 6$  and  $1 < 2$  and  $4 - 1 < 6 - 2$ ). However by truth we mean ALWAYS TRUE. Thus as soon as there is one case in which the result fails to hold it cannot always be true.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.7(L) cont'd]

To see what happened here from a more analytical point of view, we are given that  $a < b$  and  $c < d$ . This means that both  $b - a$  and  $d - c$  are positive. Now we want to investigate the relation  $a - c < b - d$ . For this relation to hold it is necessary that  $(b - d) - (a - c)$  be positive. This in turn can be rewritten as  $(b - a) + (c - d)$ . Since  $d - c$  is positive,  $c - d$  is negative. Hence we are adding a positive and a negative number in the expression  $(b - a) + (c - d)$ ; and such an expression can be either positive, negative, or zero, depending on the magnitudes of the involved numbers.

A very important by-product to this exercise is the fact that we must be very careful when we deal with the meaning of "self-evident" in inequalities. It may seem that equals subtracted from unequals should be unequal "in the same sense" but notice that the order of the inequality changes.

1.3.8

$$(a) \quad a < b \rightarrow b - a > 0$$

$$\therefore c > 0 \rightarrow c(b - a) > 0 \rightarrow cb - ca > 0 \rightarrow cb > ca \rightarrow ac < bc$$

If  $c$  is negative then  $b - a > 0 \rightarrow c(b - a) < 0$  (since negative times positive is negative).

$$\therefore cb - ca < 0$$

$$\therefore \underline{ac > bc}$$

(In other words, multiplying an inequality by a negative number reverses the order of the inequality. Failure to observe this result will result in more than one embarrassment during this course.)

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 3: Analytic Geometry

[1.3.8 cont'd]

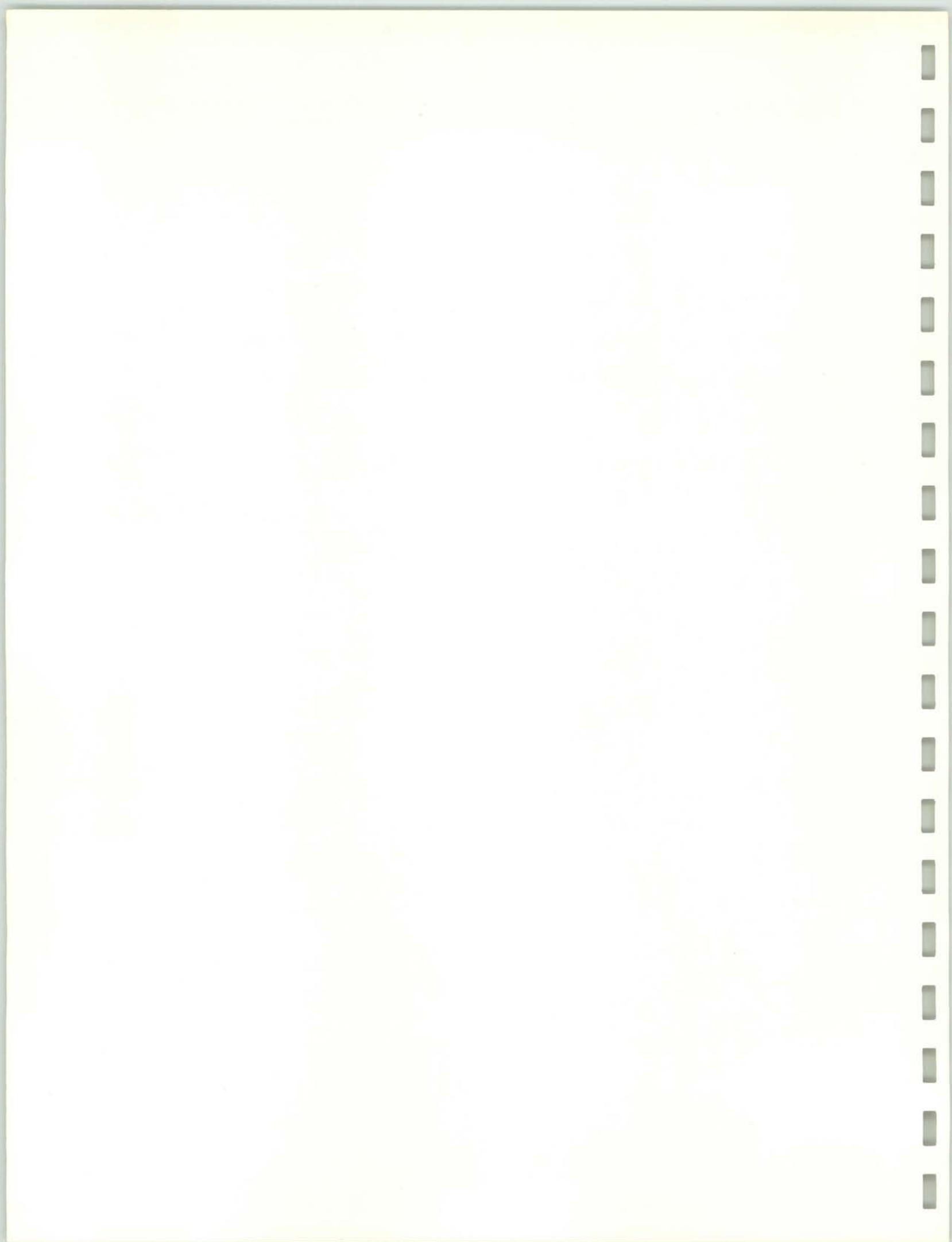
$$(b) \quad a < b \rightarrow b - a > 0$$

$$\text{Now } \frac{1}{a} - \frac{1}{b} = \frac{b - a}{ab}$$

But  $b - a > 0$ ; hence  $ab > 0 \rightarrow \frac{b - a}{ab} > 0$  (since positive  
divided by positive is positive)  $\therefore a < b$  and  $ab > 0 \rightarrow$

$$\frac{1}{a} - \frac{1}{b} > 0 \rightarrow \frac{1}{b} < \frac{1}{a}.$$

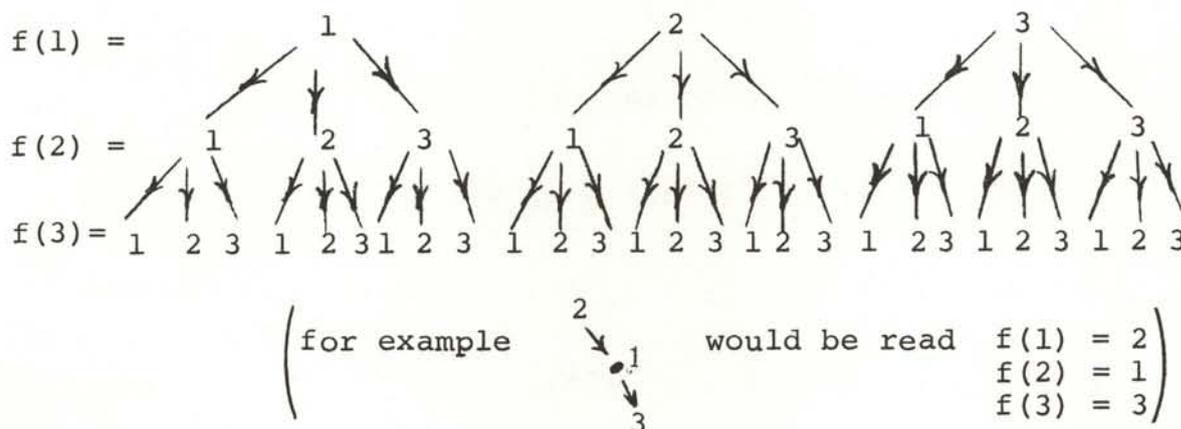
(If  $a$  and  $b$  have different signs, i.e.  $ab < 0$ , then  $\frac{b - a}{ab}$   
is negative and  $\therefore \frac{1}{a} < \frac{1}{b}$ .)



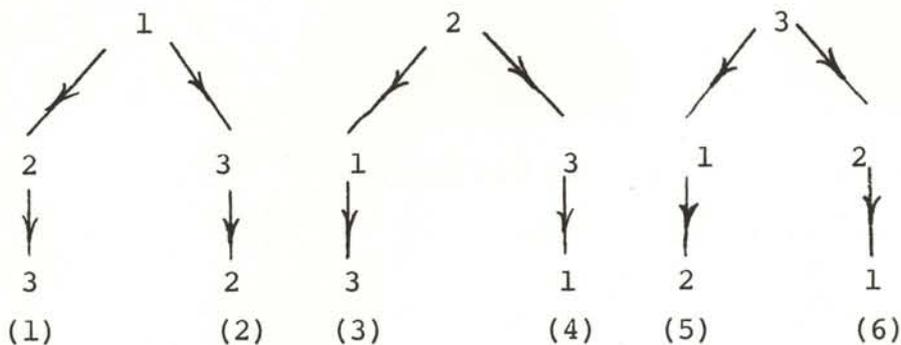
UNIT 4: Functions

1.4.1(L)

(a) Each  $a \in A$  can have any one of three possible images (1, 2, or 3), since there are three choices for  $a \in A$  (namely, again, 1, 2, or 3). We have that there are  $3^3$  or 27 functions  $f: A \rightarrow A$ . In terms of branch diagrams:



(b) The list is chopped to six as soon as we require that  $f$  be 1-1. In this case our diagram becomes



SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.1(L) cont'd]

More specifically, our six elements of  $\{f: f: A \rightarrow A, f \text{ is 1-1}\}$  are

$$\begin{array}{llll} (1) & I(1) = 1 & (2) & g(1) = 1 & (3) & h(1) = 2 & (4) & f(1) = 2 \\ & I(2) = 2 & & g(2) = 3 & & h(2) = 1 & & f(2) = 3 \\ & I(3) = 3 & & g(3) = 2 & & h(3) = 3 & & f(3) = 1 \end{array}$$

$$\begin{array}{ll} (5) & k(1) = 3 \\ & k(2) = 1 \\ & k(3) = 2 \end{array} \quad \begin{array}{ll} (6) & q(1) = 3 \\ & q(2) = 2 \\ & q(3) = 1 \end{array}$$

(c) We know that  $f^{-1}$  is characterized by

$$f^{-1}(f(1)) = 1; \quad f^{-1}(f(2)) = 2; \quad \text{and} \quad f^{-1}(f(3)) = 3 \quad (\underline{i})$$

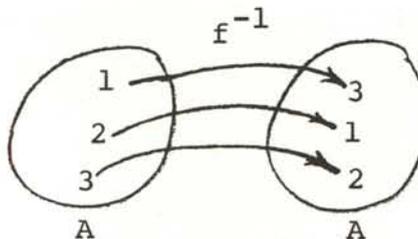
In this exercise  $f(1) = 2, f(2) = 3, f(3) = 1$ ; hence  $(\underline{i})$  becomes:

$$f^{-1}(2) = 1, \quad f^{-1}(3) = 2, \quad f^{-1}(1) = 3$$

That is

$$\left. \begin{array}{l} f^{-1}(1) = 3 \\ f^{-1}(2) = 1 \\ f^{-1}(3) = 2 \end{array} \right\}$$

Pictorially:



[If we go back to (b) we see that

$$\begin{array}{l} k(1) = 3 \\ k(2) = 1 \\ k(3) = 2 \end{array}$$

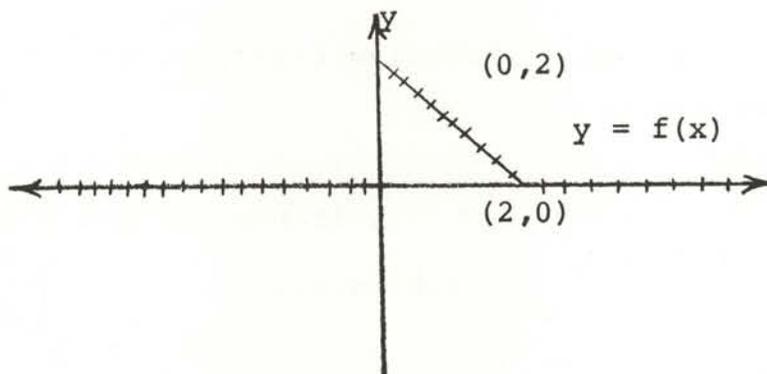
$$\therefore f^{-1}(x) = k(x) \text{ for all } x \in A \text{ and } \text{dom } f^{-1} = \text{dom } k (= A)$$

$$\therefore \underline{f^{-1} = k}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

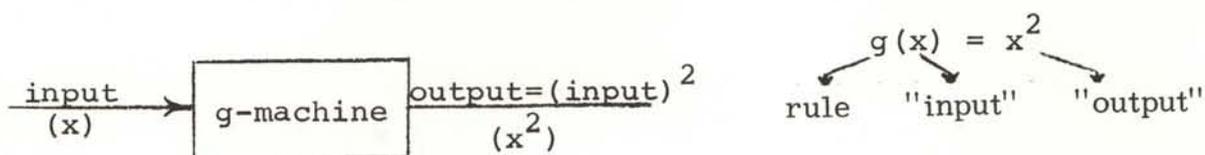
1.4.2(L)

(a)  $y = 2 - x$  represents a straight line whose slope is  $-1$  and whose  $y$ -intercept is  $2$ . Thus, our graph is this line on the interval  $0 \leq x \leq 2$ . Elsewhere, the graph is the  $x$ -axis (that is,  $y = f(x) = 0$ ). Thus, the graph is given by



Before proceeding to part (b) of this problem, a few words of review about graphs and functions might be in order.

Consider, for example, the function  $g$ , where  $g(x) = x^2$ . In terms of a picture



For the purpose of the problem we are dealing with, it is now important to note that the use of the symbol " $x$ " was not at all relevant. We could have written that  $g(y) = y^2$ ,  $g([\ ] ) = [\ ]^2$ , or  $g(3x + 4) = (3x + 4)^2$ . What is important is that our notation indicates that  $g$  is the rule which assigns to any number its square.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

With this in mind let us now return to our Exercise.

(b) (i) We are given that  $f(x) = \begin{cases} 2 - x, & \text{if } 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

In terms of a less restrictive notation, we could rewrite our rule,  $f$ , by saying:

$$f([ ]) = \begin{cases} 2 - [ ], & \text{if } 0 \leq [ ] \leq 2 \\ 0, & \text{otherwise.} \end{cases} \quad \left. \vphantom{f([ )}} \right\} \text{and this is how we shall visualize } f \text{ throughout this entire exercise.}$$

In particular, if we now replace  $[ ]$  by  $-x$ , we see that:

$$f(-x) = \begin{cases} 2 - (-x), & \text{if } 0 \leq -x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Performing the indicated algebraic operations, we see that:

$$f(-x) = \begin{cases} 2 + x, & \text{if } -2 \leq x \leq 0^* \\ 0, & \text{otherwise} \end{cases}$$

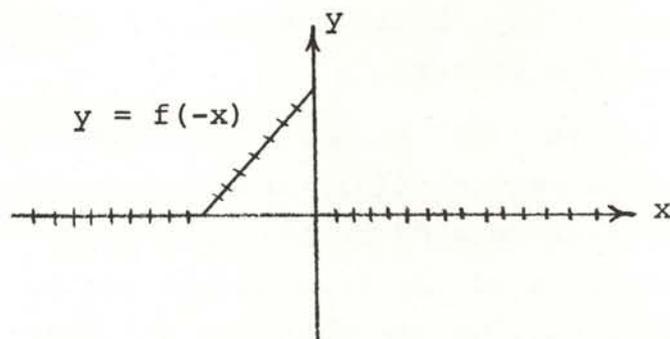
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\*Here we must be a bit careful in our use of inequalities. Observe that when we multiply both sides of an inequality by the same NEGATIVE number, we reverse the order of the inequality. See Exercise 1.3.8(a).

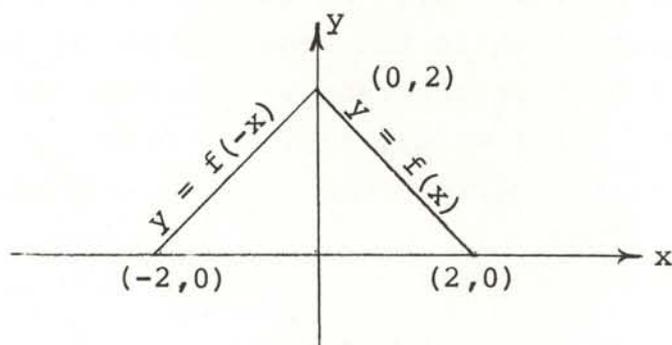
SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

If we were now to graph  $f(-x)$ , we could find that:



We could now go one step further and plot the graphs of  $f(x)$  and  $f(-x)$  in the same diagram. This yields:



and now, it is not difficult to conjecture that  $f(x)$  and  $f(-x)$  seem to be related by the fact that their graphs are the mirror image of one another with respect to the y-axis.

However, it is important to realize that while this is a nice visual interpretation for distinguishing  $f(x)$  from  $f(-x)$ , we never have to do this to find  $f(-x)$ . Rather we need only think analytically of  $-x$  replacing  $x$  as the input to the "f-machine". In summary, the graph allows us to think more "pictorially" - and,

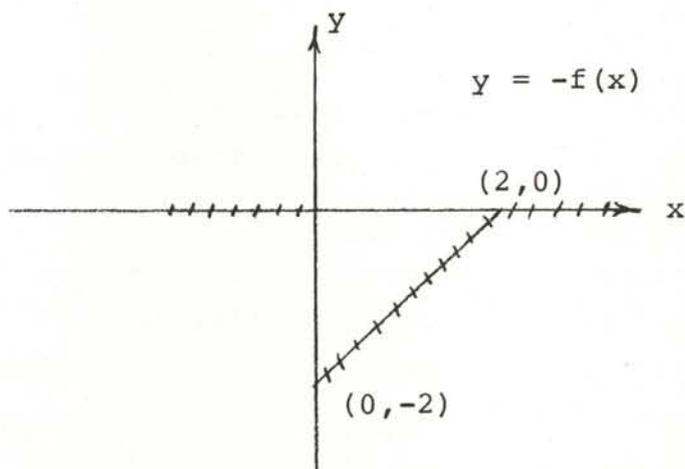
SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

hence, more "intuitively", but whenever we are in doubt as to what is happening, there is no substitute for being familiar enough with our notation so that we may obtain the desired result analytically, without the aid of a picture.

(ii) Among other things, (ii) is designed to show you that there is a basic difference between  $f(-x)$  and  $-f(x)$ . In the case of  $f(-x)$ , we see that this is related to our original function  $f(x)$  in the sense that we have changed the sign of the input. On the other hand  $-f(x)$  indicates that we are changing the sign of the output (recall that  $f(x)$  corresponds to the output of the "f-machine").

In terms of a graph, observe that the x-axis plays the role of the input, while the y-axis plays the role of the output. Thus, since  $-f(x)$  is the output which is the negative of the output  $f(x)$ , we see that  $-f(x)$  and  $f(x)$  have the same y-magnitude but are oppositely directed. In other words,  $f(x)$  and  $-f(x)$  are related in that their graphs are the mirror image of one another with respect to the x-axis. Thus:



SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

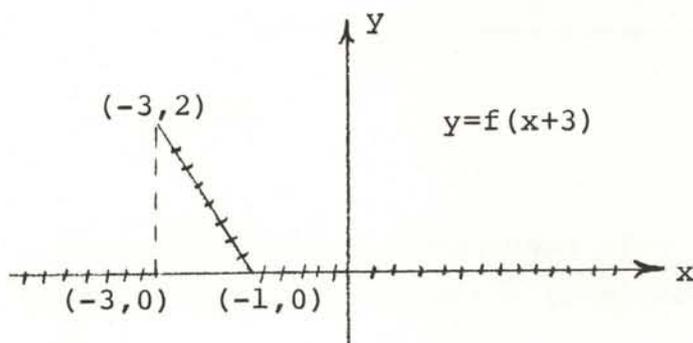
(iii) Here again we must remember that whether  $f$  yields  $2 - x$  or  $0$  as an output depends on what the input is. For example, if we change the name of the independent variable (that is, the input) from  $x$  to  $x + 3$ , our rule becomes:

$$f(x + 3) = \begin{cases} 2 - (x + 3) & \text{if } 0 \leq (x + 3) \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

That is:

$$y = f(x + 3) = \begin{cases} -x - 1 & \text{if } -3 \leq x \leq -1 \\ 0, & \text{otherwise} \end{cases}$$

This leads to the graph:



Notice that  $f(x+3)$  is another bona fide function of  $x$ . That is we could write:

$$g(x) = \begin{cases} -x - 1 & \text{if } -3 \leq x \leq -1 \\ 0, & \text{otherwise} \end{cases}$$

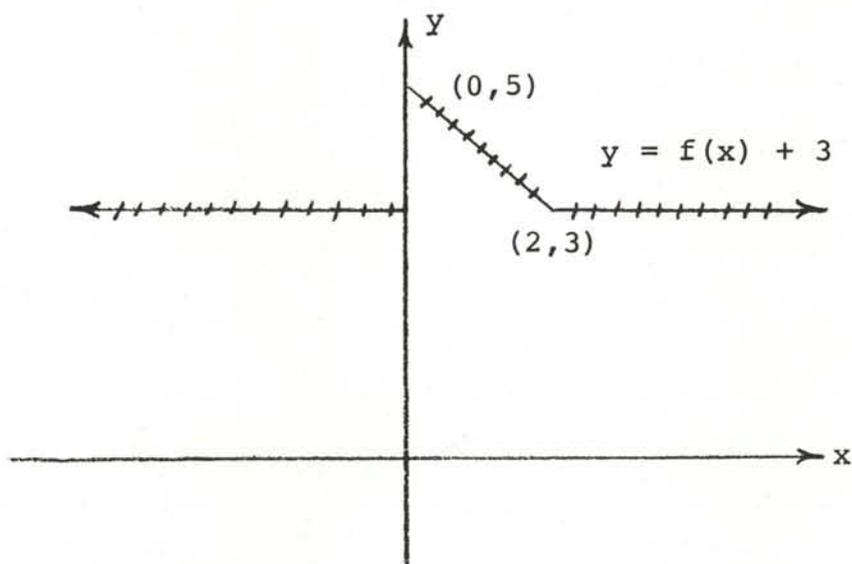
whereupon  $g(x) = f(x+3)$

Quite in general the graph of  $f(x + a)$  is the same as the graph of  $f(x)$  except that it is shifted a units to the left.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

(iv) Again we wish to show the importance of the order of operation. In this case we are going to show the difference between  $f(x + a)$  and  $f(x) + a$ . Specifically, since  $f(x)$  is graphically identified with the y-axis,  $f(x) + 3$ , is point by point 3 units higher than the graph of  $f(x)$ . Thus:



Analytically, if  $h(x) = f(x) + 3$ , then  $\text{dom } h = \text{dom } f$  but each output from the  $h$ -machine exceeds by 3 the corresponding output of the  $f$ -machine.

(v)  $f(2x)$  has the same range of outputs as does  $f(x)$ , but its graph is "compressed", compared with the graph of  $f(x)$ . Rather than tackle this idea too abstractly, let us investigate it from the point of view of our specific problem. By our definition of  $f$ , we have:

$$f(2x) = \begin{cases} 2 - (2x) & \text{if } 0 \leq 2x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

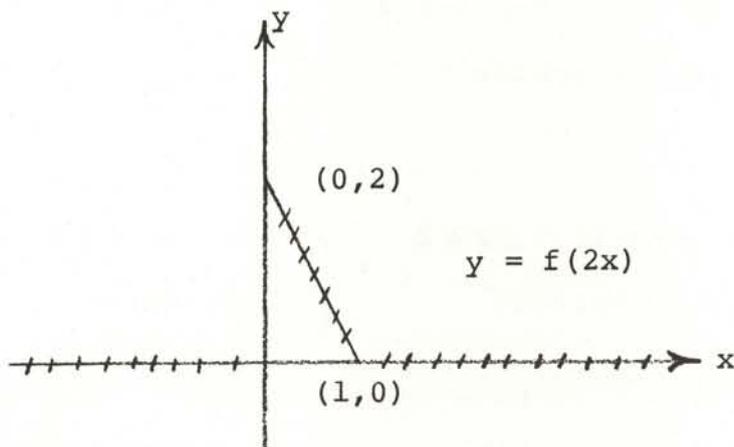
SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

That is,

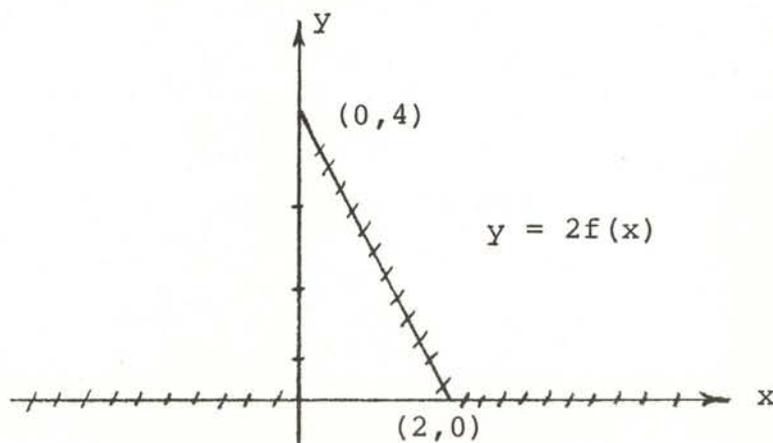
$$f(2x) = \begin{cases} 2 - 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we obtain



It is not too difficult to see that in this case the graph seems to be the original one "suitably compressed".

(vi) Again we are emphasizing the order of operation.  $f(2x)$  and  $2f(x)$  are quite different.  $2f(x)$  has the same inputs as does  $f(x)$ , but in this case each output is double the output of  $f(x)$ . Thus, our graph here is given by



SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

Actually, both (ii) and (vi) are special cases of a more general result. Namely, it should be noted that, as ominous as it may appear,  $f(x)$  is merely a number. Thus, if  $m$  is any number so also is  $mf(x)$ . In fact it is precisely that number which is  $m$  times  $f(x)$ .

In terms of our specific problem, all this means is that if

$$f(x) = \begin{cases} 2 - x & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

then

$$mf(x) = \begin{cases} m(2 - x) & \text{if } 0 \leq x \leq 2 \\ m(0), & \text{otherwise} \end{cases} = \begin{cases} 2m - mx, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

That is, as "machines,"  $f$  and  $mf$  have the same inputs, but for a given input the output of the  $mf$ -machine is  $m$  times that of the  $f$ -machine.

(vii) Finally, we indicate the effect of two of the previous types in a single problem. Specifically, our definition of  $f$  yields

$$f(2x + 3) = \begin{cases} 2 - (2x + 3), & \text{if } 0 \leq 2x + 3 \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

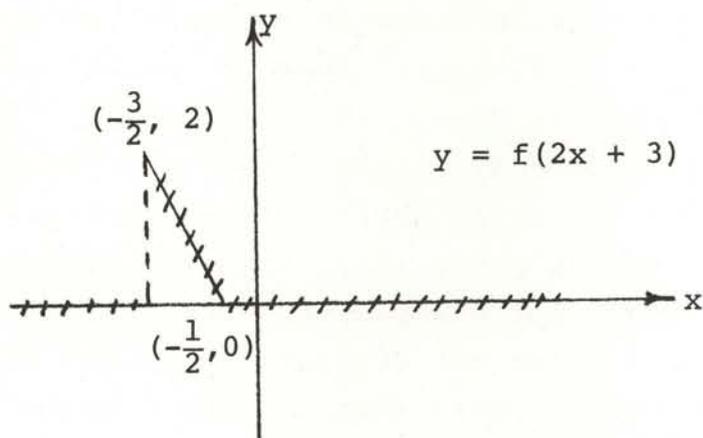
That is,

$$f(2x + 3) = \begin{cases} -2x - 1, & \text{if } -3/2 \leq x \leq -1/2 \\ 0, & \text{otherwise} \end{cases}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.2(L) cont'd]

Thus,



Observe here that we must again be careful and not decide that the  $+3$  shifts us 3 units to the left. In fact, it is easy to see from our specific problem that the shift was only  $3/2$  to the left. The point is that, analytically, we are saying that  $f(2x + 3)$  is  $f(2[x + 3/2])$ .

At any rate, the main aim of this problem is to make sure that you have the analytic tools by which one shifts, raises, lowers, compresses, etc. graphs of given functions, and that you learn to feel at home with such expressions as  $f(-x)$ ,  $f(2x)$ ,  $f(2x + 3)$ , etc.

1.4.3(L)

The major values of this problem are (1) it introduces us to the concept of CONSTRAINTS (which, among other things, are useful in the study of linear programming) and (2) it affords us another excellent example of the value of graphical techniques in mathematical analysis.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.3(L) cont'd]

To begin with, note that nowhere in this problem are we forced to view things geometrically. That is, we are given a certain SET of ordered number pairs,  $(x,y)$  and we wish to find for which of these pairs  $3x + 4y$  is the least and for which pair it is the greatest. Of course if there were no constraints (restrictions) placed on the pairs  $(x,y)$  we could make  $3x + 4y$  as large as possible just, for example, by making  $x$  and  $y$  "as large as possible". But there are constraints placed on the ordered pairs. Indeed, it appears that we must "locate" the required set of ordered pairs by solving the given system of inequalities.

Now if we wish to restate this problem in the language of sets, observe that our three inequalities define three sets of points;  $S_1$ ,  $S_2$ , and  $S_3$ , where:

$$S_1 = \{(x,y) : x + y \geq 2\}$$

$$S_2 = \{(x,y) : y \leq 3x + 2\}$$

$$S_3 = \{(x,y) : 5x \leq 10 - y\}$$

and the domain  $S$  of this problem is then simply  $S_1 \cap S_2 \cap S_3$ .

We could then proceed analytically and try to solve this problem. However, since we want to establish the benefits of a geometrical approach, we shall proceed to solve this problem pictorially.

If we think of  $S_1$ ,  $S_2$ , and  $S_3$  in geometrical terms then we are viewing ordered pairs of numbers as points in the plane (which again emphasizes the analytic geometry theme). What domain is

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.3(L) cont'd]

described by the set of points  $(x,y)$  for which  $x + y \geq 2$ ? Well,  $x + y = 2$  represents the straight line whose  $x$ - and  $y$ - intercepts are each 2. (One way of seeing this is to use the result of Exercise 1.3.1(L) and rewrite the equation as  $x/2 + y/2 = 1$ . Another way is to write the equation as  $y = -x + 2$  and use the slope-intercept formula to conclude that the line has slope equal to  $-1$  and  $y$ -intercept equal to 2. The "correct" way depends on what our mission is. As we shall soon see, in this type of problem the slope-intercept form is the more desirable.)

Now it may seem "intuitively obvious" that since  $x + y = 2$  is a straight line then  $x + y \geq 2$  must be the region which consists precisely of all those points in the plane which lie on or ABOVE this line (such a region is called a HALF PLANE. In particular, the regions above and below the  $x$ -axis, or the  $y$ -axis, are examples of half planes). (It turns out that our intuition is correct in this case, but in a way it was because of a fortunate choice of signs. For example,  $x - y = 2$  also represents a straight line; yet the region  $x - y > 2$  represents the half plane BELOW the line  $x - y = 2$ .) With regard to an earlier remark, let us observe that one of the safest ways of determining whether the half plane is above or below the line is to use the slope-intercept form of the line. That is, suppose we have the line  $y = mx + b$ . Then it is easy to see that for a given value of  $x$ ,  $y$  must be given by  $mx + b$  if the point is to be on the line; while if  $y$  is less than  $mx + b$  the point  $(x,y)$  is below the line. For example, given the line  $y = 3x + 2$  we see that  $(1,5)$  is on the line. However  $(1,4)$  which lies below the point  $(1,5)$  must therefore lie below the line. In a similar way, if  $y$  is greater than  $mx + b$  then the point  $(x,y)$  lies above the line.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.3(L) cont'd]

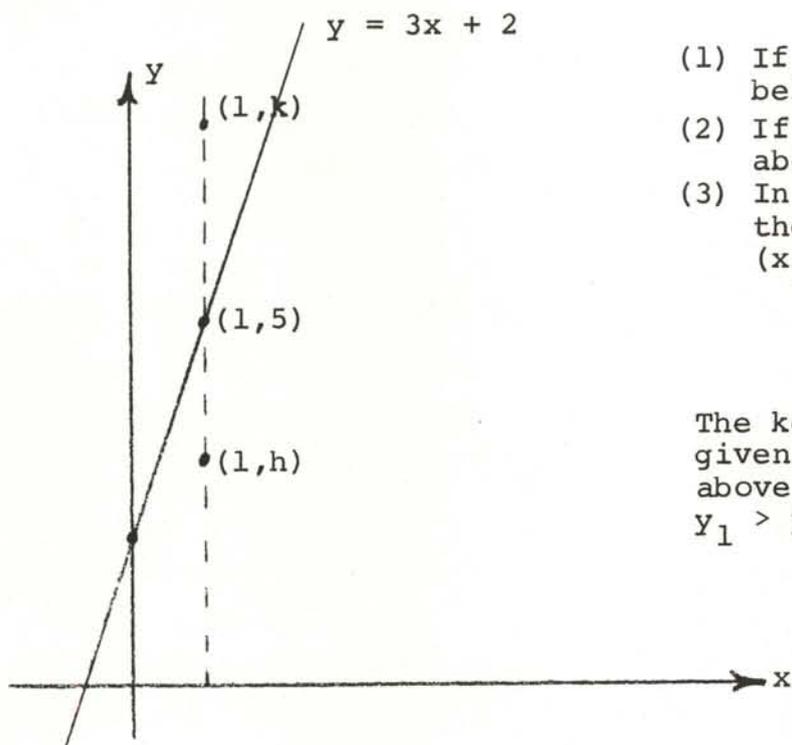


Figure 1

- (1) If  $h < 5$  then  $(1, h)$  lies below the line  $y = 3x + 2$
- (2) If  $k > 5$  then  $(1, k)$  lies above the line  $y = 3x + 2$
- (3) In general if  $(x_1, y_1)$  is on the line  $y = mx + b$  then  $(x_1, y)$  is
 

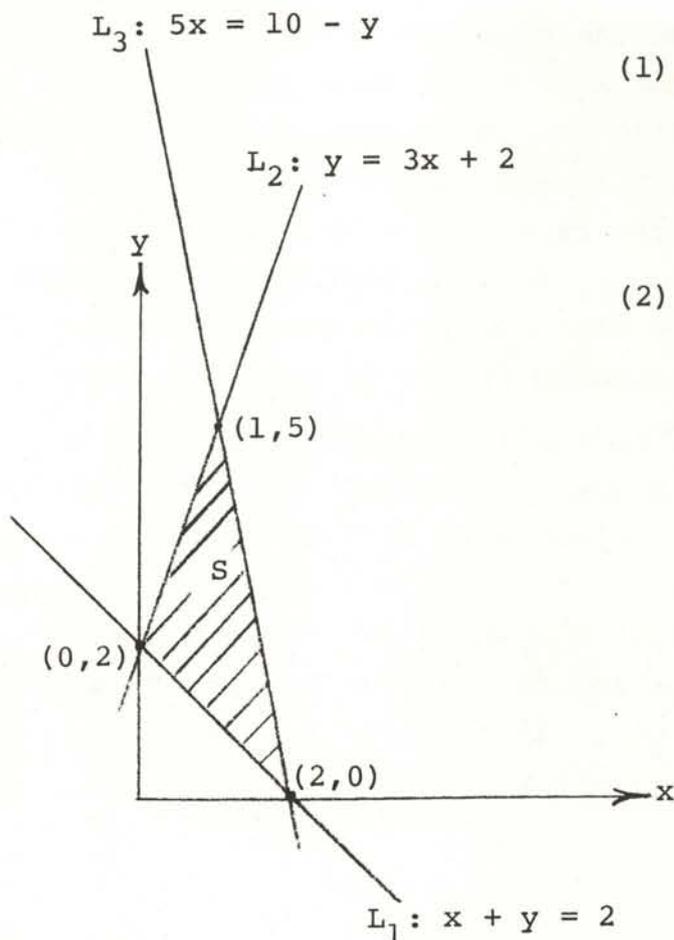
}	below the line if
	$y < mx_1 + b$
}	above the line if
	$y > mx_1 + b$

The key point is that for a given  $x_1$ ,  $(x_1, y_1)$  lies vertically above  $(x_1, y_2)$  if and only if  $y_1 > y_2$

Applying this to the example  $x - y > 2$ ; we see that  $-y > -x + 2$ . Recalling that multiplying an inequality through by a negative number reverses the sense of the inequality, (see, for example, Exercise 1.3.8(a)) we see that  $x - y > 2$  implies that  $y < x - 2$ . But if  $y < x - 2$  the point  $(x, y)$  lies below the line  $y = x - 2$  which is the same line as that given by the equation  $x - y = 2$ .

At any rate, it follows that the region  $S_1$  is the half-plane on and above the line  $x + y = 2$ ; the region  $S_2$  is the half plane on or below the line  $y = 3x + 2$ ; and the region  $S_3$  is the half plane on or below the line  $5x = 10 - y$ . Then  $S$  is the intersection of these three half planes and  $S$  is sketched below:

[1.4.3(L) cont'd]



(1) The points  $(0, 2)$ ,  $(2, 0)$  and  $(1, 5)$  are obtained as the points at which each pair of lines intersect. That is, by solving a pair of linear simultaneous equations.

(2)  $S$  is determined pictorially by the fact that a point is in  $S$  if and only if (1) it is on or above  $L_1$  AND (2) on or below  $L_2$  AND (3) on or below  $L_3$ .

Figure 2

So far, so good; we have at least determined the picture of our domain  $S$ . However, this is only part of what we must do. The next thing we must do is to decide where  $3x + 4y$  is minimum (or maximum) for the set of points  $(x, y)$  in  $S$ . Now it is clear that for a given point  $(x, y)$   $3x + 4y$  has some fixed value, say,  $k$ . There are many points in the plane which yield this same value for  $3x + 4y$ . In fact the set of such points is given by the equality

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.3(L) cont'd]

$3x + 4y = k$ , which happens to be the equation of a straight line also. (In this type of situation  $k$  is known as a parameter - a kind of "variable constant". That is, the choice of  $k$  varies with a particular choice of  $x$  and  $y$ , but once chosen in a problem  $k$  remains constant. In this context  $3x + 4y = k$  is called a one-parameter family of straight lines. In this instance, changing  $k$  does not affect the slope of the line, only its position. Thus  $3x + 4y = k$  represents a one-parameter family of parallel lines.)

Next we observe that  $(2,0)$  is the lowest point at which a member of the family  $3x + 4y = k$  meets any point in  $S$ . In this instance since  $x = 2$  and  $y = 0$ , we see that  $3x + 4y = 6$ . Moreover, if  $3x + 4y < 6$  then  $(x,y)$  is below the line  $3x + 4y = 6$ ; and this in turn means that any point  $(x,y)$  for which  $3x + 4y < 6$  lies OUTSIDE  $S$ . Thus the least value for  $3x + 4y$  for any point  $(x,y)$  in  $S$  is 6 and this occurs at  $(2,0)$ . In terms of a picture:

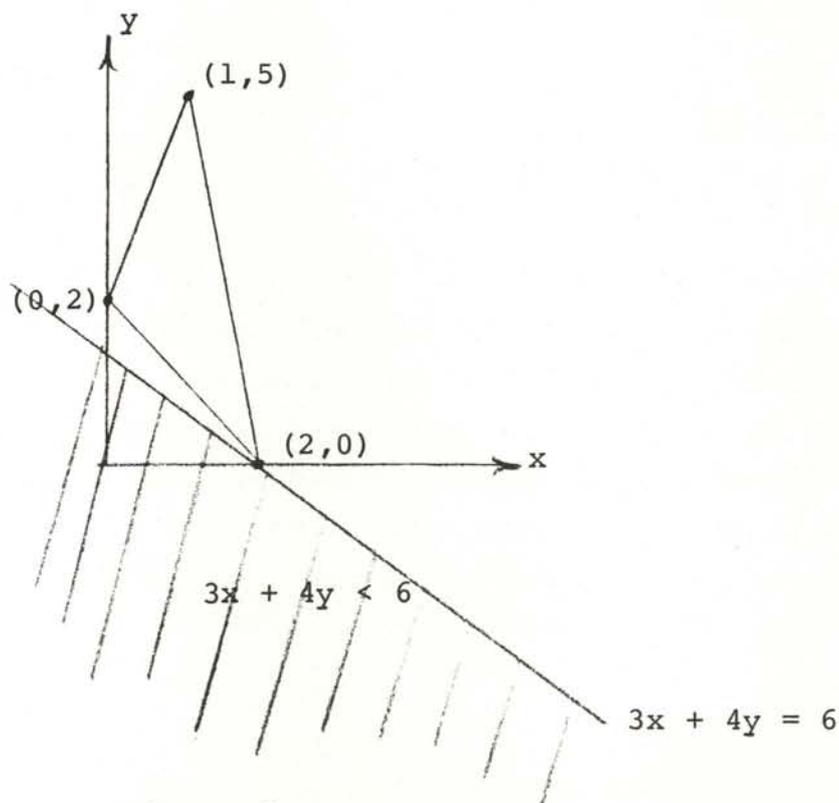


Figure 3

I.4.16

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.3(L) cont'd]

Similarly, the highest point at which a member of the family  $3x + 4y = k$  meets a point of  $S$  is  $(1,5)$ ; and for this choice of point we see that  $k = 3(1) + 4(5) = 23$ . Thus the greatest value that  $3x + 4y$  can have for any point  $(x,y)$  in  $S$  is 23 and that occurs when  $x = 1$  and  $y = 5$ . Again pictorially:

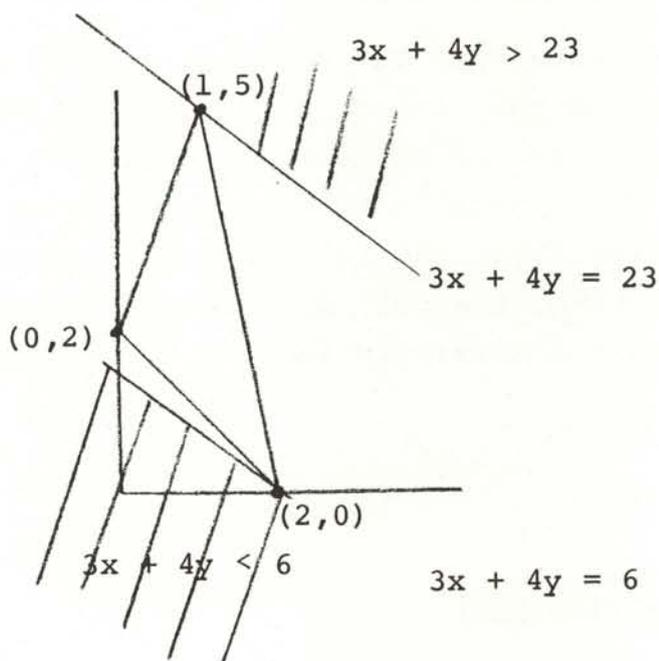


Figure 4

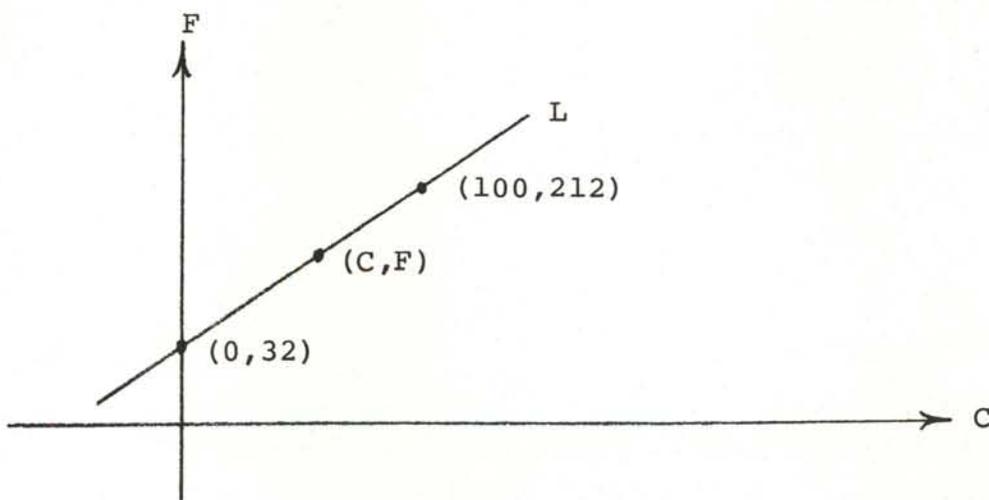
Computational complications can set in, but in general our above discussion shows how geometry may be used to solve maximizing or minimizing functions subject to systems of constraints placed on the independent variables. It is not our purpose here to explore further the idea of linear programming. All we wanted to do was to present the fundamental ideas of functions, lines, and inequalities in terms of one such problem.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

1.4.4(L)

As far as part (a) is concerned, let us first observe a rather subtle difference between pure and applied mathematics. If we were dealing with  $x$  and  $y$  instead of  $C$  and  $F$ , this would be a "pure" geometry-oriented problem. However, merely by giving physical significance to the coordinate axes, the problem suddenly becomes "applied" mathematics. We only wanted to take this opportunity to reinforce this close bond that exists in almost all of mathematics between theory and application.

At any rate, once we know that the graph is a straight line, all we need is two points on the line to determine the line; and we are given precisely this amount of information in the problem. In essence, all we want to do here is to find the equation of a line given that the points  $(0,32)$  and  $(100,212)$  are points on the curve. As we mentioned above, the only difference is that the  $y$ -axis is now labeled  $F$  and the  $x$ -axis is labeled  $C$ . Pictorially:



$$\text{The slope of } L = \frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$$

$$\therefore (C, F) \text{ is on } L \text{ if and only if } \frac{F-32}{C-0} = \frac{9}{5}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.4(L) cont'd]

$$\therefore \boxed{F = \frac{9C}{5} + 32} \quad (1)$$

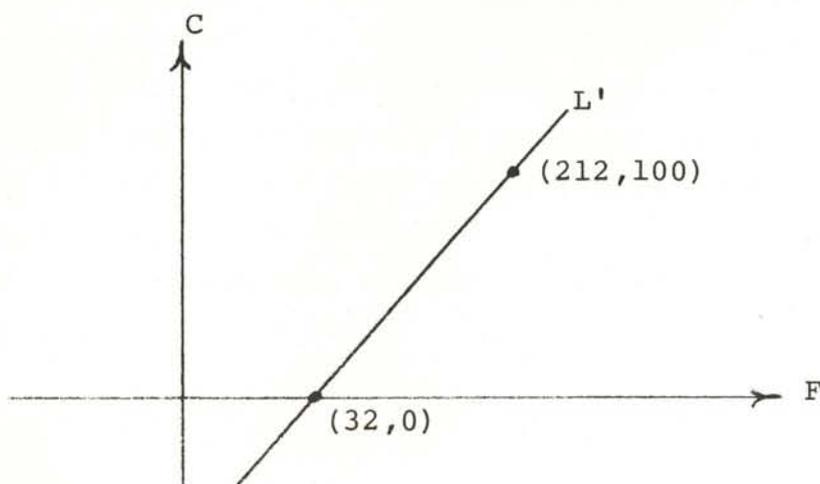
The solution to part (b) follows as a corollary to (1). Namely, in (1) we let  $C = F$  to obtain

$$C = \frac{9C}{5} + 32$$

whence  $\frac{-4C}{5} = 32$ ; or  $\boxed{C = -40}$ .

At this point let us admit that if all we want is the answer to this problem, there is nothing more for us to do. However, there are a few asides that are worth mentioning.

For one thing let us observe that there was nothing in the wording of this problem that forced us to adopt the order  $(C,F)$ . It would have been just as logical to think of the ordered pair as  $(F,C)$ . In this event, our graph would have been:



SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.4(L) cont'd]

$$\text{The slope of } L' = \frac{100-0}{212-32} = \frac{5}{9}$$

$$(F,C) \text{ is on } L' \text{ if and only if } \frac{C-0}{F-32} = \frac{5}{9}$$

∴ The equation of  $L'$  is given by:

$$\boxed{C = \frac{5}{9} (F-32)} \quad (2)$$

The thing to note is that equations (1) and (2) are INVERSES of one another. Indeed, had we obtained either equation, we could have derived the other by the usual algebraic manipulations. In still other words, (1) expresses the situation in terms of  $C$  being the input (independent variable) while (2) expresses the same relation in terms of  $F$  being the independent variable.

It should also be stressed still another time, that the relation between  $C$  and  $F$  does NOT require that we draw a picture - it is merely that the picture suggests things more directly than the analytical approach. Quite in general, if two variables say  $x$  and  $y$  are related linearly it means that there exist constants  $a$  and  $b$  such that  $y = ax + b$ ; and the graph of this algebraic expression happens to be a straight line. In other words, without recourse to any picture, we could have started with the knowledge that  $C = aF + b$  (or, as we have already mentioned, we could reverse the roles of  $C$  and  $F$  which would give us different constants - say,  $F = a'C + b'$ ). We would then have two unknown  $a$  and  $b$ , but knowing that  $C = 0$  and  $F = 32$  and that  $C = 100$  when  $F = 212$ ; we could write:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

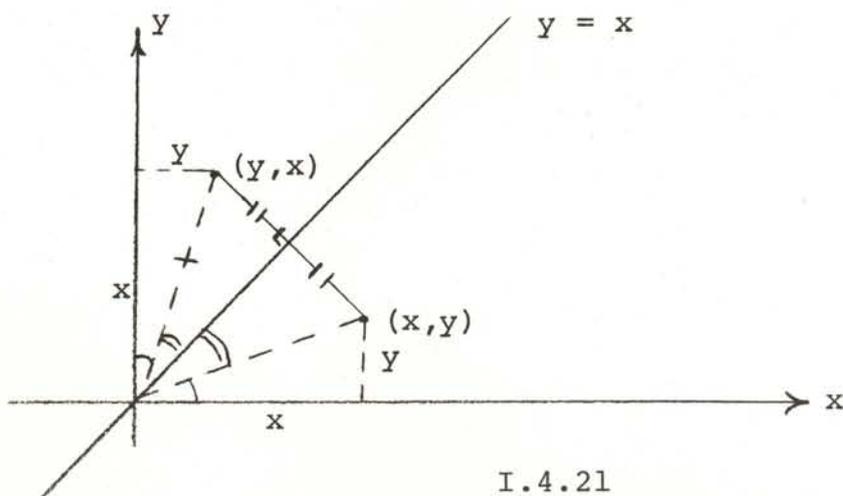
[1.4.4(L) cont'd]

$$\left. \begin{aligned} 0 &= 32a + b \\ 100 &= 212a + b \end{aligned} \right\}$$

from which we could determine  $a$  and  $b$  and thus obtain the same result as before.

Another point worth mentioning is that it is almost instantly clear from a geometric point of view that the answer to part (b) HAD TO BE in the affirmative. For the slope of the line  $C = F$  is 1 while the slope of either  $L$  or  $L'$  (the choice depending on whether we think of the order  $(C,F)$  or the order  $(F,C)$ ) is not 1. Thus the line  $C = F$  is not parallel to, say,  $L$ . Hence these two lines intersect at precisely one point; and this point, since it belongs to  $C = F$ , must be the point at which the temperature reading is the same in both units.

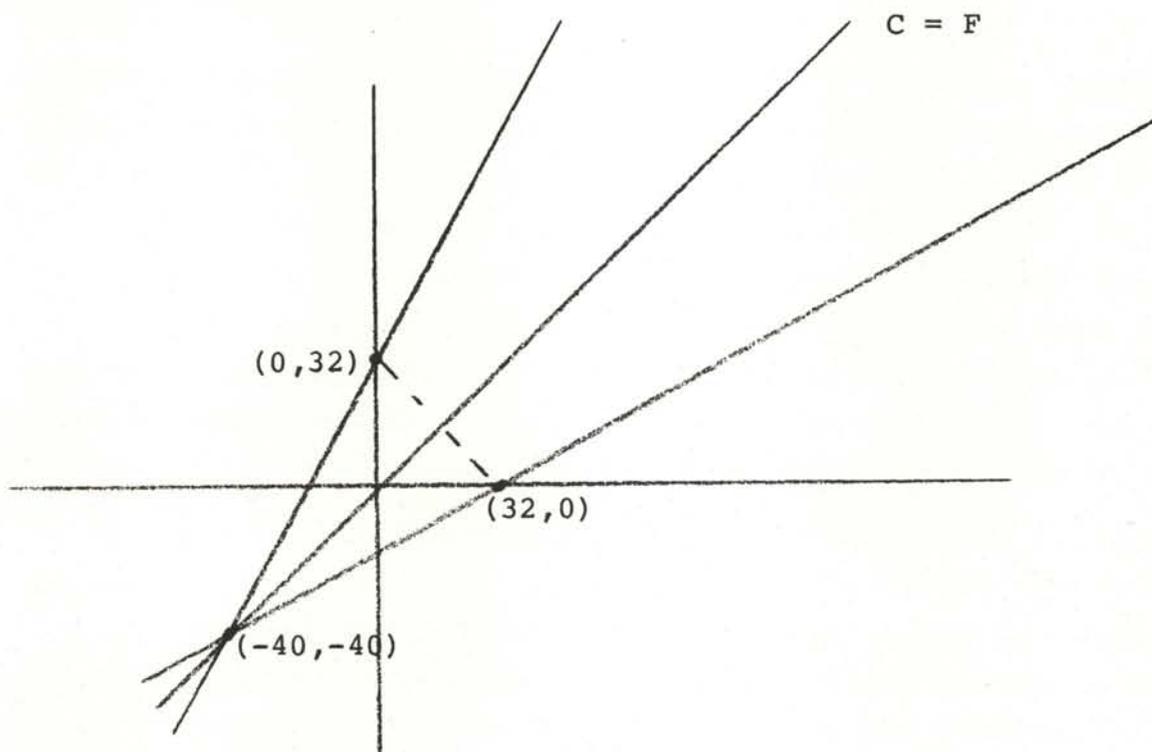
Finally, let us make an observation concerning the graphs of a function and its inverse. Namely, the two graphs are symmetric with respect to the 45 degree line. This follows from the fact that the inverse function merely reverses the role of the dependent and independent variables. Stated in terms of graphs, if  $(x,y)$  belongs to the curve  $y = f(x)$  then  $(y,x)$  belongs to the curve  $y = f^{-1}(x)$ , and conversely. However, as the following diagram shows,  $(x,y)$  and  $(y,x)$  are symmetric with respect to the line  $y = x$ :



SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.4(L) cont'd]

With regard to our problem, our equations (1) and (2) are inverse functions; hence the two graphs should be symmetric with respect to the line  $C = F$ . That this is indeed the case, may be seen from the figure below:



1.4.5(L)

There are three rather "trivial" properties of absolute values that will be of help to us in the analysis which follows:

(1)  $|a| \geq a$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.5(L) cont'd]

This follows from the fact that  $|a|$  is always positive. Thus if  $a$  is positive then  $|a| = a$ ; while if  $a$  is negative  $|a| > a$ . Since  $|a|$  is positive

$$(2) \quad |a|^2 = a^2$$

This follows from the fact that  $|a| = +\sqrt{a^2}$ . Thus:

$$|a|^2 = (+\sqrt{a^2})^2 = a^2.$$

$$(3) \quad |ab| = |a| |b| .$$

This follows from the fact that  $(+\sqrt{a^2})(+\sqrt{b^2}) = +\sqrt{a^2 b^2}$ ; and this in turn is  $+\sqrt{(ab)^2}$  or  $|ab|$ .

With these three facts we are ready to begin our solution:

(a) Let us start with the assumption that  $|a| < |b|$ . Since these two numbers are non-negative, we have already seen that we can conclude that  $|a|^2 < |b|^2$ ; but from (2) above this implies that  $a^2 < b^2$  - which is the desired result.

To complete (a) we must show that the converse is also true. That is, we must show that under the assumption that  $a^2 < b^2$ , it follows that  $|a| < |b|$ .

To this end, we know from (2) that  $a^2 < b^2$  implies that  $|a|^2 < |b|^2$ . This in turn means that  $|a|^2 - |b|^2 < 0$ . Thus  $(|a| + |b|)(|a| - |b|) < 0$ .

This can only happen if one pair of parentheses names a negative number. Since both  $|a|$  and  $|b|$  are non-negative, so also is their

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.5(L) cont'd]

sum. Hence  $(|a| + |b|)$  cannot be negative. Thus it must be  $(|a| - |b|)$  which is negative; and this says that  $|a| < |b|$ ; which completes our proof.

Now let us come to grips with a more vital question. Namely, why is it useful to know a result such as (a)? The answer lies in the fact that it gives us a way of comparing absolute values (magnitudes) without having to have recourse to the absolute value symbol. That is, a paraphrase of (a) is that we can compare the magnitudes of  $a$  and  $b$  merely by comparing those of  $a^2$  and  $b^2$ . We can illustrate this idea in the solution of (b).

(b) From the result of (a) we need only compare  $(|a| + |b|)^2$  and  $(a + b)^2$ .

$$\begin{aligned}\text{Now } (|a| + |b|)^2 &= |a|^2 + 2|a||b| + |b|^2 \\ &= |a|^2 + 2|ab| + |b|^2 \text{ [by (3)]} \\ &= a^2 + 2|ab| + b^2 \text{ [by (2)]}\end{aligned}$$

On the other hand:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$\begin{aligned}\text{Hence } (|a| + |b|)^2 - (a + b)^2 &= 2|ab| - 2ab \\ &= 2(|ab| - ab)\end{aligned}$$

But by (1)  $|ab| - ab$  is non-negative (since  $|ab| \geq ab$ )

Therefore  $(|ab| - ab)$  is non-negative.

Therefore  $(a + b)^2 \leq (|a| + |b|)^2$

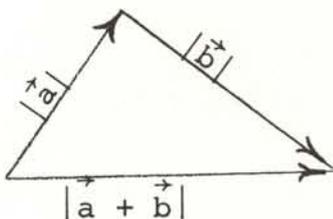
Therefore  $|a + b| \leq |a| + |b|$ ; as claimed.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.5(L) cont'd]

(To see this result more intuitively observe that if  $a$  and  $b$  have the same signs then  $a + b$  and  $|a| + |b|$  are equal. However if they have opposite signs then the magnitudes of  $a$  and  $b$  are subtracted from one another as we take the algebraic sum. On the other hand, by taking absolute values first we are insuring that all magnitudes being added are non-negative.)

This result is often referred to as THE TRIANGLE INEQUALITY. This name becomes clear if we think of numbers as lengths (vectors). Namely the triangle inequality merely says that the third side of a triangle can be no greater than the sum of the lengths of the other two sides (or the shortest distance between two points in the plane is the straight line segment which joins these points). That is:



(c) This result can be obtained intuitively in much the same way as we could do (b) intuitively. That is, for example, if  $a$  is positive and  $b$  is negative then  $a - b$  represents a SUM of two magnitudes whereas if we take absolute values first we subtract magnitudes.

From an analytical point of view (c) may be obtained as a corollary of (b) by a clever ruse. In many instances (as we shall see during the development of this course) it is wise to add 0 to an expression in a rather clever way. In (c) we observe that:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.5(L) cont'd]

$$|a| = |(a - b) + b|$$

but from (b)

$$|(a - b) + b| \leq |a - b| + |b| \therefore |a| \leq |a - b| + |b|$$

Hence:

$$|a - b| \geq |a| - |b|$$

In a similar way, we may write that

$$|b| = |(b - a) + a|$$

from which it follows that

$$|b - a| \geq |b| - |a| = -(|a| - |b|)$$

Since  $b - a$  and  $a - b$  have the same magnitude it follows that

$|a - b| = |b - a|$ ; whence

$$|a - b| \geq |a| - |b| \quad \text{AND} \quad |a - b| \geq -(|a| - |b|) \quad (1)$$

either  $|a| - |b|$  or  $-(|a| - |b|)$  equals  $\left| |a| - |b| \right|$  (why?) Therefore from (1) it follows that

$$|a - b| \geq \left| |a| - |b| \right|$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.5(L) cont'd]

Again in terms of geometry, it is easy to visualize the absolute value in terms of a picture (in fact this will be emphasized in the next exercise). In many situations, the geometry will be simpler for us to use than the analytic methods described in the solution of this exercise. However, while we do not mean to belittle the geometric approach (in fact, we encourage it wherever it is applicable) the fact remains that there are many cases in which only the analytic method is available to us (this is particularly true in the study of functions of more than two independent variables). For this reason we have elected to emphasize the analytic approach.

1.4.6(L)

Using the results of the previous exercise we can begin by observing that  $2 < |x - 3| \leq 4$  implies  $2^2 < |x - 3|^2 \leq 4^2$  which in turn implies that:

$$4 < (x - 3)^2 < 16$$

That is:

$$4 < x^2 - 6x + 9 \leq 16$$

From  $x^2 - 6x + 9 \leq 16$  we obtain

$$x^2 - 6x - 7 \leq 0$$

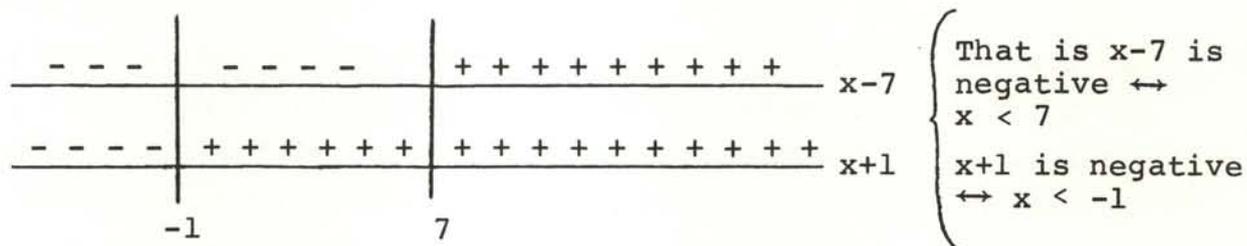
or

$$(x - 7) (x + 1) \leq 0 \quad (1)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 4: Functions

[1.4.6(L) cont'd]

(1) tells us that  $(x - 7)$  and  $(x + 1)$  have opposite signs or equal 0. A convenient graphical device here is to "draw" the sign of  $x - 7$  and  $x + 1$  in the same diagram. That is



We see at once that  $x - 7$  and  $x + 1$  have opposite signs if and only if  $-1 < x < 7$ .

Thus:

$$\{x|x^2 - 6x + 9 \leq 16\} = \{x|-1 \leq x \leq 7\}$$

Note: If  $a < b$  then  $\{x|a < x < b\}$  is called an open interval and is abbreviated by  $(a,b)$ .

Similarly  $\{x|a \leq x \leq b\}$  is called a closed interval and is abbreviated by  $[a,b]$ . Thus the difference between an open interval  $(a,b)$  and a closed interval  $[a,b]$  is that  $a$  and  $b$  belong to  $[a,b]$  but they do not belong to  $(a,b)$ .

One may also write  $(a,b]$  to abbreviate  $\{x:a < x \leq b\}$  as well as  $[a,b)$  to abbreviate  $\{x|a \leq x < b\}$ .

$$\text{Thus } (x - 3)^2 \leq 16 \leftrightarrow x \in [-1,7] \tag{2}$$

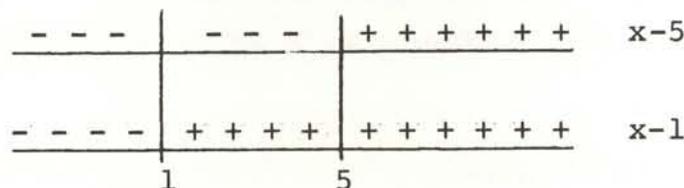
On the other hand  $4 < x^2 - 6x + 9$  implies that  $x^2 - 6x + 5 > 0$   
 or

$$(x - 5)(x - 1) > 0 \tag{3}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.6(L) cont'd]

(3) tells us that  $(x-5)$  and  $(x-1)$  must have the same sign. Thus:



or  $x$  must exceed 5 or else be less than 1.

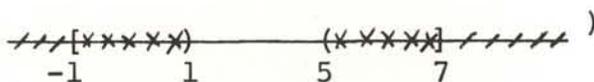
That is

$$\{x \mid 4 < x^2 - 6x + 9\} = \{x \mid x < 1\} \cup \{x \mid x > 5\} \quad (4)$$

(In terms of intervals, one often writes  $\{x \mid x < 1\}$  as  $(-\infty, 1)$  and  $\{x \mid x > 5\}$  as  $(5, \infty)$ .)

Combining (2) and (4) we see that:

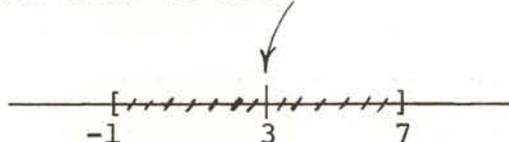
$$x \in [-1, 1) \cup (5, 7]$$

(Pictorially: )

So far we have employed (disregarding a few min or visual aids) only the analytic procedure.

To utilize graphs, let us observe that  $|x_1 - x_2|$  is simply the distance between  $x_1$  and  $x_2$  on the number line. Thus  $|x-3| \leq 4$  means that  $x$  must be no more than 4 units from 3. Pictorially:

$|x-3| \leq 4$  implies  $x$  is in here



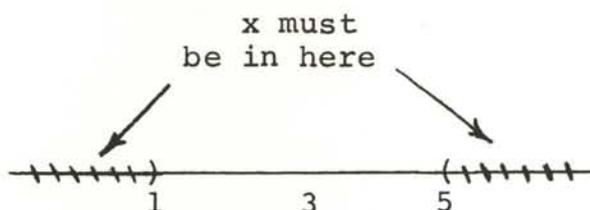
(Fig 1)

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.6(L) cont'd]

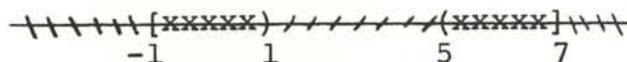
Thus we obtain the knowledge in (2) almost by inspection.

Similarly  $|x-3| > 2$  means that the distance between  $x$  and 3 exceeds 2. That is,  $x$  is more than 2 units from 3.



(Fig 2)

If we now superimpose (Fig 1) and (Fig 2) we see that:



(Fig 3)

The cross-hatched region in (Fig 3) is now the desired region. (Since xxxx is the result of being in \\\ and in //.)

1.4.7(L)

We are given

$$y = \sqrt{\frac{x}{x+1}} \quad (1)$$

Recalling that we have agreed to use single-valued functions (1) must be read as if it said:

$$y = +\sqrt{\frac{x}{x+1}}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.7(L) cont'd]

At any rate, to solve for  $x$  in terms of  $y$ , it might be advisable to square both sides of the given equation to obtain:

$$y^2 = \frac{x}{x+1} \quad (2)$$

whereupon we obtain:

$$(x+1)y^2 = x$$

It then follows that:

$$xy^2 + y^2 = x$$

or:

$$y^2 = x - xy^2 = x(1 - y^2)$$

Therefore:

$$\boxed{x = \frac{y^2}{1 - y^2}} \quad (3)$$

Equation (3) would appear to be the correct answer to our problem - EXCEPT THAT WE MUST NOW PAY ATTENTION TO OUR CONVENTION THAT WE MAKE ALL FUNCTIONS SINGLE-VALUED. In this respect, recall that we are assuming in (1) that only the positive square root is involved. (In still other words,  $a = b$  and  $a^2 = b^2$  are NOT synonyms; for  $a^2 = b^2$  means that  $a = b$  OR  $a = -b$ , that is,  $a = \pm b$ .) If we wanted the square root symbol to include both the

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.7(L) cont'd]

positive and negative roots, then (2) and (1) would have been synonyms. To get around this difficulty we replace (3) by the "amendment":

$$x = \frac{y^2}{1 - y^2} \quad (4)$$

where  $y \geq 0$ .

In any event, let us explore (4). If  $y$  is real then so also are  $y^2$  and  $1 - y^2$ . [Here we are using the "fact" that the sum, difference, or product of real numbers are again real numbers.] On the other hand, the quotient of two real numbers is real if and only if the divisor is not 0\*. Since the divisor in (3) is  $1 - y^2$  and since  $1 - y^2 = 0$  if and only if  $y = 1$  or  $-1$ ; it follows that  $y$  can assume any real values except 1 and  $-1$ . We can exclude the value of  $-1$  since (4) demands that  $y$  be non-negative. In the language of functions, we have shown that if  $f$  is defined by

$$f(x) = \sqrt{\frac{x}{x+1}}$$

then the range of  $f$  is the set of all non-negative real numbers EXCEPT for the number 1. More symbolically the range is given by  $\{x: x \geq 0 \text{ but } x \neq 1\}$ .

Lest the above seem merely like a drill exercise, let us observe that the process of going from (1) to (4) is the computational technique for constructing  $f^{-1}$  once  $f$  is known. That is,

---

\*We shall say more about this in the next unit.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.7(L) cont'd]

the function implied by (4) "undoes" the function implied by (1). In other words if we let

$$f(x) = \sqrt{\frac{x}{x+1}} \quad \text{and} \quad g(x) = \frac{x^2}{1-x^2} \quad (x \geq 0)$$

it is readily verified that  $f(g(x)) = x$ .

Of course, we should also recall from our discussion of functions that unless  $f$  is 1-1 it cannot have an inverse. Again it is not difficult to show that:

$$+\sqrt{\frac{x}{x+1}} = +\sqrt{\frac{y}{y+1}} \leftrightarrow x = y$$

(That is:  $f(x) = f(y) \rightarrow x = y$ .)

In fact:

$$\begin{aligned} +\sqrt{\frac{x}{x+1}} = +\sqrt{\frac{y}{y+1}} &\leftrightarrow \frac{x}{x+1} = \frac{y}{y+1} \\ &\leftrightarrow xy + x = xy + y \\ &\leftrightarrow x = y \end{aligned}$$

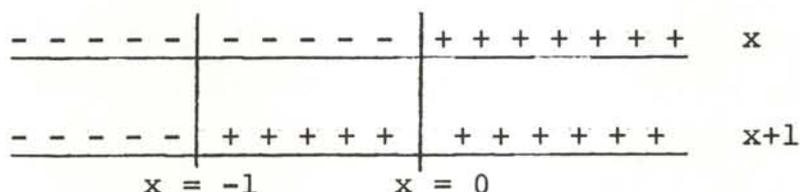
In a similar way  $-\sqrt{\frac{x}{x+1}} = -\sqrt{\frac{y}{y+1}} \rightarrow x = y$

To continue with the material required in this problem, let us return to (1). Since we are not allowed to divide by 0, it follows that  $x + 1 \neq 0$ , or that  $x \neq -1$ . But there is still more to worry about since (1) involves extracting a square root.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

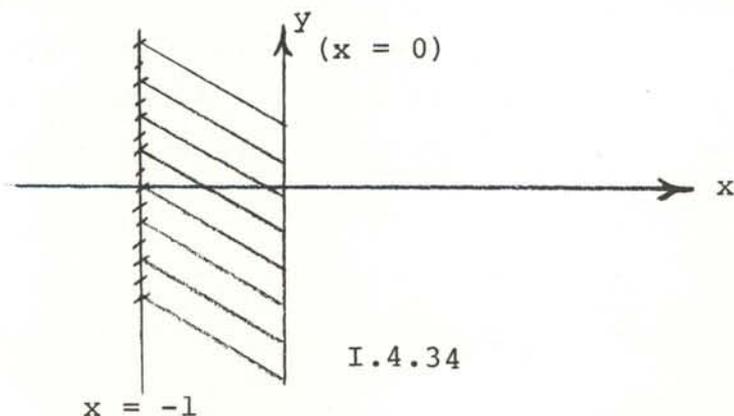
[1.4.7(L) cont'd]

In other words, from (1) if  $y$  is to be real  $x/(x+1)$  cannot be negative, since the square root of a negative number is an imaginary number. Now the only way a quotient can be negative is for numerator and denominator to have different signs. Using the technique introduced in Exercise 1.4.5(L)



whereupon it readily follows that  $x$  and  $x+1$  have different signs only if  $-1 < x < 0$ . Coupling this result with our previous result that  $x \neq -1$ , we have that  $x$  can take on all real values except for those in the interval  $[-1,0)$ .

While we have technically completed the questions asked in this exercise, it might be well to extrapolate these ideas in terms of graphs. To this end, suppose we were asked to sketch the curve whose cartesian equation was defined by (1). We would have performed the same operations as we did above, and, among other things, we would have concluded that  $x$  could not satisfy  $-1 \leq x < 0$ . Graphically this excludes any point from the shaded region shown below (since for any point in the shaded region the point  $(x,y)$  has its  $x$ -coordinate in the restricted range).

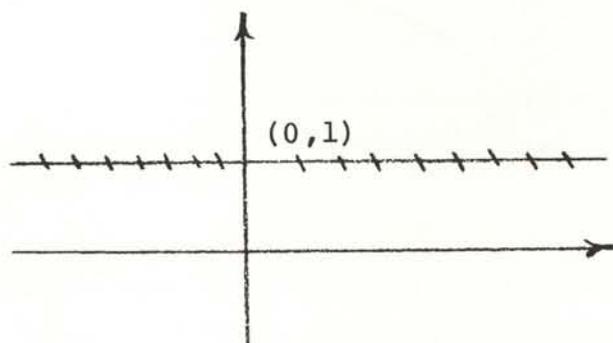


SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

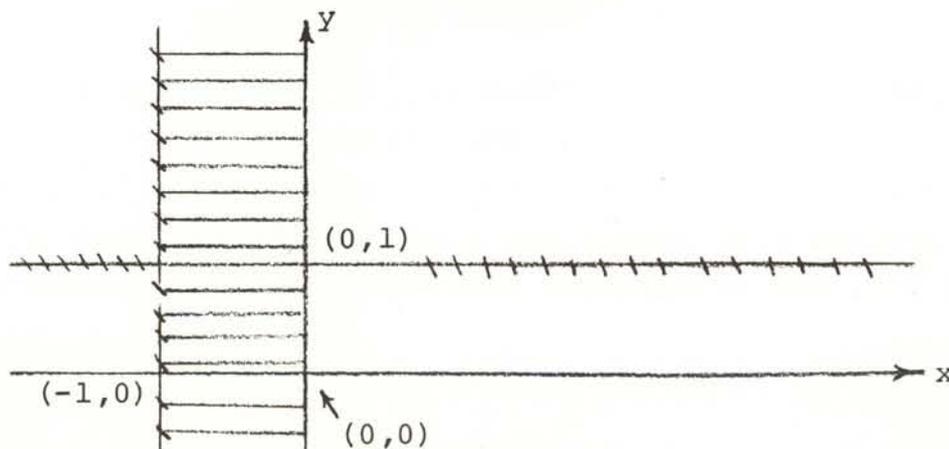
[1.4.7(L) cont'd]

[Here again we see another nice example of the difference between a point and a dot - and a possible shortcoming of our picture. Since a line has no thickness (of course the model we drew to indicate the line does have thickness) we cannot tell by looking whether points on the line are included or excluded. It is for this reason that notation such as  $[-1,0]$ ,  $(-1,0)$ ,  $[-1,0)$ , and  $(-1,0]$  is particularly useful.]

We next observe that  $y = 1$  represents the equation of a line and since we have seen that  $y \neq 1$ , we can conclude that no point in the graph can have its  $y$ -coordinate on this line. In other words:



Putting both diagrams together, we obtain the region where the graph does NOT exist. Thus:

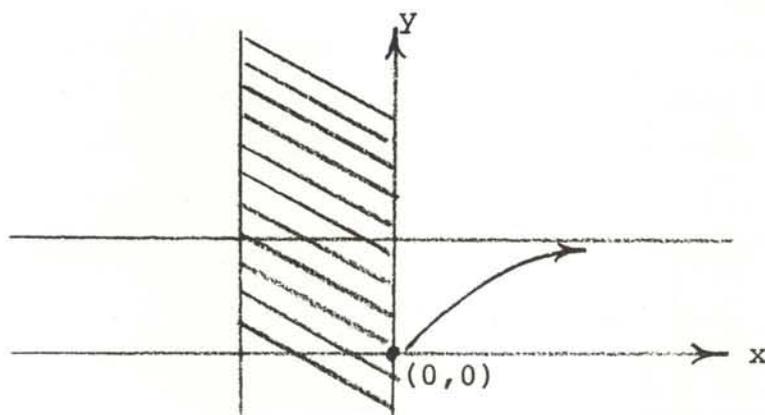


I.4.35

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.7(L) cont'd]

Of course, if we are willing to do a bit of extra work, we can obtain even more data about the curve. For example, if  $x$  is non-negative  $x/(x+1)$  starts at 0 (when  $x = 0$ ) and gets arbitrarily near 1 in value (but never equal to 1) as  $x$  gets arbitrarily large. This seems to indicate that the curve looks like:



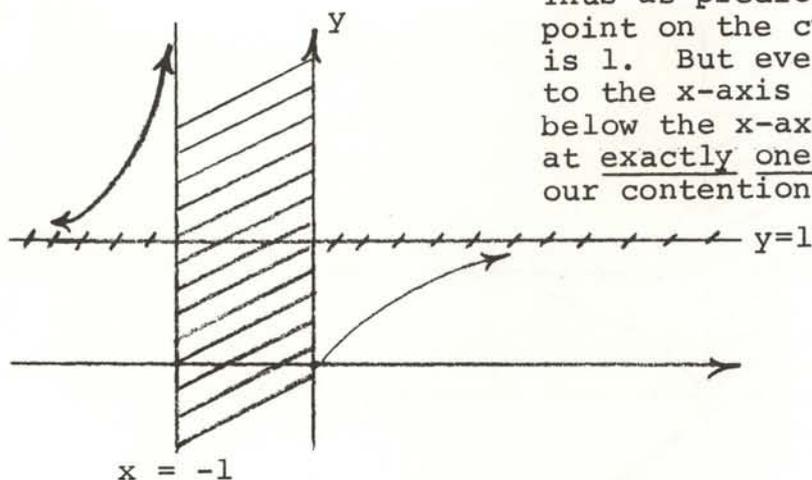
for non-negative values of  $x$ .

If on the other hand,  $x$  is negative and if we prefer working with positive numbers rather than negative ones; we can let  $x = -u$  where  $u$  is positive. For this situation (1) becomes  $y = \sqrt{-u/(-u + 1)}$ ; and if now multiply both numerator and denominator by  $-1$ , we obtain:

$$y = \sqrt{u/(u-1)}$$

If  $u$  is nearly equal to 1 (that is, if  $x$  is nearly equal to  $-1$ )  $u-1$  is nearly 0; hence  $y$  is very large. Moreover as  $u$  increases (therefore, as  $x$  decreases [becomes more negative])  $u/(u-1)$  always remains more than 1 but gets arbitrarily close to 1. Putting this new information into the hopper, we obtain

[1.4.7(L) cont'd]



Thus as predicted earlier, there is no point on the curve whose y-coordinate is 1. But every other line parallel to the x-axis (provided the line is not below the x-axis) intersects the curve at exactly one point. This bears out our contention that

$$y = +\sqrt{\frac{x}{x-1}}$$

is 1-1.

[These results could be documented by use of derivatives, but for our present purposes what we have already suffices.]

For a finale, let us now return to the question of single-valued versus multi-valued functions. Had we wished to invoke the classical meaning that the square root implied both the negative and positive roots, then the equation  $y = \sqrt{\frac{x}{x+1}}$  should have been replaced by the two equations:

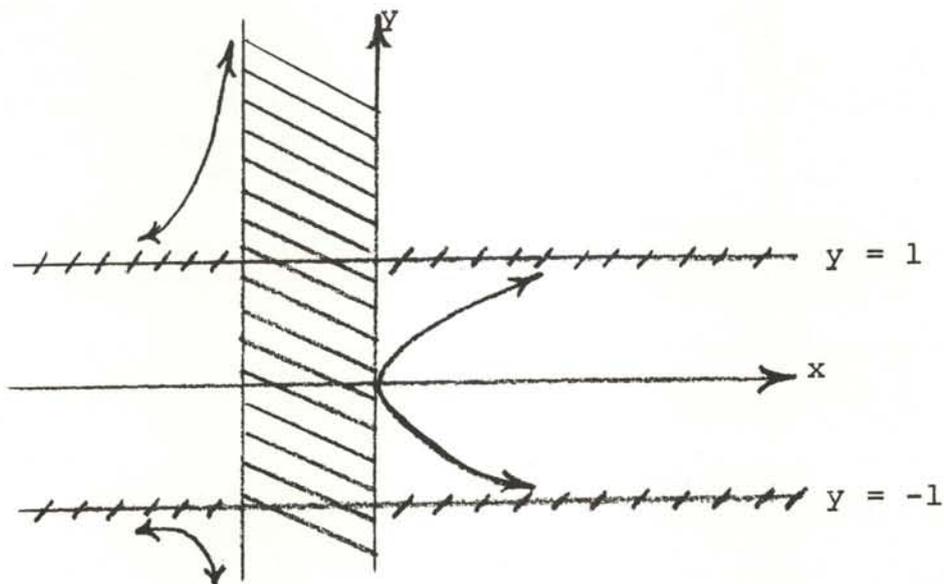
and

$$\left. \begin{array}{l} \text{(a) } y = +\sqrt{\frac{x}{x+1}} \\ \text{(b) } y = -\sqrt{\frac{x}{x+1}} \end{array} \right\} \left( \begin{array}{l} \text{These are usually abbreviated} \\ \text{by our writing } y = \pm\sqrt{\frac{x}{x+1}} \end{array} \right)$$

Our present graph is the graph of (a). Notice also that if we write (a) in the form  $y = f(x)$  then (b) has the form  $y = -f(x)$ . This means that for a given value of  $x$  the graphs in (a) and (b) are related by the fact that the y-coordinate of the point in one graph is the negative of the y-coordinate in the other. In terms of a picture, this is obtained by reflecting the graph of (a) about the x-axis. We would then obtain:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.7(L) cont'd]



and we now have a rather vivid way of seeing how a double-valued function can be viewed as the union of two single-valued functions, and that there is indeed no loss in generality when we restrict our attention to single-valued functions.

1.4.8(L)

In many respects this problem is much the same as our previous one - but with a few major differences. From a computational point of view, it may be a bit more "sophisticated" trying to solve for  $y$  as a function of  $x$  than it was in Exercise 1.4.7 (L). For another thing, we have here an excellent example of what we mean by  $y(x)$  being an IMPLICIT function of  $x(y)$ . That is, the given equation does not express either of the variables EXPLICITLY in terms of the other; but a recipe is IMPLIED whereby we can solve for  $x(y)$  once  $y(x)$  is given.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.8(L) cont'd]

By way of illustration suppose we tell our "function machine" to compute  $y$  from the relation  $x^2 + xy + y^2 = 3$  once  $x$  is given. For example, if we start with  $x = 0$ , the relation becomes  $0 + 0 + y^2 = 3$ ; whence  $y = \pm \sqrt{3}$ . In this case, it appears that we have a double-valued function. On the other hand, if we choose  $x = 2$ , the relation becomes:  $4 + 2y + y^2 = 3$ ; whence  $y^2 + 2y + 1 = 0$ ; or  $(y + 1)^2 = 0$ . In this case, the function machine cranks out the single answer  $y = -1$ ; thus for this value of  $x$  we have a single-valued function. Finally if  $x = 3$ , we obtain:  $9 + 3y + y^2 = 3$ ; in which case  $y^2 + 3y + 6 = 0$ . Now the quadratic formula (among other ways) shows us that there are no real values of  $y$  that satisfy our last equation. [An alternative to the quadratic formula is completing the square. Observe that  $y^2 + 3y + 9/4 = (y + 3/2)^2$ . Thus  $y^2 + 3y + 6 = y^2 + 3y + 9/4 + 15/4$  (since  $6 = 24/4$ )  $= (y + 3/2)^2 + 15/4 \geq 15/4$ , since  $(y + 3/2)^2 \geq 0$  and can equal 0 if and only if  $y = -3/2$ ; in other words, we have just shown that  $y^2 + 3y + 6 \geq 15/4$  and hence can never equal 0.]

The idea of the quadratic formula, however, is not a bad idea for this problem. Observe that  $x^2 + xy + y^2 = 3$  [or more suggestively  $y^2 + xy + (x^2 - 3) = 0$ ] represents a quadratic equation in  $y$ . [In thinking about the quadratic equation  $ay^2 + by + c = 0$ , we usually think of  $a$ ,  $b$ , and  $c$  as being constants; yet nothing in the proof of the quadratic formula requires this. That is, we may think of the equation  $ay^2 + by + c = 0$  with  $a = 1$ ,  $b = x$  and  $c = x^2 - 3$ .]

We then obtain that:

$$y = \frac{-x \pm \sqrt{x^2 - 4(x^2 - 3)}}{2}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.8(L) cont'd]

or:

$$y = \frac{-x \pm \sqrt{12 - 3x^2}}{2}$$

Thus, the two equations sought after in this problem are:

$$y = \frac{-x + \sqrt{12 - 3x^2}}{2} \quad (1)$$

and

$$y = \frac{-x - \sqrt{12 - 3x^2}}{2} \quad (2)$$

While this gives us the solution to this exercise we have just begun to make the comments we would like about this problem.

For a start, we observe that since we are extracting a square root,  $12 - 3x^2$  must be non-negative if  $y$  is to be real. But  $12 - 3x^2 = 3(4 - x^2) = 3(2 - x)(2 + x)$ . Utilizing the geometric approach described in previous exercises, we see that

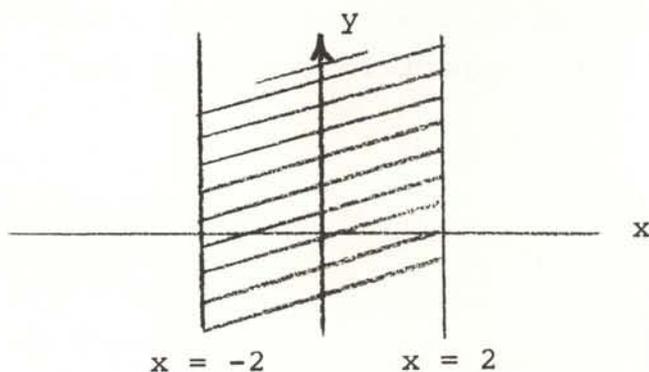
$x = -2$	$x = 2$	
+ + + + +	+ + + + +	- - - - -
↑	↑	↑
Different signs	Same sign	Different signs
↓	↓	↓
- - - - -	+ + + + +	+ + + + +
		2 - x
		2 + x

from which it follows that  $y$  is real if and only if  $-2 \leq x \leq 2$  (or in other words, if and only if  $|x| \leq 2$ ).

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

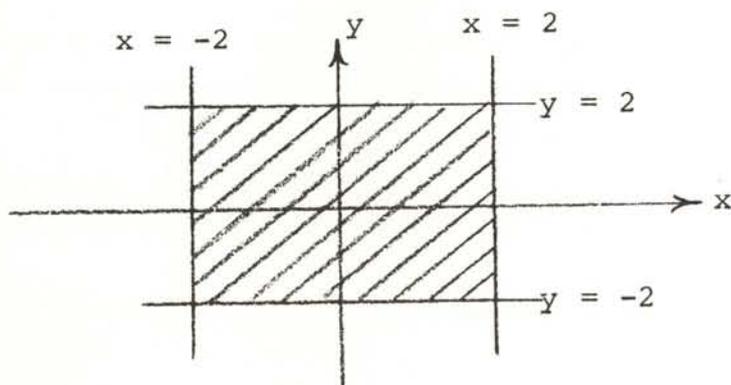
[1.4.8(L) cont'd]

If we try to graph  $x^2 + xy + y^2 = 3$ , the above information tells us that no portion of the graph can exist in the regions defined by  $x > 2$  and  $x < -2$ . That is, the graph must exist entirely within the region shaded below:



Our next remark about the equation  $x^2 + xy + y^2 = 3$  is that it has the rather uncommon property that if we replace every  $x$  by  $y$  and every  $y$  by  $x$  in the equation, we arrive at the same equation [when this happens we say the equation is SYMMETRIC in  $x$  and  $y$ ]. This in turn implies that equations (1) and (2) must remain valid, then, when  $x$  and  $y$  are interchanged. Thus we can conclude that for  $x$  to be real it must happen that  $-2 \leq y \leq 2$ .

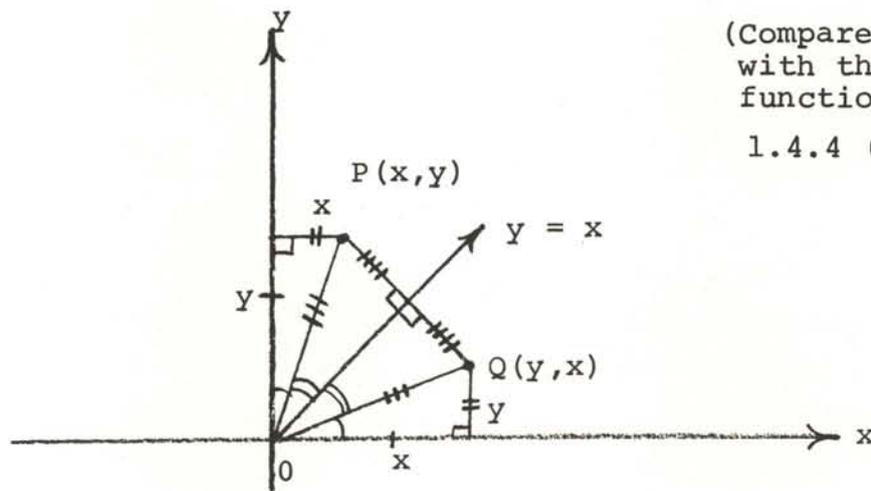
As far as our graph is concerned, we can now state that the curve lies entirely within the shaded region:



SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.8(L) cont'd]

What does it mean graphically if an equation is symmetric in  $x$  and  $y$ ? If we call the curve defined by the equation  $C$ , symmetry says that  $(x,y) \in C$  if and only if  $(y,x) \in C$ . Let  $P$  denote  $(x,y)$  and  $Q$  denote  $(y,x)$ . Then as indicated below,  $POQ$  is an isosceles triangle. Moreover the angle-bisector of  $\angle O$  is the line  $y = x$ . Hence it follows that  $P$  and  $Q$  are "mirror images" of one another with respect to the  $45^\circ$  line:

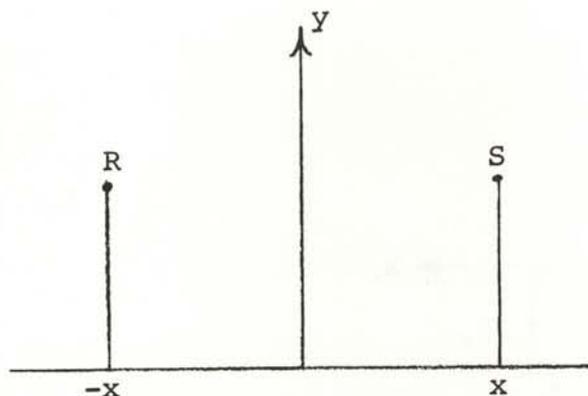


(Compare this discussion with that about inverse functions in Exercise 1.4.4 (L).)

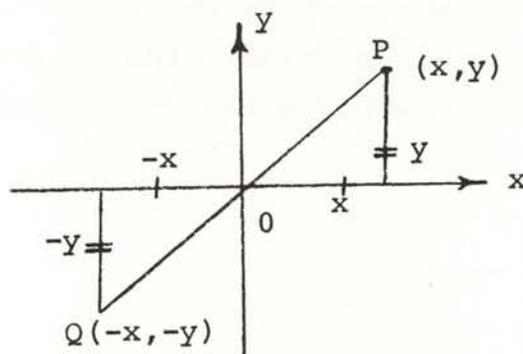
To carry this idea further, observe that if an equation remains the same when  $x$  is replaced by  $-x$ , we have symmetry with respect to the  $y$ -axis (in other words  $x$  and  $-x$  are the mirror images of each other with respect to the  $y$ -axis). Similarly we have symmetry with respect to the  $x$ -axis if the equation remains the same when  $y$  is replaced by  $-y$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.8(L) cont'd]



Finally if the equation remains the same when we simultaneously replace  $x$  by  $-x$  and  $y$  by  $-y$ , we have what is called symmetry with respect to the origin. That is:

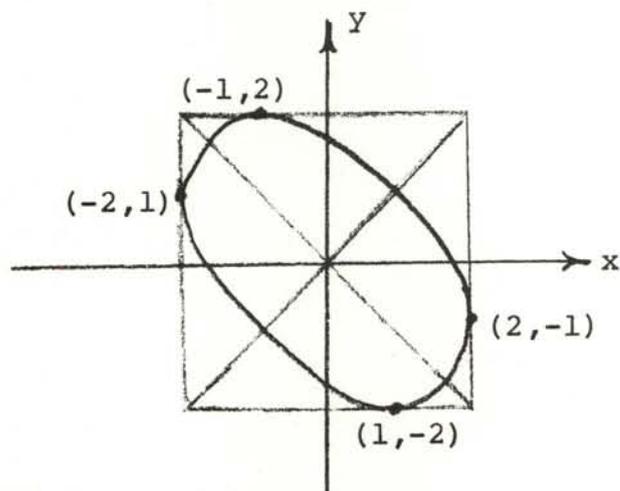


{ The line segment which joins P and Q is bisected by 0

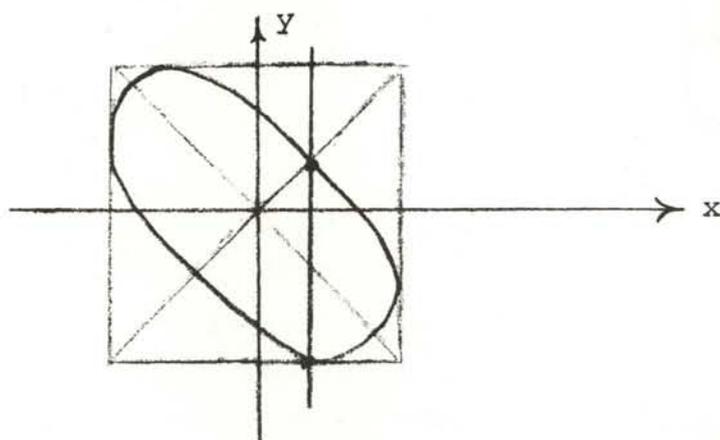
Applying these ideas to our graph we have symmetry with respect to the line  $y = x$  and also with respect to the origin. In fact, while it is not necessary for our purposes, it can be shown that our graph is an ellipse whose axes of symmetry are the lines  $y = x$  and  $y = -x$ . That is,

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 4: Functions

[1.4.8(L) cont'd]



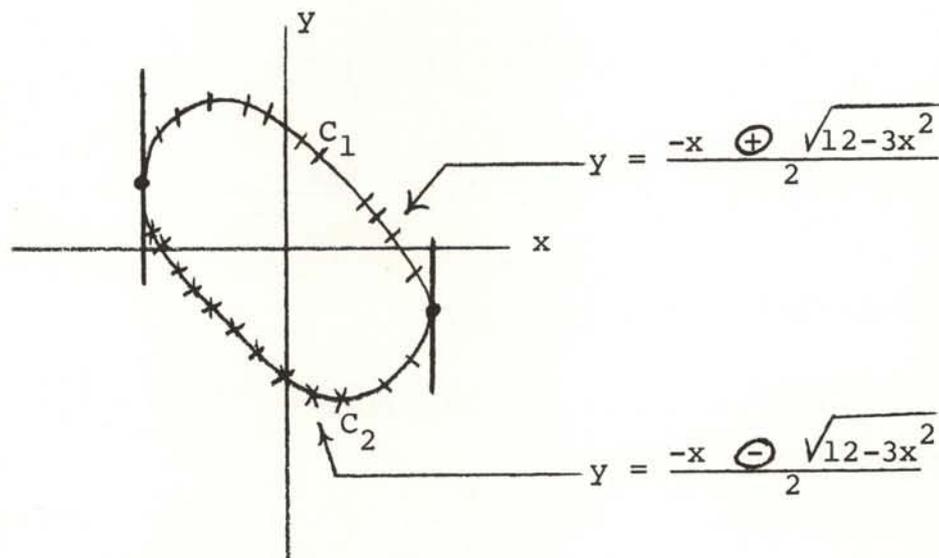
Hopefully the above diagram will supply us the necessary hints for concluding why there were two answers to this problem. Namely the graph is NOT a single-valued function. Indeed for each  $x$  such that  $-2 < x < 2$ , we have that there are two corresponding  $y$  values. That is:



The two answers correspond to the two single valued branches of the curve we obtain by drawing in the given vertical tangents:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 4: Functions

[1.4.8(L) cont'd]



In our diagram, the curve C is given as the union of the single-valued curves C<sub>1</sub> and C<sub>2</sub>; where C<sub>1</sub> is defined by  $y = \frac{-x + \sqrt{12-3x^2}}{2}$  [that is, the TOP curve] while C<sub>2</sub> is defined by the branch  $y = \frac{-x - \sqrt{12-3x^2}}{2}$ .

1.4.9(L)

We have already seen in the supplementary reading that  $f^{-1}(x) = \frac{x+7}{2}$ . This follows from the fact that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

and indeed:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 4: Functions

[1.4.9(L) cont'd]

$$f(f^{-1}(x)) = f\left(\frac{x+7}{2}\right) = 2\left(\frac{x+7}{2}\right) - 7 = x,$$

and

$$f^{-1}(f(x)) = f^{-1}(2x - 7) = \frac{(2x - 7) + 7}{2} = x$$

On the other hand  $\frac{1}{f(x)} = \frac{1}{2x-7}$ .

More symbolically, we may let  $g(x) = f^{-1}(x) = \frac{x+7}{2}$  and  $h(x) = \frac{1}{f(x)} = \frac{1}{2x-7}$ .

Among other things  $g(0) = \frac{7}{2}$  while  $h(0) = \frac{1}{-7}$ . Since  $h(0) \neq g(0)$ ,  $h \neq g$ . (Another thing is that  $\frac{7}{2}$  is not in the domain of  $h$  since  $2x - 7$  is 0 when  $x = \frac{7}{2}$  and we are not permitted to divide by 0 while  $g\left(\frac{7}{2}\right) = \frac{7/2+7}{2} = \frac{21}{4}$ . In other words  $g$  and  $h$  even have different domains.)

In any event, this exercise is meant to show the difference between  $f^{-1}(x)$  and  $\frac{1}{f(x)}$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits

UNIT 5: The Derivative as a Limit

1.5.1(L)

In this problem as well as the next two, the main idea is to emphasize that the basic definition of  $f'(x)$  in no way depends on  $f(x)$ . What does depend on  $f(x)$  is the amount of computational sophistication which is required to determine  $f'(x)$  explicitly.

Picking an arbitrary value of  $x$ , say  $x_1$ , and a number  $h \neq 0$ , we compute

$$\frac{f(x_1 + h) - f(x_1)}{h} = G(x_1, h)$$

(the quotient clearly depending on both  $x_1$  and  $h$ ). In this particular case, we obtain:

$$\begin{aligned} f(x_1 + h) &= (x_1 + h)^3 = x_1^3 + 3x_1^2h + 3x_1h^2 + h^3 \\ f(x_1) &= x_1^3 \end{aligned}$$

Hence:

$$f(x_1+h) - f(x_1) = 3x_1^2h + 3x_1h^2 + h^3 = h(3x_1^2 + 3x_1h + h^2)$$

and since  $h \neq 0$ ,

$$G(x_1, h) = 3x_1^2 + 3x_1h + h^2 \tag{1}$$

By way of review, (1) gives us the formula for computing the average rate of change of  $f(x)$  with respect to  $x$  over the interval from  $x = x_1$  to  $x = x_1 + h$ . This average is precisely what is denoted by  $G(x_1, h)$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.1(L) cont'd]

In terms of our basic definition we now obtain  $f'(x_1)$  (the instantaneous rate of change of  $f(x)$  with respect to  $x$  at  $x = x_1$ ) by letting  $h$  approach zero in (1).

That is:

$$f'(x_1) = \lim_{h \rightarrow 0} G(x_1, h) = 3x_1^2$$

Finally, since  $x_1$  denotes ANY point in the domain of  $f$ , we may write:

$$f'(x) = 3x^2 \text{ if } f(x) = x^3 \quad (2)$$

Notice that a convenient "recipe" for memorizing the correct answer is to note that we "brought the exponent down" and replaced it by one less. In fact, with the aid of the binomial theorem, we can mimic the procedure used in this exercise and deduce that if  $f(x) = x^n$  where  $n$  is any positive integer, then  $f'(x) = nx^{n-1}$ . We shall investigate such recipes in more detail later in the course. (The key to the present recipe is to observe that by use of the binomial theorem we can conclude that  $(x_1+h)^n = x_1^n + nx_1^{n-1}h + h^2(\dots)$  where the parentheses indicate that we can factor the common factor  $h^2$  from each of the remaining terms in the expansion. If we then subtract  $x_1^n$  and divide by  $h$ , we obtain:  $nx_1^{n-1} + h(\dots)$  and the result follows when we let  $h$  approach zero.) For now we only wish to indicate that by use of our basic definition for  $f'(x)$  we can often find convenient recipes that allow us to write down  $f'(x)$  very quickly for certain special cases of  $f(x)$ . At the same time

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 5: The Derivative as  
a Limit

[1.5.1(L) cont'd]

we wish to caution that: (1) the validity of the short cut depends on the "long" way, and (2) for certain functions there will be no recourse other than the long way. For example, the fact that we have a recipe for finding  $f'(x)$  if  $f(x) = x^n$  does little if anything to find  $f'(x)$  if, say,  $f(x) = \sin x$ .

1.5.2(L)

As we said before, when we've seen one such problem, we've seen them all. In every case

$$f'(x) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

It's just that computationally some expressions are nastier to handle than others.

In our present example  $f(x) = \sqrt{3x}$ ; hence  $f(x+\Delta x) = \sqrt{3(x+\Delta x)}$ .

Hence:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{3(x+\Delta x)} - \sqrt{3x}}{\Delta x} \quad (1)$$

If we replace  $\Delta x$  by 0 in the bracketed portion of (1), we obtain our old friend  $0/0$  - but this was obtained illegally since  $\lim_{\Delta x \rightarrow 0}$  means that while we let  $\Delta x$  get close to 0, we never let it equal 0.

Somehow or other, we would like to rewrite the bracketed expression in (1) so that its numerator contains  $\Delta x$  as a factor.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.2(L) cont'd]

(Then since  $\Delta x \neq 0$ , we could cancel it from both numerator and denominator.) To this end we "rationalize the numerator." That is, we utilize the identity that

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$$

Thus:

$$\begin{aligned} \frac{\sqrt{3(x+\Delta x)} - \sqrt{3x}}{\Delta x} &= \left( \frac{\sqrt{3(x+\Delta x)} - \sqrt{3x}}{\Delta x} \right) \left( \frac{\sqrt{3(x+\Delta x)} + \sqrt{3x}}{\sqrt{3(x+\Delta x)} + \sqrt{3x}} \right) * \\ &= \frac{3(x+\Delta x) - 3x}{\Delta x(\sqrt{3(x+\Delta x)} + \sqrt{3x})} = \frac{3\Delta x (1)}{\Delta x(\sqrt{3(x+\Delta x)} + \sqrt{3x})} \end{aligned}$$

$$\therefore \text{ If } x \neq 0, \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{3}{\sqrt{3(x+\Delta x)} + \sqrt{3x}} \quad (2)$$

Now letting  $\Delta x \rightarrow 0$  in (2), we see that

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x+\Delta x) - f(x)}{\Delta x} \right] = \frac{3}{\sqrt{3(x+0)} + \sqrt{3x}} = \frac{3}{\sqrt{3x} + \sqrt{3x}} \\ &= \frac{3}{2\sqrt{3x}} \end{aligned}$$

---

\*The only time  $\frac{a}{a} \neq 1$  is when  $a = 0$ . Thus our above operation requires that  $\sqrt{3(x+\Delta x)} + \sqrt{3x}$  is not 0. Since we "accept" only non-negative roots,  $\sqrt{3(x+\Delta x)} + \sqrt{3x}$  can be 0 only if  $\sqrt{3(x+\Delta x)}$  and  $\sqrt{3x}$  are both 0. But this can't happen since  $\sqrt{3x} = 0$  implies  $x = 0$  but when  $x = 0$ ,  $\sqrt{3(x+\Delta x)} = \sqrt{3\Delta x}$  and  $\Delta x \neq 0$ .  $\therefore \sqrt{3\Delta x} \neq 0$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.2(L) cont'd]

A Note on "Rationalizing Denominators"

In most elementary algebra courses, one talks about rationalizing denominators.

For example, consider

$$a = \frac{1}{\sqrt{2}} \quad (3)$$

We are taught to remove radicals from the denominator; and in this context we would rewrite (1) as:

$$a = \left(\frac{1}{\sqrt{2}}\right) \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2} \quad (4)$$

We are not usually taught to "rationalize the numerator" (as we did in this exercise).

Now  $\frac{1}{\sqrt{2}}$  and  $\frac{\sqrt{2}}{2}$  are synonyms. Why should one be preferred to the other? The answer depends on what we are trying to do!

For example in most engineering applications we would write  $\sqrt{2}$  as a decimal. In this context if  $\sqrt{2} = 1.4142$ , (3) becomes  $\frac{1}{1.4142}$  while (4) becomes  $\frac{1.4142}{2}$ . It is clear that (4) is easier to calculate as a decimal than (3) is (at least if computers are not involved).

Thus it is quite likely that the topic of rationalizing denominators was born out of consideration for having to divide by "messy" decimals. In other words, we can often convert the "messy" decimal into the dividend rather than have to treat it

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.2(L) cont'd]

as the divisor by use of rationalizing denominators. On the other hand, to eliminate a factor of  $\Delta x$  in computing, say,  $f'(x)$  in this exercise, it was to our advantage to know how to rationalize numerators.

The important thing is that

$$a - b = (\sqrt{a} + \sqrt{b}) (\sqrt{a} - \sqrt{b})$$

and how we use this result is best determined by the "real life" situation we are facing at a given moment.

1.5.3

$$f(x) = \frac{1}{\sqrt{2x}} \qquad f(x+\Delta x) = \frac{1}{\sqrt{2(x+\Delta x)}}$$

$$\begin{aligned} \therefore \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{\frac{1}{\sqrt{2(x+\Delta x)}} - \frac{1}{\sqrt{2x}}}{\Delta x} \\ &= \frac{\sqrt{2x} - \sqrt{2(x+\Delta x)}}{\Delta x \sqrt{2(x+\Delta x)} \sqrt{2x}} \\ &= \left[ \frac{\sqrt{2x} - \sqrt{2(x+\Delta x)}}{\Delta x \sqrt{2(x+\Delta x)} \sqrt{2x}} \right] \left[ \frac{\sqrt{2x} + \sqrt{2(x+\Delta x)}}{\sqrt{2x} + \sqrt{2(x+\Delta x)}} \right] \\ &= \frac{2x - [2(x+\Delta x)]}{\Delta x \sqrt{2(x+\Delta x)} \sqrt{2x} (\sqrt{2x} + \sqrt{2(x+\Delta x)})} \end{aligned}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.3 cont'd]

$$= \frac{-2\Delta x}{\Delta x \sqrt{2(x+\Delta x)} \sqrt{2x} (\sqrt{2x} + \sqrt{2(x+\Delta x)})}$$

$$= \frac{-2}{\sqrt{2(x+\Delta x)} \sqrt{2x} (\sqrt{2x} + \sqrt{2(x+\Delta x)})} \quad (\text{since } \Delta x \neq 0)$$

$$\therefore \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-2}{\sqrt{2x} \sqrt{2x} (\sqrt{2x} + \sqrt{2x})}$$

$$= \frac{-2}{2x(2\sqrt{2x})} = \frac{-1}{2x\sqrt{2x}} = \frac{-1}{2x\sqrt{2}x^{1/2}}$$

$$\therefore f'(x) = \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x+\Delta x) - f(x)}{\Delta x} \right] = \frac{-1}{2\sqrt{2} x^{3/2}} = -\frac{x^{-3/2}}{2\sqrt{2}} = \frac{-\sqrt{2} x^{-3/2}}{4}$$

1.5.4(L)

In the truest sense there is little here to make this problem worthy of being called a learning exercise except that it helps us to reinforce the connection between analysis and geometry. Specifically, the slope of the curve  $y = f(x)$  at the point  $(x_1, y_1)$  is precisely  $f'(x_1)$ . In fact, if we now refer to the solution of Exercise 1.5.1(L) we observe that  $G(x_1, h)$  can be interpreted as the slope of the line which joins the points  $(x_1, f(x_1))$  and  $(x_1+h, f(x_1+h))$  on the curve  $y = f(x)$ .

Thus the slope of the tangent line to the curve at  $(x_1, f(x_1))$  is  $3x_1^2$  just as in our solution to Exercise 1.5.1(L).

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

1.5.5

In 1.5.2(L) we found that if  $f(x) = \sqrt{3x}$ ,  $f'(x) = \frac{3}{2\sqrt{3x}}$ . Therefore, the slope of  $y = \sqrt{3x}$  at  $(x_1, y_1)$  is given by:

$$m = \frac{3}{2\sqrt{3x_1}}$$

If  $x_1 = 12$ ,  $m = \frac{3}{2\sqrt{36}} = \frac{3}{12} = \frac{1}{4}$  ans.

1.5.6(L)

Here we see a rather elementary problem involving equations of straight lines provided that we know the slope of the tangent line.

Now given  $y = x^2 - 2$ , we can easily show that  $\frac{dy}{dx} = 2x$  and hence that the slope of the curve at  $(2, 2)$  is 4. Thus the tangent line we seek has its slope equal to 4 and  $(2, 2)$  is on the line. Thus the equation is:

$$\frac{y - 2}{x - 2} = 4$$

or

$$\underline{y = 4x - 6} \quad (1)$$

Knowing the equation of the line we find where it crosses the x-axis by letting  $y = 0$ . Letting  $y = 0$  in (1) we obtain:

$$0 = 4x - 6 \quad \text{or} \quad x = 3/2$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.6(L) cont'd]

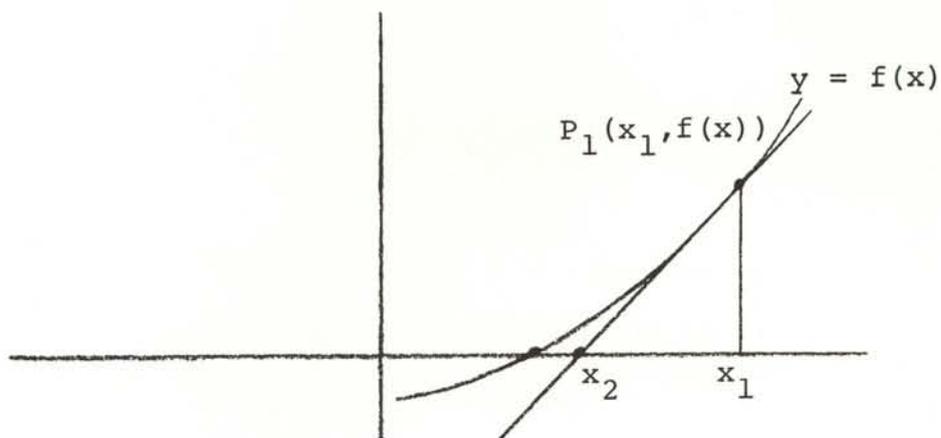
Notice that (and this happens very often in the use of calculus) we used calculus for essentially one step (to find the slope of our line), after which the problem proceeded in a manner independent of calculus. That is, once we found the slope, the rest was a pre-calculus problem.

As elementary as this problem is, it has a very useful generalization known as NEWTON'S METHOD FOR APPROXIMATING ROOTS.

The technique is as follows.

Suppose we wish to find a root of  $f(x) = 0$ . This is equivalent to finding where the curve  $y = f(x)$  crosses the x-axis.

While there are certain complications that may occur, the general idea is that we choose any value  $x = x_1$  which we call our first approximation. We assume that  $f(x)$  is differentiable and we draw the line tangent to  $y = f(x)$  at  $(x_1, f(x_1))$ .



SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.6(L) cont'd]

We then let  $x_2$  denote the  $x$ -coordinate of the point at which this line meets the  $x$ -axis.  $x_2$  becomes our next approximation and we continue in this way, obtaining, it is hoped, better and better approximations.

In any event the tangent line to the curve  $y = f(x)$  at  $x = x_1$ , passes through  $(x_1, f(x_1))$  and has slope  $f'(x_1)$ . Hence its equation is:

$$\frac{y - f(x_1)}{x - x_1} = f'(x_1) \quad (2)$$

We find  $x_2$  by setting  $y = 0$  in (2).

We obtain:

$$\frac{0 - f(x_1)}{x_2 - x_1} = f'(x_1)$$

$$\therefore x_2 - x_1 = \frac{-f(x_1)}{f'(x_1)}$$

$$\therefore \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}} \quad (3)$$

Equation (3) tells us how to obtain  $x_2$  (the next approximation) from  $x_1$ . We can then use (3) again with  $x_2$  as our "new"  $x_1$  to obtain the next approximation.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.6(L) cont'd]

In terms of our specific exercise, observe that a root of  $x^2 - 2 = 0$  is  $\sqrt{2}$ . Thus  $x_1 = 2$  was our first approximation to  $\sqrt{2}$ . Our answer  $x_2 = \frac{3}{2}$  was the next approximation. In particular:

$$f(x) = x^2 - 2 \rightarrow f'(x) = 2x$$

Hence using (3) with  $x_1 = 2$  we obtain:

$$x_2 = 2 - \frac{2}{4} = \frac{3}{2}$$

If we now use (3) with  $x_1 = \frac{3}{2}$ , we obtain:

$$x_2 = \frac{3}{2} - \frac{\left(\frac{3}{2}\right)^2 - 2}{3} = \frac{3}{2} - \frac{1}{12} = \frac{17}{12}$$

(and  $\left(\frac{17}{12}\right)^2 = \frac{289}{144} = 2 + \frac{1}{144}$ ). In this way we can "converge on"  $\sqrt{2}$  rather quickly.

(The interested reader can find additional discussion of Newton's method in Thomas, Chapter 10.3.)

1.5.7(L)

$$A = \pi r^2 = A(r)$$

$$\therefore A(r+\Delta r) = \pi(r+\Delta r)^2 = \pi r^2 + 2\pi r\Delta r + \pi(\Delta r)^2$$

$$\therefore A(r+\Delta r) - A(r) = 2\pi r\Delta r + \pi(\Delta r)^2$$

$$\therefore \frac{A(r+\Delta r) - A(r)}{\Delta r} = \frac{(2\pi r + \pi\Delta r)\Delta r}{\Delta r}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.7(L) cont'd]

$$= 2\pi r + \pi\Delta r, \text{ since } \Delta r \neq 0.$$

$$\begin{aligned} \therefore \frac{dA}{dr} &= \lim_{\Delta r \rightarrow 0} \left[ \frac{A(r+\Delta r) - A(r)}{\Delta r} \right] = \lim_{\Delta r \rightarrow 0} [2\pi r + \pi\Delta r] \\ &= \underline{2\pi r} \quad \underline{\text{answer.}} \end{aligned}$$

Notice that up to this point the problem is no different from 1.5.1, 1.5.2, or 1.5.3 (except that it's a bit less computational). We see at once, however, that  $A = \pi r^2$  suggests that we are studying the area of a circle. Notice also that our answer,  $2\pi r$ , suggests the circumference of a circle.

In fact, what we have proved is that the rate of change of the area of a circle with respect to its radius at any instant is numerically equal to its circumference at that instant.

Here we see an application of calculus to elementary geometry and we have arrived at a rather "nice" piece of information by this analysis that might not have been so obvious in terms of our intuition.

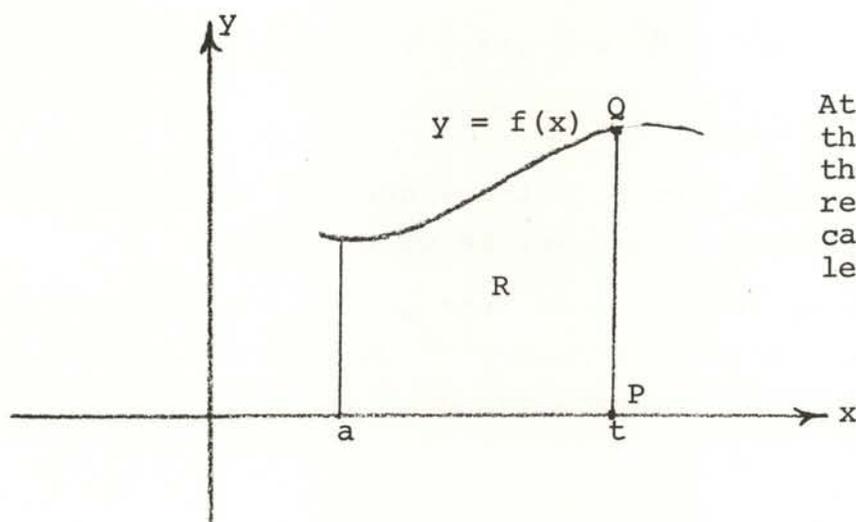
The result obtained in this section has a rather far-reaching generalization that will be discussed later in this course. In essence what will be shown then is the following.

If  $R$  denotes the area of the region bounded above by the curve  $y = f(x)$ , where  $f(x) \geq 0$  for all  $x$ , below by the  $x$ -axis, on the left by the line  $x = a$  and on the right by the line  $x = t$  then the area of  $R$ ,  $A_R$ , is a function of  $t$ . The amazing fact that will be proven is that

$$\frac{dA_R}{dt} = f(t)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.7(L) cont'd]



At the instant  $x = t$ , the rate of change of the area of  $R$  with respect to  $x$  is numerically equal to the length of  $PQ$ .

In this way we shall have exhibited a rather amazing relationship between the rate of change of an area ( $A_R$ ) and the length of a line ( $PQ$ ).

1.5.8

$$V(r) = \frac{4}{3} \pi r^3 \quad V(r+\Delta r) = \frac{4}{3} \pi (r+\Delta r)^3$$

$$\frac{V(r+\Delta r) - V(r)}{\Delta r} = \frac{\frac{4}{3} \pi (r+\Delta r)^3 - \frac{4}{3} \pi r^3}{\Delta r} = \frac{4}{3} \pi \frac{[(r^3 + 3r^2\Delta r + 3r\Delta r^2 + \Delta r^3) - r^3]}{\Delta r}$$

$$= \frac{4}{3} \pi \left[ \frac{(3r^2 + 3r\Delta r + \Delta r^2) \Delta r}{\Delta r} \right]$$

$$= \frac{4}{3} \pi (3r^2 + 3r\Delta r + \Delta r^2), \text{ since } \Delta r \neq 0$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.8 cont'd]

$$\therefore \lim_{\Delta r \rightarrow 0} \frac{V(r+\Delta r) - V(r)}{\Delta r} = \frac{dV}{dr} = \frac{4}{3} \pi (3r^2 + 3r0 + 0^2) = \underline{4\pi r^2} \text{ ans.}$$

Note:  $\frac{4}{3}\pi r^3$  denotes the volume of the sphere of radius  $r$ .  $4\pi r^2$  denotes the surface area of the sphere of radius  $r$ .

Thus in a manner analogous to that in 1.5.7(L) we see that at a given instant the rate of change of volume of a sphere with respect to its radius is numerically equal to its surface area at that instant.

1.5.9(L)

The main point of this exercise is for us to learn the difference between being able to compute  $\frac{dh}{dt}$  and knowing how to use the information contained in  $\frac{dh}{dt}$ .

The key concept - and this concept plays a great role in calculus as we shall see many times throughout this course - is that at its maximum height the particle can be neither rising nor falling (for if it were falling it would have already passed through its maximum height, and if it were rising it would not yet have reached its maximum height).

Thus the particle reaches its maximum height at the instant that it is neither rising nor falling, but this means that the particle has zero speed at this time. In other words since  $\frac{dh}{dt}$  denotes the speed of the particle, the particle attains its maximum height when  $\frac{dh}{dt} = 0$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 5: The Derivative as  
a Limit

[1.5.9(L) cont'd]

With this as motivation, we are led to compute  $\frac{dh}{dt}$ . Recalling once again that  $\frac{dh}{dt} = \lim_{\Delta t \rightarrow 0} \frac{h(t+\Delta t) - h(t)}{\Delta t}$  and omitting the details, we see that:

$$\frac{dh}{dt} = 128 - 32t \quad (1)$$

$\frac{dh}{dt} = 0$  if and only if  $t = 4$ . Hence the particle reaches its maximum height when  $t = 4$  seconds and since at any time  $t$ , the height,  $h$ , is given by

$$h = 128t - 16t^2 \quad (2)$$

we compute the maximum height by letting  $t = 4$  in (2). This yields

$$\begin{aligned} h_{\max} = h_{t=4} &= 128(4) - 16(4)^2 \\ &= 512 - 256 \\ &= \underline{256 \text{ feet}} \quad \text{ans.} \end{aligned}$$

The result of this exercise can be generalized as follows.

If a particle is projected vertically upward with an initial speed of  $v_0$  ft/sec it attains a height of  $h$  feet in  $t$  seconds given by

$$h = v_0 t - 16t^2$$

Thus the speed of the particle is given by

$$v = \frac{dh}{dt} = v_0 - 32t$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 5: The Derivative as a Limit

[1.5.9(L) cont'd]

Hence the maximum height is attained when  $t = \frac{v_0}{32}$ . Therefore the maximum height is given by:

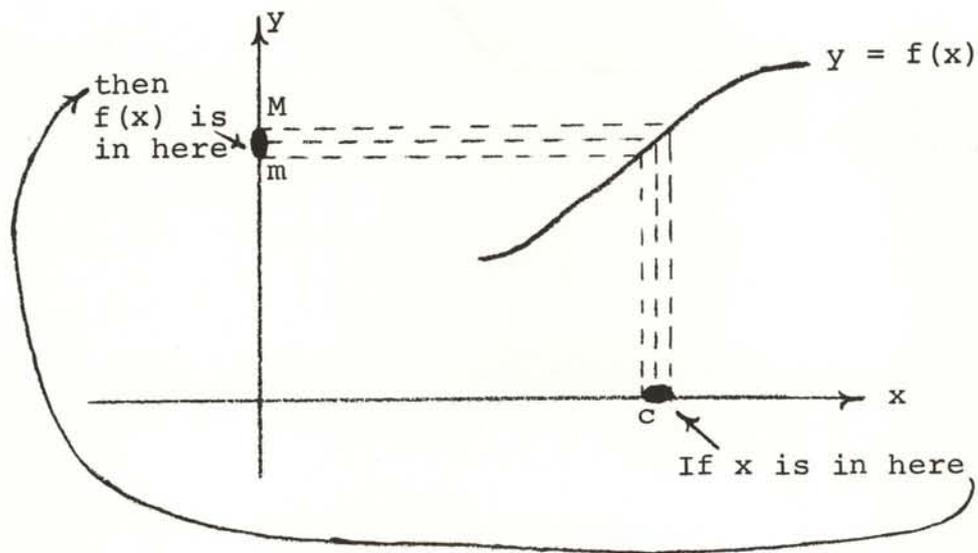
$$\begin{aligned}h_{\max} &= v_0 \left( \frac{v_0}{32} \right) - 16 \left( \frac{v_0}{32} \right)^2 \\&= \frac{v_0^2}{32} - \frac{v_0^2}{64} \\&= \frac{v_0^2}{64}\end{aligned}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits

UNIT 6: A More Rigorous Approach to Limits I

1.6.1(L)

What this problem is telling us in terms of more intuitive language is that if  $\lim_{x \rightarrow c} f(x) = L$  we can always find a deleted neighborhood of  $c$  so that in this neighborhood the values of  $f(x)$  can be "squashed in" as close to  $L$  as we desire. (Recall that  $c$  is itself removed from consideration. Among other things  $\lim_{x \rightarrow c} f(x) = L$  doesn't even guarantee that  $f(c)$  exists.) In still other words, we may think of a dot centered at  $y = L$  on the  $y$ -axis. Then we can find another dot, centered at  $x = c$  on the  $x$ -axis such that for all values of  $x$  in the latter dot, the values of  $f(x)$  are in the former dot. Pictorially:

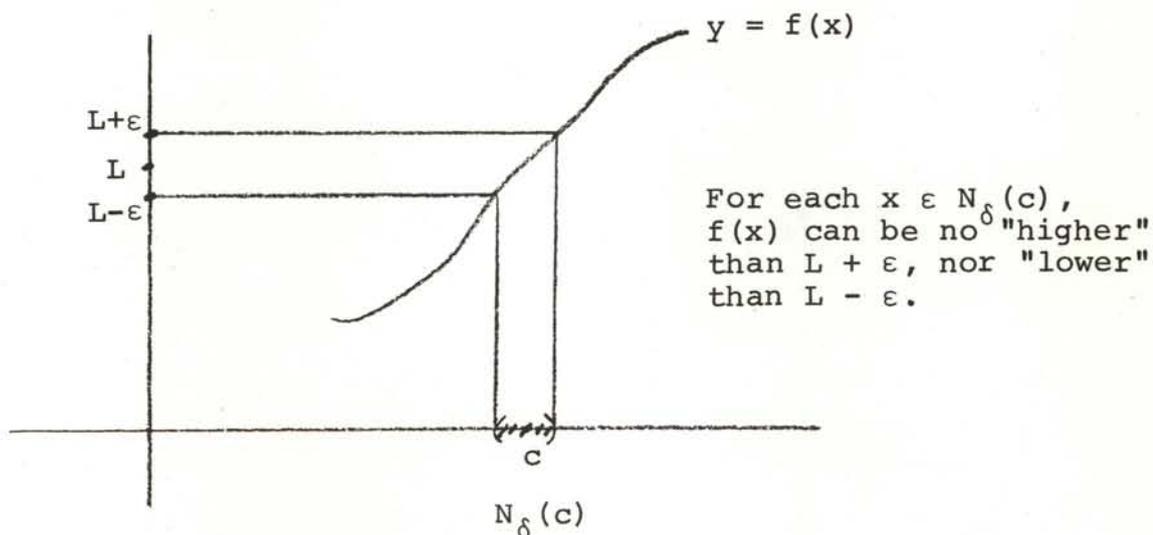


To see how we arrive at this result more formally, let us recall that by definition  $\lim_{x \rightarrow c} f(x) = L$  means that for any  $\epsilon > 0$  we can find  $\delta > 0$  such that  $|f(x) - L| < \epsilon$  provided that  $0 < |x - c| < \delta$ . That is, if  $0 < |x - c| < \delta$  then  $L - \epsilon < f(x) < L + \epsilon$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 6: A More Rigorous  
Approach to Limits I

[1.6.1(L) cont'd]

Notice that the only restriction on  $\epsilon$  is that it is positive. Consequently by choosing  $\epsilon$  sufficiently small in magnitude we can make  $L - \epsilon$  and  $L + \epsilon$  as nearly equal as we wish, and the more nearly equal they are the more easily we may think of the "dot" which joins them as being a point. This also agrees with our intuitive notion that we are interested in what happens "near"  $L$ . In any event, then, for this choice of  $\epsilon$  we can find a deleted neighborhood  $N_\delta(c)$  such that  $L - \epsilon$  is a LOWER BOUND and  $L + \epsilon$  is an UPPER BOUND of  $f(x)$  for all  $x \in N_\delta(c)$ . Again in terms of a graph:

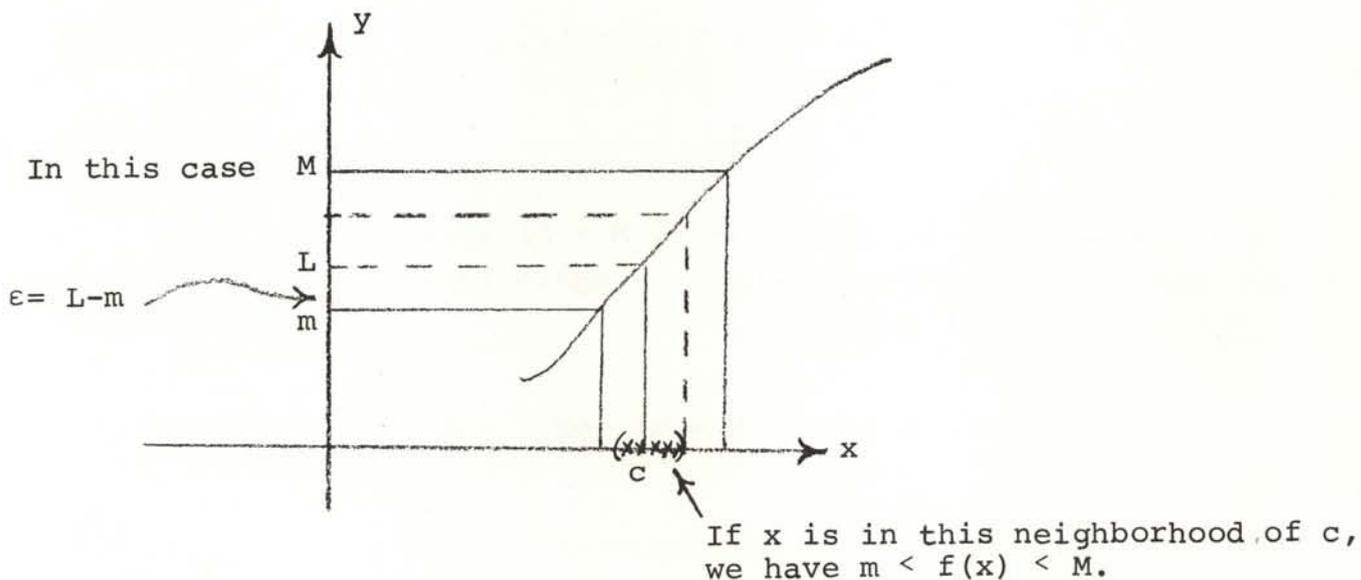


In our problem we do not require that  $m$  and  $M$  be symmetrically located with respect to  $L$  (in less geometric language, we are saying we do not require that  $L - m$  equal  $M - L$ ). In this event we merely choose  $\epsilon$  to be the minimum of  $L - m$  and  $M - L$ . For this choice of  $\epsilon$  we then determine the value of  $\delta$  such that for  $0 < |x - c| < \delta$ ,  $L - \epsilon < f(x) < L + \epsilon$  and this is equivalent to saying that  $m < f(x) < M$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
 Functions, and Limits - Unit 6: A More Rigorous  
 Approach to Limits I

[1.6.1(L) cont'd]

In terms of a picture:



If we wish to be even more rigorous and establish our proof without having to make reference to a graph, we proceed as follows.

Since  $m < L < M$ , both  $L - m$  and  $M - L$  are positive. Let  $\epsilon$  equal the minimum of  $L - m$  and  $M - L$  (abbreviated as  $\epsilon = \min(L - m, M - L)^*$ ). By the definition of  $\lim_{x \rightarrow c} f(x) = L$ , we are guaranteed that for this choice of  $\epsilon$  we can find  $\delta$  such that  $0 < |x - c| < \delta$  implies that

$$L - \epsilon < f(x) < L + \epsilon. \quad (1)$$

\*Quite in general,  $\min(a, b)$  denotes the smaller of the two numbers  $a$  or  $b$ . From a structural point of view, the important fact is that  $\min(a, b) \leq a$  and  $\min(a, b) \leq b$ . (That is, the lesser of the two numbers,  $a$  and  $b$ , cannot exceed either  $a$  or  $b$ , or else it wouldn't be the lesser of the two.)

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.1(L) cont'd]

But  $\epsilon = \min(L - m, M - L)$  implies that  $\epsilon \leq L - m$  and also  $\epsilon \leq M - L$ . Hence, in particular,

$$L + \epsilon \leq L + (M - L),$$

or

$$L + \epsilon \leq M \tag{2}$$

In a similar way,  $\epsilon = \min(L - m, M - L)$  implies that  $-\epsilon \geq -(L - m)$  and that  $-\epsilon \geq -(M - L)$  (where we again recall that  $a < b$  if and only if  $-a > -b$ ). Therefore:

$$L - \epsilon \geq L - (L - m), \text{ or } L - \epsilon \geq m$$

or

$$m \leq L - \epsilon \tag{3}$$

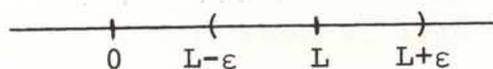
If we now introduce the results of (2) and (3) into (1) we obtain:  $0 < |x - c| < \delta$  implies that  $m \leq L - \epsilon < f(x) < L + \epsilon \leq M$  and, therefore,  $m < f(x) < M$  and this establishes the proof.

Notice that we would like to use both the intuitive and the rigorous aspects of the situation. The rigorous aspect (and notice how the rigorous demonstration was modeled after the fact that we sensed intuitively what was happening) shows us that the result is an inescapable consequence of our already-accepted definitions and hence is more than just a conjecture. On the other hand, the intuitive aspects are what we tend to invoke when we are setting up a problem. From the intuitive point of view what we are really saying is that if  $\lim_{x \rightarrow c} f(x) = L$  then in a sufficiently small neighborhood of  $c$ ,  $f(x)$  "behaves like"  $L$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.1(L) cont'd]

itself. For example of  $\lim_{x \rightarrow c} f(x) = L$  and  $L$  is positive then we can conclude that there is a neighborhood of  $c$  in which  $f(x)$  is always positive. Namely if  $L > 0$  we may choose, say,  $\epsilon = L/2$ . This choice is not critical; what is critical is that we choose  $\epsilon$  to be less than  $L$ . In terms of a picture:



The important thing is to choose  $\epsilon$  so that  $0 \notin (L-\epsilon, L+\epsilon)$

In this event, we now have that we can find  $\delta$  such that  $0 < |x - c| < \delta$  implies that  $L - \epsilon < f(x) < L + \epsilon$  but each element of  $(L - \epsilon, L + \epsilon)$  is a positive number by our choice of  $\epsilon$ . Therefore for this neighborhood of  $c$ ,  $f(x)$  cannot be negative.

As the course progresses, we will find ourselves often (we hope) being able to use limits in a very intuitive (natural) way but at the same time always being able to demonstrate the validity of our beliefs more rigorously whenever we are called upon to do so (as for example, in those cases where the situation is of sufficient complexity so that we are not at all positive about the truth of what we feel intuitively to be correct).

1.6.2(L)

$\lim_{x \rightarrow c} h(x) = 0$  means that for each  $\epsilon > 0$  we can find  $\delta > 0$  such that  $0 < |x - c| < \delta \rightarrow |h(x) - 0| < \epsilon$ .

Now  $h(x) = f(x)g(x)$ .

Hence, we must show that given  $\epsilon > 0$  we can exhibit  $\delta > 0$  such that  $0 < |x - c| < \delta \rightarrow |f(x)g(x)| < \epsilon$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.2(L) cont'd]

The fact that  $g(x)$  is bounded means that there exists a number  $K > 0$  such that  $|g(x)| < K$ .

This means that  $|f(x)g(x)| = |f(x)| |g(x)| < |f(x)|K$ . In turn, the fact that

$$|f(x)g(x)| < |f(x)|K \quad (1)$$

guarantees that  $|f(x)g(x)|$  will be less than  $\epsilon$  as soon as  $|f(x)| < \frac{\epsilon}{K}$ . (Notice here the "convenience" of the fact that  $K \neq 0$ .)

Recalling that  $\lim_{x \rightarrow c} f(x) = 0$ , we have that for  $\epsilon_1 > 0$  we can find  $\delta_1 > 0$  such that  $0 < |x - c| < \delta_1 \rightarrow |f(x) - 0| < \epsilon_1$ . If we choose  $\epsilon_1 = \frac{\epsilon}{K}$ , we have that there exists a number  $\delta_1 > 0$  such that

$$0 < |x - c| < \delta_1 \rightarrow |f(x) - 0| < \frac{\epsilon}{K}, \text{ or } |f(x)| < \frac{\epsilon}{K}$$

Putting this result into (1) we have:

$$\text{Given } \epsilon > 0 \text{ there exists } \delta_1 > 0 \text{ such that } 0 < |x - c| < \delta_1 \rightarrow |f(x)g(x)| < |f(x)|K < \left(\frac{\epsilon}{K}\right)K = \epsilon.$$

That is:

Given  $\epsilon > 0$ , there exists  $\delta_1 > 0$  such that  $0 < |x - c| < \delta_1$  implies that  $|f(x)g(x) - 0| < \epsilon$ .

By definition, then:

$$\lim_{x \rightarrow c} f(x)g(x) = \lim_{x \rightarrow c} h(x) = 0$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.2(L) cont'd]

The practical value of this result is that if we know that  $\lim_{x \rightarrow c} f(x) = 0$ , we can conclude that  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$  as soon as we know that  $g(x)$  is bounded.

This result is rather analogous to the result that any number times zero is zero. That is, if  $f(x)$  approaches 0 then so also goes  $f(x)g(x)$  if  $g(x)$  is bounded. The situation wherein  $g(x)$  is not bounded is analogous to the problem of "infinity" times zero which is indeterminate. ( $\infty \times 0$  is of the same ilk as  $0/0$ .) For example, for very large values of  $n$ ,  $2n$  is very large, while  $\frac{1}{n}$  is very small. Yet  $(2n) \left(\frac{1}{n}\right) = 2$  which is neither very small nor very large.

Another very important point is that it is not necessary that  $g(x)$  be bounded everywhere. Rather it is sufficient that  $g(x)$  be bounded in a neighborhood of  $x = c$ . In essence, finding a limit is a "local" thing that depends only on what is happening "nearby". In still other words if  $|g(x)| < K$  held only in some interval  $0 < |x - c| < \delta_2$  we would merely choose  $\delta = \min(\delta_1, \delta_2)$  and for this  $\delta$ , both  $|g(x)| < K$  and  $|f(x)| < \frac{\epsilon}{K}$  would both be true.

1.6.3(L)

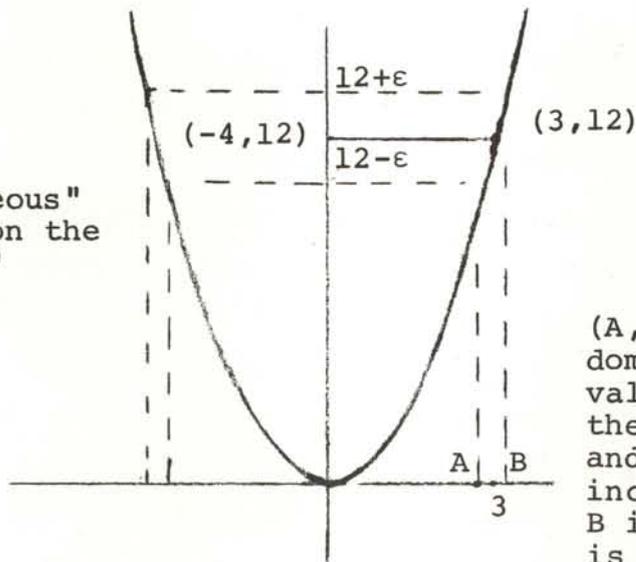
In disguised form, this exercise is asking us to prove that  $\lim_{t \rightarrow 3} t^2 + t = 12$ . Namely given  $\epsilon > 0$  we are being asked to exhibit  $\delta > 0$  such that  $0 < |t - 3| < \delta$  implies  $|t^2 + t - 12| < \epsilon$ .

Now, the fact that we want  $|t^2 + t - 12| < \epsilon$  means that  $-\epsilon < t^2 + t - 12 < \epsilon$ . We can then use the quadratic formula to locate the bounds on  $t$ . Pictorially:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.3(L) cont'd]

Our "extraneous" values are on the curve "near"  $t = -4$



(A,B) determines the domain of the required values for  $t$ . Since the curve rises more and more rapidly as  $t$  increases, notice that B is closer to 3 than A is to 3. Hence the largest value of  $\delta$  is the length of the segment from 3 to B.

However we will offer an alternative approach that leads to the same solution. Once we are at the stage where:

$$-\epsilon < t^2 + t - 12 < \epsilon \quad (1)$$

we observe that in terms of "completing the square",  $t^2 + t - 12$  may be written as  $t^2 + t + 1/4 - 49/4$ ; and this in turn may be written as  $(t + 1/2)^2 - 49/4$ .

Thus we have:

$$-\epsilon < (t + 1/2)^2 - 49/4 < \epsilon$$

$$49/4 - \epsilon < (t + 1/2)^2 < 49/4 + \epsilon$$

$$\frac{49-4\epsilon}{4} < (t + 1/2)^2 < \frac{49+4\epsilon}{4} \quad (2)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.3(L) cont'd]

Now recall from our discussion of inequalities that if  $a^2 < b^2 < c^2$  then all we can be sure of is that  $|a| < |b| < |c|$ . However if  $a, b,$  and  $c$  are all positive then  $a^2 < b^2 < c^2$  does imply that  $a < b < c$ . (By way of an illustration to help refresh any reluctant memories observe that  $-4 < -3$  but that  $(-4)^2 > (-3)^2$ .)

With this in mind let us revisit (2). For one thing we know that  $\epsilon$  is positive hence the right hand side of this equation is a positive number [and therefore it has a real (positive) square root]. Secondly, we are, by the nature of this problem interested in the case where  $t$  is "nearly" equal to 3. In particular, then, we may assume that  $t + 1/2$  is "near"  $7/2$  - and therefore, is at least positive. Thirdly, since we are interested primarily in small values of  $\epsilon$  we may also assume that  $49 - 4\epsilon$  is positive. Thus (2) may be rewritten as:

$$\left[ \frac{\sqrt{49-4\epsilon}}{2} \right]^2 < (t + 1/2)^2 < \left[ \frac{\sqrt{49+4\epsilon}}{2} \right]^2$$

and since all of the numbers being squared may be assumed to be positive, we obtain:

$$\frac{\sqrt{49-4\epsilon}}{2} < t + 1/2 < \frac{\sqrt{49+4\epsilon}}{2} \quad (3)$$

$\therefore$  Since  $t - 3 = (t + 1/2) - 7/2$ , (3) yields:

$$\frac{1}{2} \sqrt{49-4\epsilon} - 7/2 < t - 3 < \frac{1}{2} \sqrt{49+4\epsilon} - 7/2 \quad (4)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.3(L) cont'd]

(In terms of our earlier drawn graph  $\frac{1}{2}\sqrt{49+4\epsilon} - 7/2$  corresponds to the distance between 3 and B, while  $7/2 - \frac{1}{2}\sqrt{49-4\epsilon}$  ( $= |\frac{1}{2}\sqrt{49-4\epsilon} - 7/2|$ ) corresponds to the distance between A and 3. In terms of our previous observations about this graph, we have a pictorial proof [since B is closer to 3 than A is] that

$$\frac{1}{2}\sqrt{49+4\epsilon} - 7/2 < |\frac{1}{2}\sqrt{49-4\epsilon} - 7/2|$$

[a fact which is not so trivial to verify algebraically].)

At any rate, from (4), if we now pick any  $\delta$  such that

$$0 < \delta < \min \left\{ \frac{1}{2}\sqrt{49+4\epsilon} - 7/2, 7/2 - \frac{1}{2}\sqrt{49-4\epsilon} \right\} \quad (5)$$
$$\left( = \frac{1}{2}\sqrt{49+4\epsilon} - 7/2 \right)$$

we have solved the problem.

While (5) supplies us with an excellent solution, there is no denying the fact that this solution is computationally cumbersome. That is, it might not be too convenient given a particular value for  $\epsilon$  to compute  $\sqrt{49+4\epsilon}$ .

We can avoid much of the difficulty of this approach if we next observe that there is nothing sacred about finding the greatest value of  $\delta$  which will work.

With this in mind we can proceed as follows: We want  $|t^2+t - 12| < \epsilon$ . This in turn means that  $|(t+4)(t-3)| < \epsilon$ ; and since we have already demonstrated that the absolute value of a product is the product of absolute values, we may in turn write that:

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 6: A More Rigorous  
Approach to Limits I

[1.6.3(L) cont'd]

$$|t + 4| |t - 3| < \epsilon \quad (6)$$

Now for the purposes of our present problem, we know that  $t$  must be "near" 3, hence  $t + 4$  (which, of course is equal to  $|t + 4|$  when  $t$  is "near" 3) is "near" 7; or in terms of a more concrete inequality,  $t + 4$  being near 7 on either side, must lie between 6 and 8.\* Thus:

$$|t + 4| < 8 \quad (7)$$

If we combine the result of (7) with (6) we see that if  $|t - 3| < \epsilon/8$  it is SUFFICIENT to guarantee that  $|t + 4| |t - 3| < \epsilon$ . (We say sufficient rather than NECESSARY because since  $t + 4$  is less than 8 the inequality in (6) could hold even if  $|t - 3|$  were to exceed  $\epsilon/8$ . However, if we make sure that  $t - 3 < \epsilon/8$  then (6) certainly holds, even if we could have got away with less!)

This approach allows us to obtain an answer much more readily - namely  $0 < \delta < \epsilon/8$ .

While the problem is now solved, we would still like to tie together a few loose ends.

For one thing we assumed that we were interested in small values of  $\epsilon$ . Among other places we used this fact to assume that  $49 - 4\epsilon$  was positive. If we were not sure that  $\epsilon$  was this small (for example if  $\epsilon$  exceeded  $49/4$ ) the problem of finding  $\delta$  would be trivial

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\*If  $\delta > 1$  then  $t$  being within  $\delta$  of 3 would not guarantee that  $t + 4$  lies between 6 and 8. However in this case we can settle for a smaller value, say,  $\delta_1 < 1$ . In other words, as we have mentioned before, if it is more convenient we can always replace our neighborhood by a smaller one.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.3(L) cont'd]

since the "tolerance limits" were so great). In other words, the flavor of this problem is in no way lost by the assumption that  $\epsilon$  is small, even though nothing in the problem says we have to make this assumption (again, the main reason we make this assumption is to capture the reality of the situation. In dealing with limits we want to know what is happening "very near" the point in question. In modern dress, if we are making a trip and we want to know how fast we were going when we were near Albany, our speed when we were in Buffalo would be of little importance now).

Secondly, it might be interesting to see how  $\epsilon/8$  and  $\frac{1}{2}\sqrt{49+4\epsilon} - 7/2$  compare.

An application of the binomial theorem tells us that  $(a + b)^n \approx a^n + na^{n-1}b$  if  $b \ll a$  (that is, if  $b$  is very small compared with  $a$ ). Thus for  $\epsilon$  sufficiently small:

$$\begin{aligned}\sqrt{49+4\epsilon} &= (49+4\epsilon)^{1/2} \approx (49)^{\frac{1}{2}} + \frac{1}{2}(49)^{-\frac{1}{2}}(4\epsilon) \\ &\approx 7 + \frac{2\epsilon}{7}\end{aligned}$$

$\therefore$  For sufficiently small  $\epsilon$ ,

$$\frac{1}{2}\sqrt{49+4\epsilon} - 7/2 \approx \frac{1}{2}\left(7 + \frac{2\epsilon}{7}\right) - 7/2 = \frac{\epsilon}{7}$$

So it would appear that  $\frac{\epsilon}{7}$  is "close" to a least upper bound.

As a partial check let us look at  $t^2 + t - 12$  when  $t = 3 + \frac{\epsilon}{7}$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.3(L) cont'd]

We obtain:

$$\begin{aligned} & \left(3 + \frac{\epsilon}{7}\right)^2 + \left(3 + \frac{\epsilon}{7}\right) - 12 = \\ & 9 + \frac{6\epsilon}{7} + \frac{\epsilon^2}{49} + 3 + \frac{\epsilon}{7} - 12 = \\ & \epsilon + \frac{\epsilon^2}{49}, \end{aligned}$$

and while this exceeds  $\epsilon$  by  $\frac{\epsilon^2}{49}$ , we see that we are "pretty close" when  $\epsilon$  is small. (In fact, for small  $\epsilon$ ,  $\frac{\epsilon^2}{49}$  is much smaller.)

To generalize this approach, let us put  $t = 3 + \frac{\epsilon}{c}$  ( $c > 0$ ).  
Then:  $t^2 + t - 12 =$

$$\begin{aligned} & \left(3 + \frac{\epsilon}{c}\right)^2 + \left(3 + \frac{\epsilon}{c}\right) - 12 = \\ & 9 + \frac{6\epsilon}{c} + \frac{\epsilon^2}{c^2} + 3 + \frac{\epsilon}{c} - 12 = \\ & \frac{7\epsilon}{c} + \frac{\epsilon^2}{c^2} \end{aligned}$$

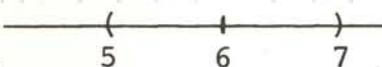
Thus for sufficiently small  $\epsilon$ , we need only be sure that  $c > 7$ . (Indeed  $c = 8$  meets this challenge, but, for example, so does  $c = 7.001$ .)

1.6.4

$$|x^2 - 5x - 6| < \epsilon \rightarrow |x - 6| |x + 1| < \epsilon$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.4 cont'd]

We are interested in values of  $x$  near 6. In particular if  $x$  is within 1 unit of 6,  ,  $x + 1$  is between  $5 + 1$  and  $7 + 1$ . That is:

$$6 < |x + 1| < 8$$

$$\therefore |x + 1| < 8$$

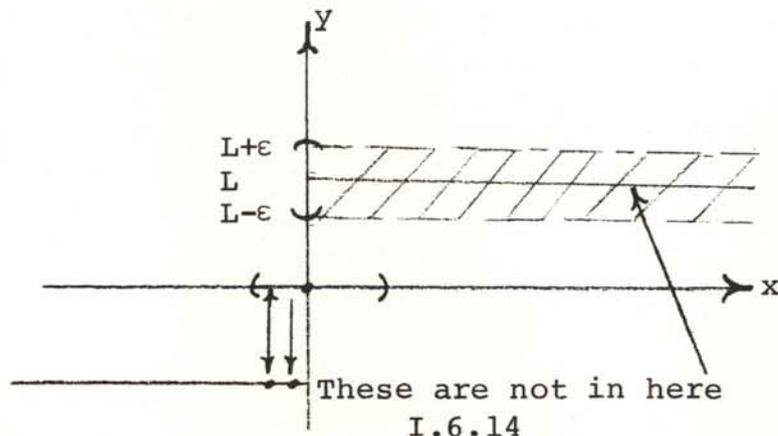
$$\therefore |x - 6| < \frac{\epsilon}{8} \rightarrow |x - 6| |x + 1| < \frac{\epsilon}{8} 8 = \epsilon$$

Thus, we may pick  $\delta = \frac{\epsilon}{8}$

1.6.5(L)

At first glance it might appear that this problem had two answers, 1 and -1. Surely as  $x$  approaches 0 through positive values,  $f(x)$  approaches 1. In fact, for all positive values of  $x$ ,  $f(x)$  is exactly equal to 1. In a similar way for each negative  $x$ ,  $f(x) = -1$ , hence it would appear that as  $x$  approaches 0 through negative values,  $f(x)$  approaches -1.

However, neither 1 nor -1 pass the epsilon-delta test. Pictorially the case against  $L = 1$  is given by:



If we surround  $L = 1$  by an  $\epsilon$ -neighborhood that doesn't include -1,  $f(x)$  for  $x < 0$  doesn't fall into the required range.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 6: A More Rigorous Approach to Limits I

[1.6.5(L) cont'd]

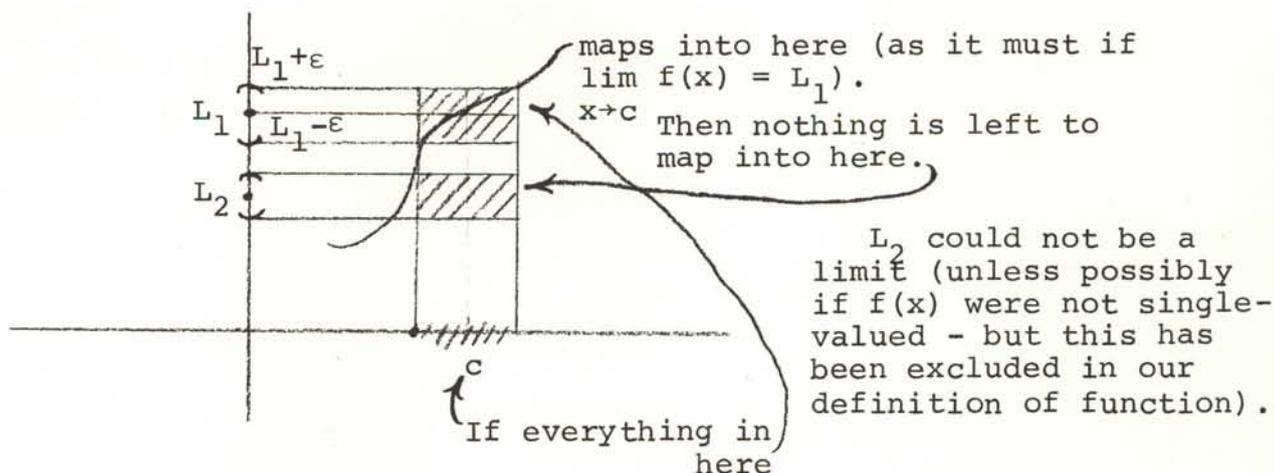
and a similar treatment holds for  $L = -1$ .

In fact, for any  $\delta > 0$ , the fact that  $0 < |x| < \delta$  means that  $x$  is within  $\delta$  of 0 and this means that  $x$  is in the interval  $(-\delta, \delta)$ , and in this way the pictorial situation is readily translated into the analytic equivalent.

This problem motivates the mathematical notion of such expressions as  $\lim_{x \rightarrow c}^+ f(x)$  and  $\lim_{x \rightarrow c}^- f(x)$ .

That is, we often talk about limits as  $x$  approaches  $c$  through values greater than  $c$  or through values less than  $c$ . In this context,  $\lim_{x \rightarrow c} f(x)$  existing means that both  $\lim_{x \rightarrow c}^+ f(x)$  and  $\lim_{x \rightarrow c}^- f(x)$  exist and that  $\lim_{x \rightarrow c}^+ f(x) = \lim_{x \rightarrow c}^- f(x) = \lim_{x \rightarrow c} f(x)$ .

As a final aside to this problem, let us observe that our definition of limit makes it impossible for a limit to have more than one value. Roughly speaking if  $\lim_{x \rightarrow c} f(x) = L_1$  and also  $\lim_{x \rightarrow c} f(x) = L_2$  and if  $L_1 \neq L_2$ , we could choose a value of  $\epsilon$  such that  $0 < \epsilon < |L_1 - L_2|$  and establish a contradiction. Namely:



SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 6: A More Rigorous  
Approach to Limits I

[1.6.5(L) cont'd]

What happens in our present problem is that  $\lim_{x \rightarrow 0} f(x)$  does not exist but that  $\lim_{x \rightarrow 0^+} f(x)$  and  $\lim_{x \rightarrow 0^-} f(x)$  each exist. In fact,  $\lim_{x \rightarrow 0^+} f(x) = 1$  while  $\lim_{x \rightarrow 0^-} f(x) = -1$ .

Perhaps the most important thing about this problem is that it shows us how once we accept a particular definition logic allows us to deduce inescapable consequences of the definition regardless of how strange the consequences might be. In this case the fact that we might feel that the limit should exist requires either that we change the definition of limit or else agree that our definition has some "unnatural" consequences.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits

UNIT 7: A More Rigorous Approach to Limits II

1.7.1(L)

a. This exercise is known more familiarly as "the limit of a product is the product of the limits". To prove that  $\lim_{x \rightarrow c} [f(x)g(x)] = L_1L_2$  we must show that for a given  $\epsilon > 0$  we can find  $\delta > 0$  such that  $0 < |x - c| < \delta$  implies  $|f(x)g(x) - L_1L_2| < \epsilon$ .

Now, what do we know about this problem? Well, we know that  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} g(x) = L_2$ . This, in turn, means that we have a good hold on both  $|f(x) - L_1|$  and  $|g(x) - L_2|$ . Namely, in a sufficiently small neighborhood of  $x = c$  we can make both of these expressions as small as we wish. The problem is to somehow or other manage to get such expressions from  $|f(x)g(x) - L_1L_2|$ . A neat trick in such cases is to add zero in a "clever" way. For example, starting with  $f(x)g(x)$  we observe that if we added  $-L_1g(x)$  to this we would have:  $f(x)g(x) - L_1g(x) = [f(x) - L_1]g(x)$ . So if we add this and then subtract it we obtain:

$$\begin{aligned} |f(x)g(x) - L_1L_2| &= \\ |f(x)g(x) - L_1g(x) + L_1g(x) - L_1L_2| &= \\ |(f(x) - L_1)g(x) + L_1(g(x) - L_2)| &\leq \\ |f(x) - L_1| |g(x)| + |L_1| |g(x) - L_2| &\quad (1) \end{aligned}$$

In (1) we see that both  $|f(x) - L_1|$  and  $|g(x) - L_2|$  can be made as small as we wish and  $g(x)$  and  $L_1$  are both bounded in a suitable neighborhood of  $x = c$  ( $L_1$  because it is a constant and  $g(x)$  because  $\lim_{x \rightarrow c} g(x)$  exists [see Exercise 1.6.1(L)]).

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.1(L) cont'd]

Thus (1), and hence  $|f(x)g(x) - L_1L_2|$ , can be made arbitrarily small, and this seems to confirm the result sought in this exercise. The next problem is to make the details more rigorous.

To this end, we observe that for a given  $\epsilon$  it is sufficient to make  $|f(x) - L_1| |g(x)| + |L_1| |g(x) - L_2| < \epsilon$  since:

$$|f(x)g(x) - L_1L_2| \leq |f(x) - L_1| |g(x)| + |L_1| |g(x) - L_2|$$

In turn this can be accomplished if each of the terms  $|f(x) - L_1| |g(x)|$  and  $|L_1| |g(x) - L_2|$  are less than  $\epsilon/2$ . Now by virtue of the fact that  $g(x)$  is bounded in a neighborhood of  $c$ , we can say that there exists a number  $\delta_1$  such that if  $0 < |x - c| < \delta_1$ ,  $|g(x)| < K$ , for some number  $K$ . By virtue of the fact that  $\lim_{x \rightarrow c} f(x) = L_1$ , we can find another number  $\delta_2$  such that  $0 < |x - c| < \delta_2$  implies that  $|f(x) - L_1| < \epsilon/2K$ . (Notice here that we are using the idea that if  $|f(x) - L_1| < \epsilon/2K$  and  $|g(x)| < K$ , then  $|f(x) - L_1| |g(x)| < (\epsilon/2K) K = \epsilon/2$ .)

In a similar way, since  $\lim_{x \rightarrow c} g(x) = L_2$ , we can find a number  $\delta_3$  such that  $0 < |x - c| < \delta_3$  implies that  $|g(x) - L_2| < \epsilon/2 |L_1|$  (the only trouble is if  $L_1 = 0$ , but in this case the result is true by the result of exercise 1.6.2(L)).

Now if we choose  $\delta = \min(\delta_1, \delta_2, \delta_3)$  all three of the above results hold. Hence:  $0 < |x - c| < \delta$  implies that  $|f(x)g(x) - L_1L_2| \leq |f(x) - L_1| |g(x)| + |L_1| |g(x) - L_2|$  which in turn is less than  $(\frac{\epsilon}{2K})(K) + |L_1| (\frac{\epsilon}{2|L_1|}) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ . That is,  $0 < |x - c| < \delta$  implies that  $|f(x)g(x) - L_1L_2| < \epsilon$  and the result follows.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.1(L) cont'd]

This exercise has many aspects which recommend it for study. For one thing, it was not quite as obvious in this case as to how one determines delta in terms of epsilon as it was in some of our earlier examples presented in the previous unit.

Another very important result of this exercise is that we can now, once for all, compute the limit of a product by merely taking the product of the limits. The point is that in many problems we are only concerned with whether the limit exists and if it does what is its value; and we are not concerned with being able to exhibit a specific delta for a given epsilon.

Up to now we have stressed the epsilon-delta technique both because we wanted to emphasize the definition of a limit and also because there were times when we might want to know specifically the size of a particular neighborhood. What we wish to now is to exploit the other side of the coin and see how we may conveniently determine limits without having to make recourse to epsilons and deltas (and by the way this is a common situation in most sciences. We have one definition that is best suited for qualitative purposes and other criteria which lend themselves more readily to quantitative situations).

Thus the main aim of part (b) is to utilize the result of part (a) and determine the limit directly. At the same time we will take the opportunity to point out that even in this case, something is lost if we completely abandon the epsilon-delta ideas.

To begin with, let us assume that  $\lim_{x \rightarrow c} \sqrt{f(x)}$  exists and let us call this limit G.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.1(L) cont'd]

Now  $\lim_{x \rightarrow 0} \sqrt{f(x)} = G$  implies by the result of 1.7.1 (L) that  
 $\lim_{x \rightarrow 0} (\sqrt{f(x)})^2 = G^2$ ; but since  $(\sqrt{f(x)})^2 = f(x)$ , we have that:

$$G^2 = L$$

and the result follows.

Notice, however, that we have only proven that if there is a limit, it is  $\sqrt{L}$ . We have not proven that the limit exists.

To prove that a limit exists, we often have no recourse other than the epsilon-delta approach. In this case, we would want to show that we could make  $\sqrt{f(x)} - \sqrt{L}$  arbitrarily small, and we could accomplish this by writing:

$$\sqrt{f(x)} - \sqrt{L} = \frac{f(x) - L}{\sqrt{f(x)} + \sqrt{L}}$$

We could then argue that the numerator can be made small and unless  $L = 0$ , the denominator is bounded away from 0, and hence that the quotient can be made small. A little refinement handles the case  $L = 0$ , but this is not our main purpose in this exercise. The major point is that there is a difference between a limit existing and knowing its value if it does exist.

Finally, to complete this exercise, we can use mathematical induction to show that the limit of a product is the product of the limits. We would then write  $\lim_{x \rightarrow c} f(x)$  as  $\lim_{x \rightarrow c} \left[ (\sqrt[n]{f(x)})^n \right]$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.1(L) cont'd]

Letting  $\lim_{x \rightarrow c} f(x) = G$  and invoking the product theorem, we obtain:

$$L = \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left[ \left( \sqrt[n]{f(x)} \right)^n \right] = \left( \lim_{x \rightarrow c} f(x) \right)^n = G^n$$

whereupon it follows that  $G = L$ .

Thus if  $\lim_{x \rightarrow c} \sqrt[n]{f(x)}$  exists it is equal to  $\sqrt[n]{\lim_{x \rightarrow c} f(x)}$ .

1.7.2(L)

In the previous unit we proved that  $\lim_{t \rightarrow 3} (t^2 + t) = 12$  and  $\lim_{x \rightarrow 6} (x^2 - 5x) = 6$  by the epsilon=delta method. The main purpose of this exercise is to reinforce the remarks made in Exercise 1.7.1(L) and show that we can now find these limits much more conveniently. Namely, by invoking such results as the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, etc., we obtain:

$$\begin{aligned} \lim_{t \rightarrow 3} (t^2 + t) &= \lim_{t \rightarrow 3} t^2 + \lim_{t \rightarrow 3} t \\ &= \lim_{t \rightarrow 3} (t \times t) + \lim_{t \rightarrow 3} t \\ &= \left[ \left( \lim_{t \rightarrow 3} t \right) \left( \lim_{t \rightarrow 3} t \right) \right] + \lim_{t \rightarrow 3} t \\ &= [(3) (3)] + 3 \\ &= 12 \end{aligned}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.2(L) cont'd]

(Equivalently we could have shown that  $\lim_{t \rightarrow 3} (t^2 + t - 12) = 0$  by writing  $t^2 + t - 12 = (t + 4)(t - 3)$ .)

Then

$$\begin{aligned}\lim_{t \rightarrow 3} (t^2 + t - 12) &= \lim_{t \rightarrow 3} [(t + 4)(t - 3)] = \lim_{t \rightarrow 3} (t + 4) \lim_{t \rightarrow 3} (t - 3) \\ &= 7 \times 0 \\ &= 0.\end{aligned}$$

In a similar way, we observe that:

$$\begin{aligned}\lim_{x \rightarrow 6} (x^2 - 5x) &= \lim_{x \rightarrow 6} x^2 + \lim_{x \rightarrow 6} (-5x) \\ &= (\lim_{x \rightarrow 6} x) (\lim_{x \rightarrow 6} x) + [\lim_{x \rightarrow 6} -5] (\lim_{x \rightarrow 6} x) \\ &= (6) (6) + (-5) (6) \\ &= 36 - 30 = 6\end{aligned}$$

The important point is that we could compute these limits without specific recourse to epsilons and deltas - yet we must keep in mind that it was the use of the epsilon-delta approach that allowed us to invoke these limit theorems validly.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

1.7.3

$$\begin{aligned}\lim_{x \rightarrow 1} [(x + 1)^5 (x^2 + x + 2)] &= [\lim_{x \rightarrow 1} (x + 1)^5] [\lim_{x \rightarrow 1} (x^2 + x + 2)] \\ &= [\lim_{x \rightarrow 1} (x + 1)]^5 [\lim_{x \rightarrow 1} x^2 + \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 2] \\ &= [\lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 1]^5 [\lim_{x \rightarrow 1} x)^2 + \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 2] \\ &= (1 + 1)^5 [1^2 + 1 + 2] \\ &= (2^5) (4) = (32) (4) \\ &= 128\end{aligned}$$

Notice, again, the advantage of not being required to exhibit a specific  $\delta$  for a given  $\epsilon$  in order to find the required limit.

1.7.4(L)

In essence what we must do here is to show that  $\left| \frac{1}{f(x)} - \frac{1}{L} \right|$  becomes arbitrarily small as  $x$  gets arbitrarily close to  $c$ . (The restriction that  $L \neq 0$  is obvious in the sense that  $L = 0$  implies that  $1/L$  doesn't exist). From a semi-rigorous point of view, we observe that:

$$\frac{1}{f(x)} - \frac{1}{L} = \frac{L - f(x)}{Lf(x)} \quad (1)$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.4(L) cont'd]

In (1) we observe that since  $\lim_{x \rightarrow c} f(x) = L$ , then: (a)  $L - f(x)$  can be made arbitrarily small, and (b)  $f(x)$  is bounded in a neighborhood of  $x = c$  (see solution to Exercise 1.6.1(L)). In other words, the numerator of (1) may be made as small as we wish while the denominator is bounded but not zero. Thus the quotient exists and can be made as small as we wish.

To handle this problem more formally, we can prove that  $\lim_{x \rightarrow c} \frac{L - f(x)}{L} = 0$  and that  $\frac{1}{f(x)}$  is bounded. We could then invoke the result of Exercise 1.6.2(L) to conclude:

$$\lim_{x \rightarrow c} \left\{ \frac{1}{f(x)} - \frac{1}{L} \right\} = \lim_{x \rightarrow c} \left[ \left( \frac{L - f(x)}{L} \right) \left( \frac{1}{f(x)} \right) \right] = 0$$

$$\therefore \lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{1}{L}$$

We may observe that this exercise together with Exercise 1.7.1(L) (a) gives us an interesting way of proving the quotient theorem for limits. Namely suppose that  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} g(x) = L_2$ , with  $L_2 \neq 0$ . Then we may write  $\frac{f(x)}{g(x)}$  as  $f(x) \left[ \frac{1}{g(x)} \right]$ .

We then have:

---

\*Note that  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} [f(x) - L] = 0$  are equivalent statements. To prove this, merely write what the epsilon-delta definition for each says and see that they are identical.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.4(L) cont'd]

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow c} \left[ f(x) \right] \left[ \frac{1}{g(x)} \right] \\ &= \lim_{x \rightarrow c} f(x) \lim_{x \rightarrow c} \frac{1}{g(x)} \\ &= L_1 \left( \frac{1}{L_2} \right) \\ &= \frac{L_1}{L_2} \\ &= \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}\end{aligned}$$

1.7.5(L)

In more intuitive language this problem is asking us to verify that studying what happens for large values of  $x$  is equivalent to studying what happens for small values of  $1/x$ . That is, as  $x$  gets large,  $1/x$  gets small. From a more analytical point of view, the significance of the result lies in the fact that our limit theorems involve the fact that when we say  $x$  approaches  $c$ ,  $c$  is a real number. On the other hand  $\infty$  is not a real number. Thus, what we may do, once we prove the stated result, is: whenever we see  $\lim_{x \rightarrow \infty} f(x)$  we may replace it by  $\lim_{x \rightarrow 0^+} g(x)$ , where  $g(x) = f(\frac{1}{x})$ ; and thus reduce the "new" problem to a previously-solved type.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.5(L) cont'd]

So much for the motivation of this exercise, let us now turn our attention to a formal proof. What we must do is show that either property leads to the other. For example, suppose we start with the fact that  $\lim_{x \rightarrow \infty} f(x) = L$ . We must show that this implies that  $\lim_{x \rightarrow 0^+} f(\frac{1}{x}) = L$ . That is, given  $\epsilon > 0$  we must show that there exists  $\delta > 0$  such that  $0 < x < \delta$  implies that  $|f(\frac{1}{x}) - L| < \epsilon$ . (Notice here that we are saying  $0 < x < \delta$  NOT  $0 < |x| < \delta$ . This is why we write  $\lim_{x \rightarrow 0^+}$  rather than  $\lim_{x \rightarrow 0}$ . This change comes about from the fact that  $x \rightarrow \infty$  implies that, among other things,  $x$  is positive.)

At any rate, from the definition of  $\lim_{x \rightarrow \infty} f(x) = L$ , we know that given any  $\epsilon > 0$  we can find  $M > 0$  such that  $t > M$  implies that  $|f(t) - L| < \epsilon$ . (The change from  $x$  to  $t$  will clarify our notation later in the problem. For now observe that no change occurs by replacing  $x$  by  $t$ , since the use of either  $t$  or  $x$  is as a "dummy" variable. In still other words, in the notation  $\lim_{x \rightarrow c} f(x)$ , we may think of  $x$  as denoting any variable. That is,  $\lim_{x \rightarrow c} f(x)$  may be viewed  $\lim_{[] \rightarrow c} f([])$ .)

If we now use a change of variable and let  $t$  be replaced by  $\frac{1}{x}$ , we have that  $\lim_{x \rightarrow \infty} f(x) = L$  means that given  $\epsilon > 0$  we can find  $M > 0$  such that if  $\frac{1}{x} > M$  then  $|f(\frac{1}{x}) - L| < \epsilon$ . (Notice here that  $x > 0$  since if  $x = 0$ ,  $1/x$  would be undefined.) Next, since both  $x$  and  $M$  are positive, it follows that  $\frac{1}{x} > M$  if and only if  $x < \frac{1}{M}$ . Thus, if we now let  $\delta$  denote  $\frac{1}{M}$ , we have arrived at:

$\lim_{x \rightarrow \infty} f(x) = L$  means that given  $\epsilon > 0$  we can find  $\delta > 0$  such that  $0 < x < \delta$  implies that  $|f(\frac{1}{x}) - L| < \epsilon$ ; and this is precisely what it means to say that  $\lim_{x \rightarrow 0^+} f(\frac{1}{x}) = L$ . Thus we have proved that

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.5(L) cont'd]

$\lim_{x \rightarrow \infty} f(x) = L$  implies that  $\lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = L$ . While we spare you the details, it is not hard to show that reversing the steps in our above derivation establishes that the converse is also true, and the result of the exercise follows.

Again from a practical point of view,  $\lim_{x \rightarrow \infty} f(x)$  merely means to study  $f(x)$  for large values of  $x$ , and the fact that this is equivalent to studying  $f\left(\frac{1}{x}\right)$  for values of  $x$  near 0 allows us to utilize all of our previous limit theorems in the investigation of  $\lim_{x \rightarrow \infty} f(x)$ .

1.7.6(L)

From the result of Exercise 1.7.5(L) we have:

$$\begin{aligned} \lim_{x \rightarrow \infty} \left[ \frac{3x^2 - 7x + 1}{4x^2 + 5x - 7} \right] &= \lim_{x \rightarrow 0^+} \left[ \frac{3\left(\frac{1}{x}\right)^2 - 7\left(\frac{1}{x}\right) + 1}{4\left(\frac{1}{x}\right)^2 + 5\left(\frac{1}{x}\right) - 7} \right] \\ &= \lim_{x \rightarrow 0^+} \left[ \frac{(3 - 7x + x^2)/x^2}{(4 + 5x - 7x^2)/x^2} \right] \end{aligned}$$

Since  $x \rightarrow 0^+$  implies that  $x \neq 0$ , we finally have:

$$\lim_{x \rightarrow \infty} \left[ \frac{3x^2 - 7x + 1}{4x^2 + 5x - 7} \right] = \lim_{x \rightarrow 0^+} \left[ \frac{3 - 7x + x^2}{4 + 5x - 7x^2} \right]$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.6(L) cont'd]

Now we may employ our previous limit theorems (observing first that the theorems remain valid if  $\lim_{x \rightarrow c}$  is replaced by  $\lim_{x \rightarrow c}^+$  or  $\lim_{x \rightarrow c}^-$ ) to obtain:

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left[ \frac{3 - 7x + x^2}{4 + 5x - 7x^2} \right] &= \frac{\lim_{x \rightarrow 0^+} 3 - 7 \lim_{x \rightarrow 0^+} x + (\lim_{x \rightarrow 0^+} x)^2}{\lim_{x \rightarrow 0^+} 4 + 5 \lim_{x \rightarrow 0^+} x - 7(\lim_{x \rightarrow 0^+} x)^2} = \frac{3 - 7(0) + 0^2}{4 + 5(0) - 7(0)} \\ &= \frac{3}{4} \end{aligned}$$

Hence:

$$\lim_{x \rightarrow \infty} \left[ \frac{3x^2 - 7x + 1}{4x^2 + 5x - 7} \right] = \frac{3}{4}$$

Before we leave this exercise, let us point out that we may have decided that there was a quicker way to arrive at the same answer. For example when  $x$  is very large,  $x$  is "small" compared with  $x^2$ . Thus we might feel that for very large values of  $x$ ,  $3x^2 - 7x + 1$  "behaves like"  $3x^2$  while  $4x^2 + 5x - 7$  "behaves like"  $4x^2$ . Thus:

$$\frac{3x^2 - 7x + 1}{4x^2 + 5x - 7}$$

"behaves like"  $\frac{3x^2}{4x^2}$  or  $\frac{3}{4}$  for large values of  $x$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.6(L) cont'd]

There is no denying, as we have just shown, that this approach yields the correct answer, and that it does it without the fuss of our first method. There is, however, a great deal of subjectivity involved with the expression "behaves like" and it is possible that results which seem plausible may in fact turn out to be incorrect in situations where great delicacy is necessary. We shall indicate this in more specific detail in the next exercise.

1.7.7(L)

We might be tempted to say that for large values of  $x$ ,  $x^2 + x$  "behaves like"  $x^2$ . We might then decide  $\sqrt{x^2 + x} - x$  behaves like  $\sqrt{x^2} - x = x - x = 0$ . Yet, surprising as it may seem  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$  is NOT equal to 0. To obtain some sort of empirical evidence that  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$  is not zero, we might compute  $\sqrt{x^2 + x} - x$  for certain values and see what happens when  $x$  gets large. That is, letting  $f(x) = \sqrt{x^2 + x} - x$ , we see that:

$$f(1) = \sqrt{2} - 1 = 0.41^+$$

$$f(2) = \sqrt{6} - 2 = 0.44^+$$

$$f(3) = \sqrt{12} - 3 = 2\sqrt{3} - 3 = 0.46^+$$

$$f(100) = \sqrt{(100^2 + 100)} - 100 = \sqrt{10,100} - 100 = 100.49^+ - 100 = 0.49^+$$

and while we haven't really proven anything we begin to feel that  $f(x)$  increases with  $x$  and seems to begin to "behave like"  $\frac{1}{2}$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.7(L) cont'd]

To show that  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$  is indeed  $\frac{1}{2}$ , we may observe that:

$$\begin{aligned}\sqrt{x^2 + x} - x &= \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} \\ &= \frac{(\sqrt{x^2 + x})^2 - x^2}{\sqrt{x^2 + x} + x} \\ &= \frac{x}{\sqrt{x^2 + x} + x} \quad (\text{in essence, we rationalized the numerator})\end{aligned}$$

$$\therefore \sqrt{x^2 + x} - x = \frac{x}{x(\sqrt{1 + \frac{1}{x}} + 1)}$$

Since we are interested in large values of  $x$  ( $x \rightarrow \infty$ ) we may assume  $x \neq 0$ . Hence, for  $x > 0$ :

$$\sqrt{x^2 + x} - x = \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} \quad (1)$$

Looking at the right hand side of (1) we sense that as  $x \rightarrow \infty$ ,  $\frac{1}{x} \rightarrow 0$  and hence:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x) = \lim_{x \rightarrow \infty} \left[ \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} \right] = \frac{1}{1 + 1} = \frac{1}{2}.$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.7(L) cont'd]

If we prefer to invoke the result of Exercise 1.7.1(L) we  
have:

$$\lim_{x \rightarrow \infty} \left[ \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} \right] = \lim_{x \rightarrow 0^+} \left[ \frac{1}{\sqrt{1 + x} + 1} \right]$$

This, in turn, becomes:

$$\frac{\left[ \lim_{x \rightarrow 0^+} 1 \right]}{\left[ \lim_{x \rightarrow 0^+} \sqrt{1 + x} \right] + \left[ \lim_{x \rightarrow 0^+} 1 \right]} = \frac{1}{\left[ \lim_{x \rightarrow 0^+} \sqrt{1 + x} \right] + 1}$$

Now from the result of Exercise 1.7.1(b), since  $\lim_{x \rightarrow 0^+} 1 + x = 1$ ,  
it follows that  $\lim_{x \rightarrow 0^+} \sqrt{1 + x} = \sqrt{1} = 1$ .

$$\therefore \lim_{x \rightarrow 0^+} \left[ \frac{1}{\sqrt{1 + x} + 1} \right] = \frac{1}{1 + 1} = \frac{1}{2}$$

Of course we could have done this problem by going at once  
from  $\lim_{x \rightarrow \infty} \left[ \sqrt{x^2 + x} - x \right]$  to

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.7(L) cont'd]

$$\lim_{x \rightarrow 0^+} \left[ \sqrt{\left(\frac{1}{x}\right)^2 + \frac{1}{x}} - \frac{1}{x} \right]$$

and then proceeding as described above.

The main feeling that we hope comes about from Exercises 1.7.2(L) and 1.7.3(L) is that we learn to appreciate the sensitive balance between intuition and rigor. We shall not object to such terms as "behaves likes" provided that whenever we are challenged to do so, we can show more objectively that our intuitive feeling can be "documented."

1.7.8(L)

We observe that as  $x \rightarrow 1^+$ ,  $x - 1$  becomes a very small but positive number, hence  $\frac{1}{x - 1}$  gets arbitrarily large. In other words we can make  $\frac{1}{x - 1}$  as large as we wish by choosing  $x$  ( $>1$ ) sufficiently close to 1.

This in turn implies that if  $L$  is any finite number it is impossible that  $\lim_{x \rightarrow 1^+} \frac{1}{x - 1} = L$ .

Indeed given any  $\epsilon > 0$  we can find  $\delta > 0$  such that  $1 < x < 1 + \delta$  (equivalently:  $0 < x - 1 < \delta$ ) implies  $\left| \frac{1}{x - 1} - L \right| > \epsilon$  (since  $\frac{1}{x - 1}$  gets as large as we wish by an appropriate choice of  $\delta$ , while  $L$  remains fixed).

To indicate the fact that  $\frac{1}{x - 1}$  increases without bound as  $x \rightarrow 1^+$ , we write

$$\lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.8(L) cont'd]

Quite in general,

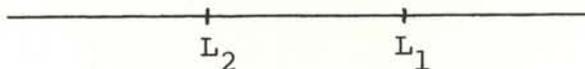
$$\lim_{x \rightarrow c} f(x) = \infty$$

means that given any real number  $M > 0$  (no matter how large), we can find  $\delta > 0$  such that whenever  $0 < |x - c| < \delta$  then  $|f(x)| > M$ . This is the formal way of saying that  $f(x)$  increases without bound as  $x$  approaches  $c$ .

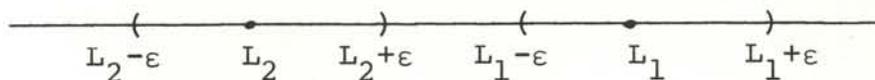
1.7.9(L)

$$\text{Let } \lim_{x \rightarrow c} f(x) = L_1 \text{ and } \lim_{x \rightarrow c} g(x) = L_2.$$

We shall show that  $L_1 > L_2$  implies a contradiction. Namely, in terms of a picture:



Choose any  $\varepsilon > 0$  such that  $\varepsilon < \frac{1}{2} |L_1 - L_2|$ . This insures the fact that we can find  $\varepsilon$ -neighborhoods of both  $L_1$  and  $L_2$  which do not intersect. Thus:



Now by definition of  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} g(x) = L_2$  for the above  $\varepsilon$ , there exist numbers  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 7: A More Rigorous  
Approach to Limits II

[1.7.9(L) cont'd]

$$0 < |x - c| < \delta_1 \rightarrow L_1 - \varepsilon < f(x) < L_1 + \varepsilon$$

$$0 < |x - c| < \delta_2 \rightarrow L_2 - \varepsilon < g(x) < L_2 + \varepsilon$$

Thus if  $\delta = \min\{\delta_1, \delta_2\}$ , we have that

$$0 < |x - c| < \delta \rightarrow \begin{cases} L_1 - \varepsilon < f(x) < L_1 + \varepsilon \\ L_2 - \varepsilon < g(x) < L_2 + \varepsilon \end{cases}$$

Recalling that we chose  $\varepsilon$  so that  $L_2 + \varepsilon < L_1 - \varepsilon$ , we have

$$0 < |x - c| < \delta \rightarrow g(x) < L_2 + \varepsilon < L_1 - \varepsilon < f(x); \text{ or} \\ g(x) < f(x).$$

But,  $g(x) < f(x)$  gives us the desired contradiction since we are given that  $f(x) < g(x)$ .

Notes:

(1) Observe that it is not important that  $f(x) < g(x)$  for all  $x$ . Rather it is sufficient that  $f(x) < g(x)$  for all  $x$  in some deleted neighborhood of  $x = c$ . For example if  $f(x) < g(x)$  only if  $0 < |x - c| < \delta_3$ , we need only let  $\delta_4 = \min\{\delta, \delta_3\}$  to adjust the result of this exercise to the new situation.

(2) Notice that  $f(x) \not\leq g(x)$  for all  $x$  doesn't prevent the fact that  $\lim_{x \rightarrow c} f(x)$  might equal  $\lim_{x \rightarrow c} g(x)$ .

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Unit 7: A More Rigorous Approach to Limits II

[1.7.9(L) cont'd]

For example  $\frac{x^2}{x^2 + 1} < 1$  for all  $x$ , yet  $\lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 1} = 1$ .

(3) An important corollary to this exercise is that if  $f(x) < g(x) < h(x)$  and  $\lim_{x \rightarrow c} f(x)$ ,  $\lim_{x \rightarrow c} g(x)$ , and  $\lim_{x \rightarrow c} h(x)$  all exist, then:

$$\lim_{x \rightarrow c} f(x) < \lim_{x \rightarrow c} g(x) < \lim_{x \rightarrow c} h(x)$$

In particular if  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) (=L)$  then

$$L < \lim_{x \rightarrow c} g(x) < L$$

or

$$\lim_{x \rightarrow c} g(x) = L.$$

In other words one way to compute  $\lim_{x \rightarrow c} g(x)$  is to find two functions  $f$  and  $h$  such that

$$f(x) < g(x) < h(x)$$

but where  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x)$

In this case  $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} f(x)$  (or  $\lim_{x \rightarrow c} h(x)$ ). This "sandwiching" device is a very powerful technique for determining limits - and we shall use this device several times in this course.



UNIT 8: Mathematical Induction

1.8.1

We let  $P(n)$  denote the proposition that

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Since  $1 = 1 \frac{(2)}{2}$  we see that  $P(1)$  is true.

We next assume that  $P(k)$  is true and try to show that because of this  $P(k+1)$  is also true.

That is, we try to show that

$$1 + \dots + k = \frac{k(k+1)}{2}$$

implies that

$$1 + \dots + (k+1) = \frac{(k+1)(k+2)}{2}$$

To this end:

$$\begin{aligned} 1 + \dots + (k+1) &= (1 + \dots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \quad \text{[Since our assumption} \\ &\quad \text{is } 1 + \dots + k = \frac{k(k+1)}{2}] \\ &= (k+1) \left( \frac{k}{2} + 1 \right) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

[1.8.1 cont'd]

$$\therefore \begin{cases} P(1) \text{ is true} \\ \text{The truth of } P(k) \text{ implies the truth of } P(k+1) \end{cases}$$

$\therefore$  By mathematical induction  $P(n)$  is true for all positive integers,  $n$ .

1.8.2

Letting  $P(n)$  denote  $1^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$ , we see that

$$P(1) \text{ means } 1^3 = \frac{1^2(1+1)^2}{4} \text{ or } 1 = 1 \quad \therefore P(1) \text{ is true}$$

$$P(2) \text{ means } 1^3 + 2^3 = \frac{2^2(2+1)^2}{4} \text{ or } 9 = 9 \quad \therefore P(2) \text{ is true}$$

Assume  $P(k)$  is true. That is,  $1^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4}$

$$\begin{aligned} \text{Now: } 1^3 + \dots + (k+1)^3 &= (1^3 + \dots + k^3) + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad (\text{by the assumption} \\ &\quad \text{that } P(k) \text{ is true}) \\ &= (k+1)^2 \left[ \frac{k^2}{4} + (k+1) \right] = (k+1)^2 \left[ \frac{k^2 + 4k + 4}{4} \right] \\ &= \frac{(k+1)^2 (k+2)^2}{4} \end{aligned}$$

but this is precisely the statement that  $P(k+1)$  is true.

[1.8.2 cont'd]

∴ By Mathematical Induction  $P(n)$  is true for all  $n$ .

Interesting Aside:

Note that  $\frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$  and that  $\frac{n(n+1)}{2} = 1 + \dots + n$ .

$$\therefore 1^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2 = (1 + \dots + n)^2 \quad (1)$$

Equation (1) indicates the interesting fact that the sum of the first  $n$  cubes is the square of the sum of the first  $n$  numbers. For example.

$$1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100 = (1+2+3+4)^2$$

### 1.8.3

Letting  $P(n)$  denote  $1 + 3 + \dots + (2n - 1) = n^2$  we see at once that  $P(1)$  is true. Namely  $1 = 1^2$ .

Next  $P(k+1)$  means that  $1 + 3 + \dots + (2[k+1] - 1) = (k+1)^2$

Now if we assume that  $P(k)$  is true we obtain:

$$\begin{aligned} 1 + 3 + \dots + [2(k+1)-1] &= [1 + 3 + \dots + (2k-1)] + [2(k+1)-1] \\ &= k^2 + [2(k+1)-1] \quad (\text{since we assume } P(k) \text{ is true}) \\ &= k^2 + (2k+1) \\ &= (k+1)^2 \end{aligned}$$



SOLUTIONS: Calculus of a Single Variable - Block I: Sets,  
Functions, and Limits - Unit 8: Mathematical Induction

[1.8.4 cont'd]

∴ Let's assume  $|a_1 + \dots + a_k| \leq |a_1| + \dots + |a_k|$  is true.  
Then the required result follows as soon as we can prove that  
 $|a_1 + \dots + a_{k+1}| \leq |a_1| + \dots + |a_{k+1}|$

Well:

$$\begin{aligned} |a_1 + \dots + a_{k+1}| &= |(a_1 + \dots + a_k) + a_{k+1}| \\ &\leq |a_1 + \dots + a_k| + |a_{k+1}|^* \\ &\leq (|a_1| + \dots + |a_k|) + |a_{k+1}| \quad (\text{by the} \\ &\hspace{15em} \text{assumption} \\ &\hspace{15em} \text{that } P(k) \text{ is} \\ &\hspace{15em} \text{true}) \end{aligned}$$

and since

$$\begin{aligned} (|a_1| + \dots + |a_k|) + |a_{k+1}| &= \\ |a_1| + \dots + |a_k| + |a_{k+1}| &\quad (\text{the associative} \\ &\hspace{15em} \text{rule for addition}) \end{aligned}$$

the required result follows.

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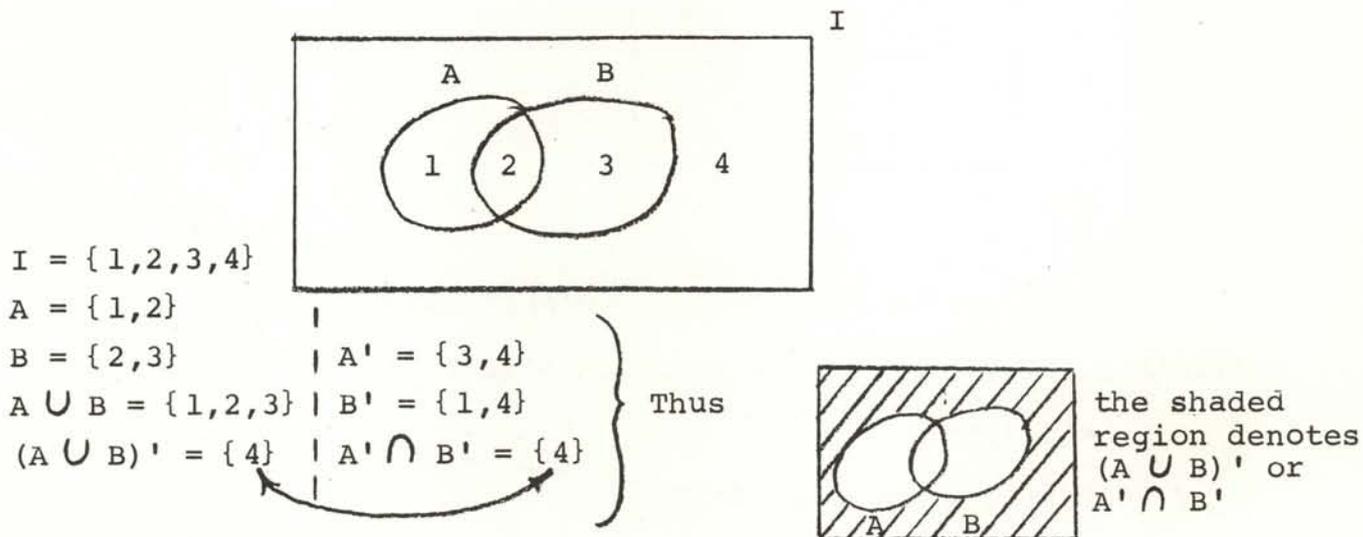
\*Here we utilize the fact that  $a_1 + \dots + a_k$  is a single number,  $A_k$ . Then  $|(a_1 + \dots + a_k) + a_{k+1}| = |A_k + a_{k+1}|$  and by the truth of  $P(2)$ , this in turn cannot exceed  $|A_k| + |a_{k+1}|$ .



SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits

QUIZ

1. (a)



(b) Let  $P(n)$  denote the proposition that

$$(A_1 \cup \dots \cup A_n)' = A_1' \cap \dots \cap A_n' \quad n \geq 2$$

Then we already know from part (a) that  $P(2)$  is true. So we assume  $P(k)$  is true and show that this implies the truth of  $P(k+1)$ .

That is, we must show that if  $(A_1 \cup \dots \cup A_k)' = A_1' \cap \dots \cap A_k'$  then  $(A_1 \cup \dots \cup A_{k+1})' = A_1' \cap \dots \cap A_{k+1}'$

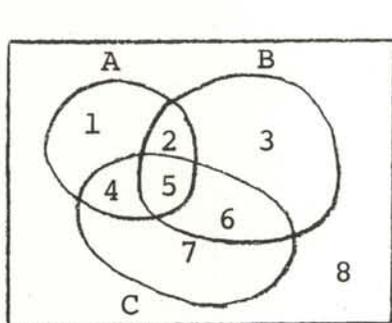
Now:

$$\begin{aligned}
 (A_1 \cup \dots \cup A_{k+1})' &= ([A_1 \cup \dots \cup A_k] \cup A_{k+1})' \\
 &= [A_1 \cup \dots \cup A_k]' \cap A_{k+1}' \quad (\text{since } P(2) \text{ is true}) \\
 &= [A_1' \cap \dots \cap A_k'] \cap A_{k+1}' \quad (\text{since } P(k) \text{ is assumed true}) \\
 &= A_1' \cap \dots \cap A_k' \cap A_{k+1}'
 \end{aligned}$$

and the proof by induction is complete.

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Quiz

[1. cont'd]



$$I = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$A = \{1, 2, 4, 5\}$$

$$B = \{2, 3, 5, 6\}$$

$$C = \{4, 5, 6, 7\}$$

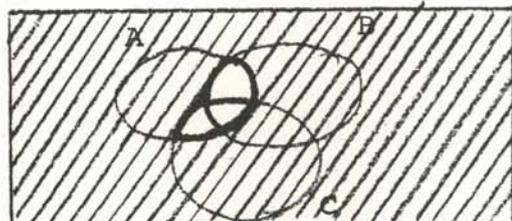
$$B \cup C = \{2, 3, 4, 5, 6, 7\}$$

$$A \cap (B \cup C) = \{1, 2, 4, 5\} \cap \{2, 3, 4, 5, 6, 7\} = \{2, 4, 5\}$$

$$\therefore [A \cap (B \cup C)]' = \{1, 3, 6, 7, 8\}$$

$$B' \cap C' = \{1, 4, 7, 8\} \cap \{1, 2, 3, 8\} = \{1, 8\}$$

$$A' \cup (B' \cap C') = \{3, 6, 7, 8\} \cup \{1, 8\} = \{1, 3, 6, 7, 8\}$$



shaded region is  $[A \cap (B \cup C)]'$   
or  $A' \cup (B' \cap C')$

2. (a) We know that

$$f^{-1}(f(1)) = 1$$

$$f^{-1}(f(2)) = 2$$

$$f^{-1}(f(3)) = 3$$

$$f^{-1}(f(4)) = 4$$

Hence by the definition of  $f$ ,

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Quiz

[2. cont'd]

$$f^{-1}(2) = 1$$

$$f^{-1}(4) = 2$$

$$f^{-1}(1) = 3$$

$$f^{-1}(3) = 4$$

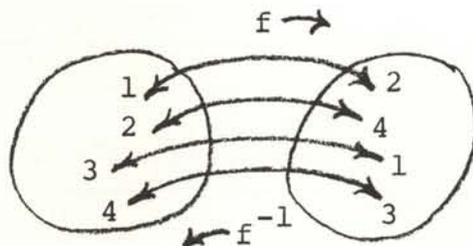
or:

$$f^{-1}(1) = 3$$

$$f^{-1}(2) = 1$$

$$f^{-1}(3) = 4$$

$$f^{-1}(4) = 2$$



(b) If  $y = \frac{3x - 4}{5}$  then  $5y = 3x - 4$  or  $x = \frac{5y + 4}{3}$ .

Let  $g(x) = \frac{5x + 4}{3}$ . Then  $g = f^{-1}$ . That is:

$$f(g(x)) = \frac{3 \left[ \frac{5x + 4}{3} \right] - 4}{5} = x$$

and

$$g(f(x)) = \frac{5 \left( \frac{3x - 4}{5} \right) + 4}{3} = x.$$

3. Since  $y = x^3 - 2x^2$  for  $(x, y)$  to belong to  $C$ , we have that

$$\begin{cases} y_1 = x_1^3 - 2x_1^2 \\ y_1 + \Delta y = (x_1 + \Delta x)^3 - 2(x_1 + \Delta x)^2 \end{cases}$$

SOLUTIONS: Calculus of a Single Variable - Block I: Sets, Functions, and Limits - Quiz

[3. cont'd]

(a) Now the slope of PQ is  $\frac{\Delta y}{\Delta x}$

$$\begin{aligned} \therefore \text{slope of PQ} &= \frac{[(x_1 + \Delta x)^3 - 2(x_1 + \Delta x)^2] - [x_1^3 - 2x_1^2]}{\Delta x} \\ &= \frac{x_1^3 + 3x_1^2 \Delta x + 3x_1 \Delta x^2 + \Delta x^3 - 2x_1^2 - 4x_1 \Delta x - 2\Delta x^2 - x_1^3 + 2x_1^2}{\Delta x} \\ &= \frac{3x_1^2 \Delta x + 3x_1 \Delta x^2 + \Delta x^3 - 4x_1 \Delta x - 2\Delta x^2}{\Delta x} \\ &= \frac{(3x_1^2 - 4x_1) \Delta x + \Delta x^2 (3x_1 + \Delta x - 2)}{\Delta x} \\ &= \underline{\underline{3x_1^2 - 4x_1 + \Delta x (3x_1 + \Delta x - 2)}} \quad (\text{since } \Delta x \neq 0) \end{aligned}$$

(b) The slope of the curve at P is by definition  $\lim_{Q \rightarrow P} (\text{slope of PQ})$ . That is, slope of the line tangent to C at P( $x_1, y_1$ ) is:

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} [(3x_1^2 - 4x_1) + \Delta x (3x_1 + \Delta x - 2)] &= \\ 3x_1^2 - 4x_1 + 0(3x_1 - 2) &= 3x_1^2 - 4x_1 \end{aligned}$$

Hence the equation of the tangent line to C at P is:

$$\left. \begin{aligned} \frac{y - y_1}{x - x_1} &= 3x_1^2 - 4x_1 \\ \text{or: } y &= (3x_1^2 - 4x_1)(x - x_1) + y_1 \end{aligned} \right\}$$

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Functions, and Limits - Quiz

[3. cont'd]

(c) In particular when  $x_1 = 3$ ,  $3x_1^2 - 4x_1 = 3(9) - 12 = 15$ . Hence the line tangent to C at (3,9) is given by:

$$\frac{y-9}{x-3} = 15$$

or  $y-9 = 15x - 45$ .

$$\therefore y = 15x - 36$$

(Note that  $y = (3x_1^2 - 4x_1)(x - x_1) + y_1$  [the answer to part (b)] reduces to this when  $x_1 = 3$  and  $y_1 = 9$ .)

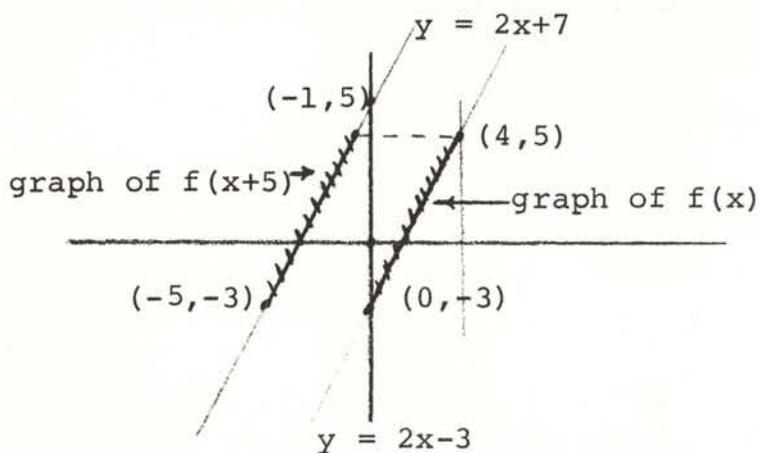
At any rate  $y = 15x - 36$  meets the x-axis when  $y = 0$ .

$$\therefore 0 = 15x - 36$$

$$\therefore x = \frac{36}{15} = \frac{12}{5}$$

$\therefore$  The line meets the x-axis at  $(\frac{12}{5}, 0)$

4. (a) and (b) The graph (see below) of  $f(x+5)$  is that of  $f(x)$ , shifted 5 (horizontal) units to the left.



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[4. cont'd]

More abstractly  $f$  is defined by

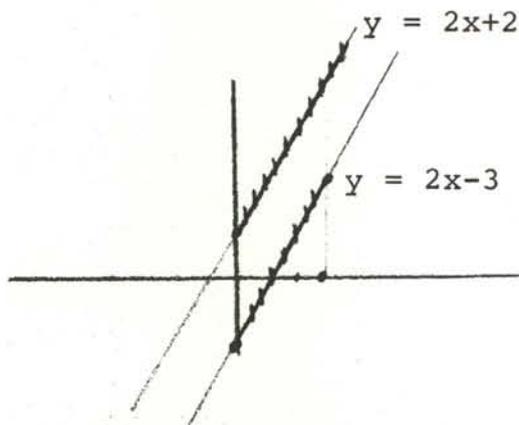
$$f([ ]) = 2[ ] - 3, 0 \leq [ ] \leq 4$$

$$\text{Hence } f(x+5) = 2[x+5] - 3, 0 \leq x + 5 \leq 4$$

$$\therefore f(x+5) = 2x+7, -5 \leq x \leq -1$$

$$\begin{aligned} \text{(c) } f(x) + 5 &= (2x-3) + 5 \quad 0 \leq x \leq 4 \\ &= 2x + 2 \end{aligned}$$

Thus the graph of  $f(x) + 5$  is that of  $f(x)$  raised 5 units.



(b) and (c) show us that  $f(x+5)$  and  $f(x)+5$  are different.

$$\begin{aligned} \text{(d) } f(4x) &= 2(4x) - 3, 0 \leq 4x \leq 4 \\ &= \underline{8x - 3}, 0 \leq x \leq 1 \end{aligned}$$

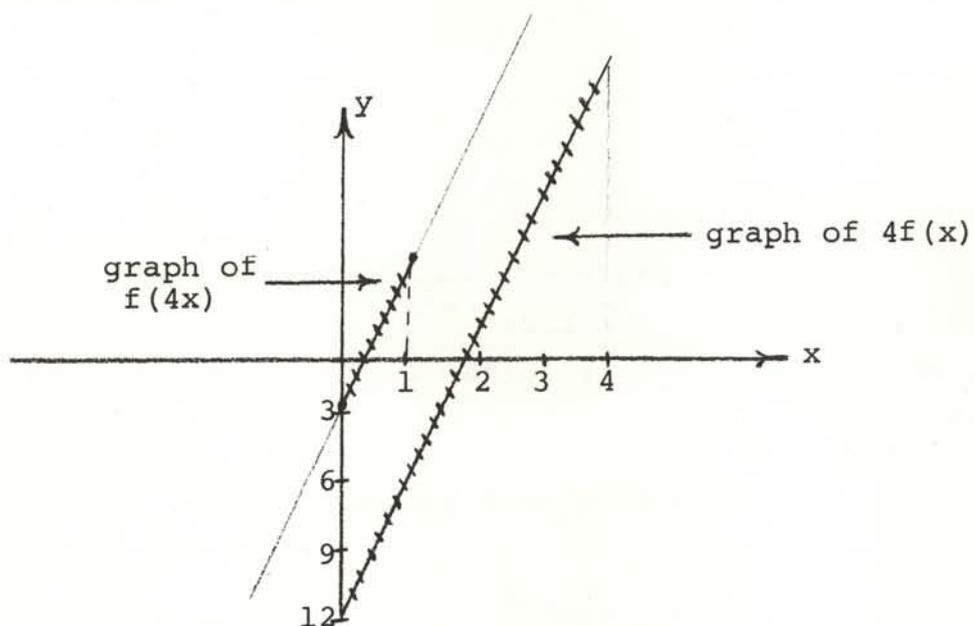
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[4. cont'd]

(e) On the other hand

$$\begin{aligned}4f(x) &= 4(2x-3), \quad 0 \leq x \leq 4 \\ &= \underline{8x-12} \quad 0 \leq x \leq 4\end{aligned}$$

Thus, in a single diagram:

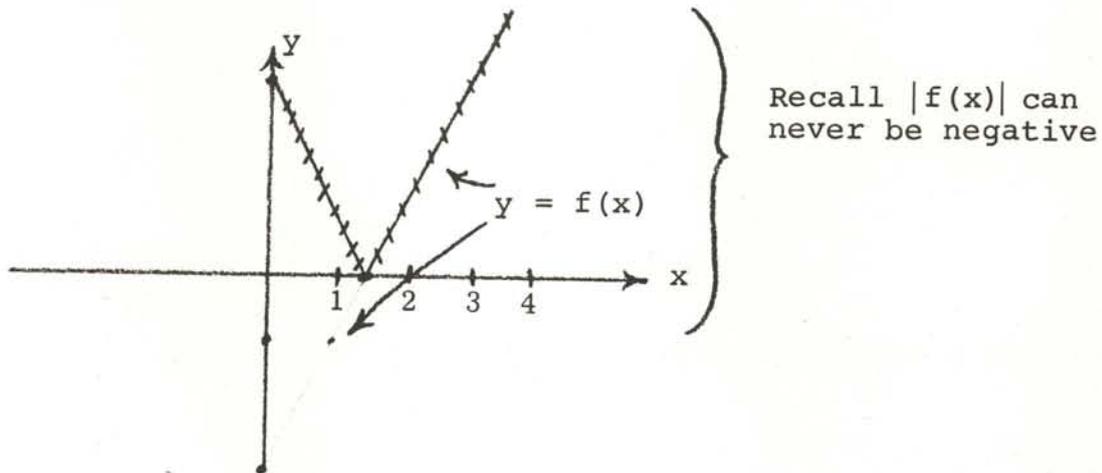


(f) Since  $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) \leq 0 \end{cases}$

the graph of  $y = |f(x)|$  is the same as that of  $y = f(x)$  except that anything below the x-axis ( $-|f(x)|$ ) is reflected above the x-axis. Thus

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[4. cont'd]



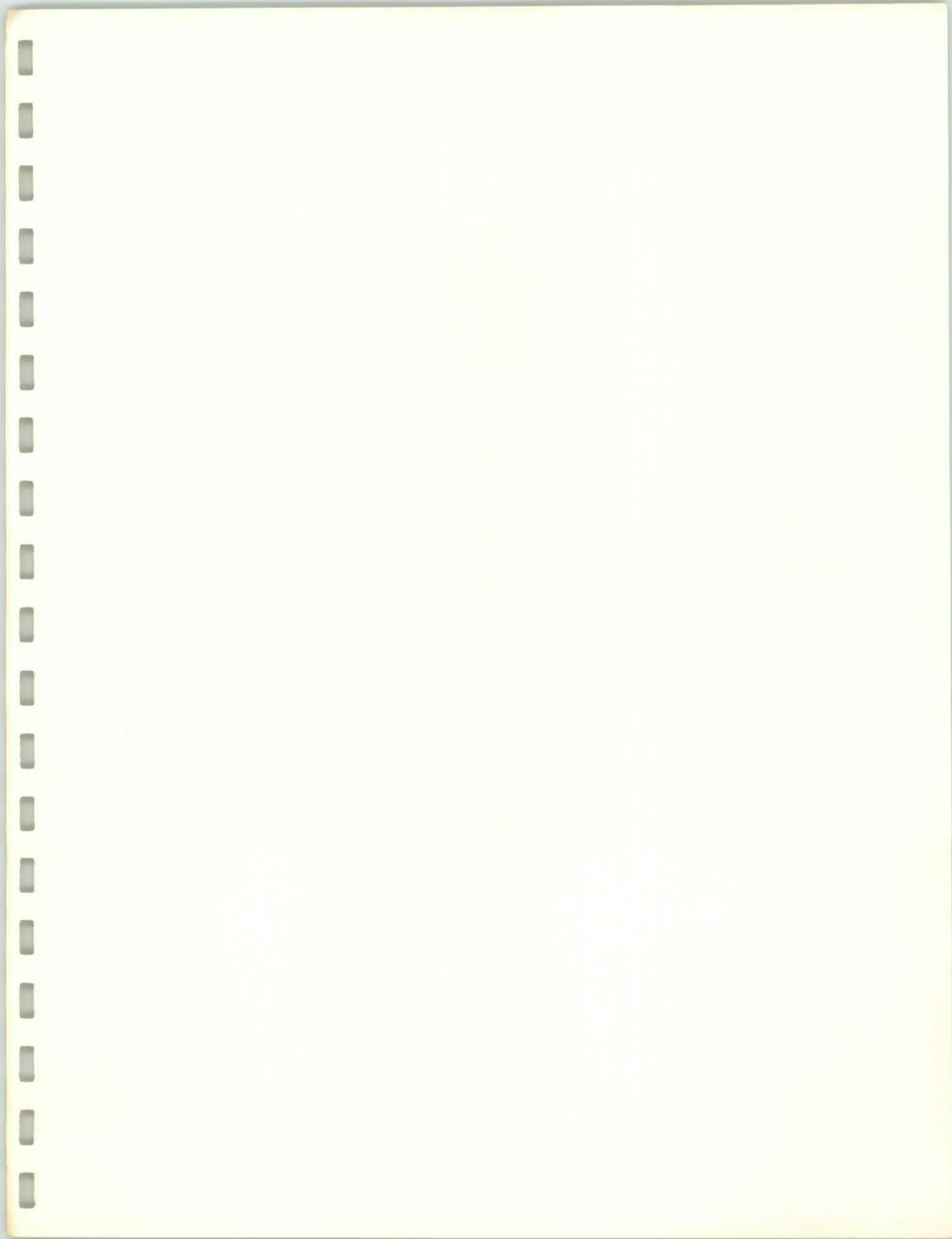
5. (a) We want  $\frac{1}{\sqrt{x-1}} > M$ .

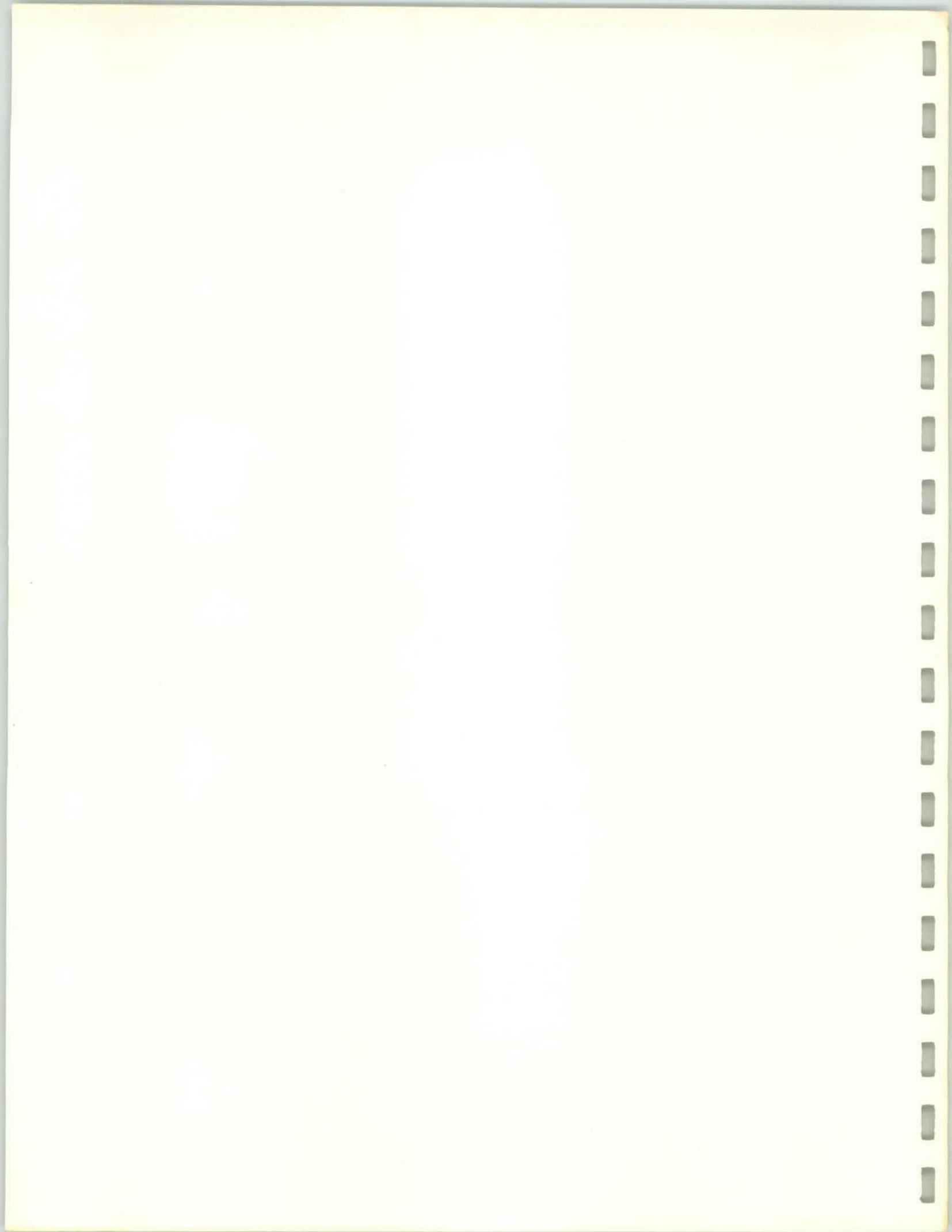
Since both  $\frac{1}{\sqrt{x-1}}$  and  $M$  are positive,  $\frac{1}{\sqrt{x-1}} > M$  is equivalent to  $\sqrt{x-1} < \frac{1}{M}$  which, in turn, implies that  $(\sqrt{x-1})^2 < (\frac{1}{M})^2$  - that is,  $x-1 < \frac{1}{M^2}$ , or:  $x < 1 + \frac{1}{M^2}$ . In other words  $x$  must not exceed 1 by more than  $\frac{1}{M^2}$ .

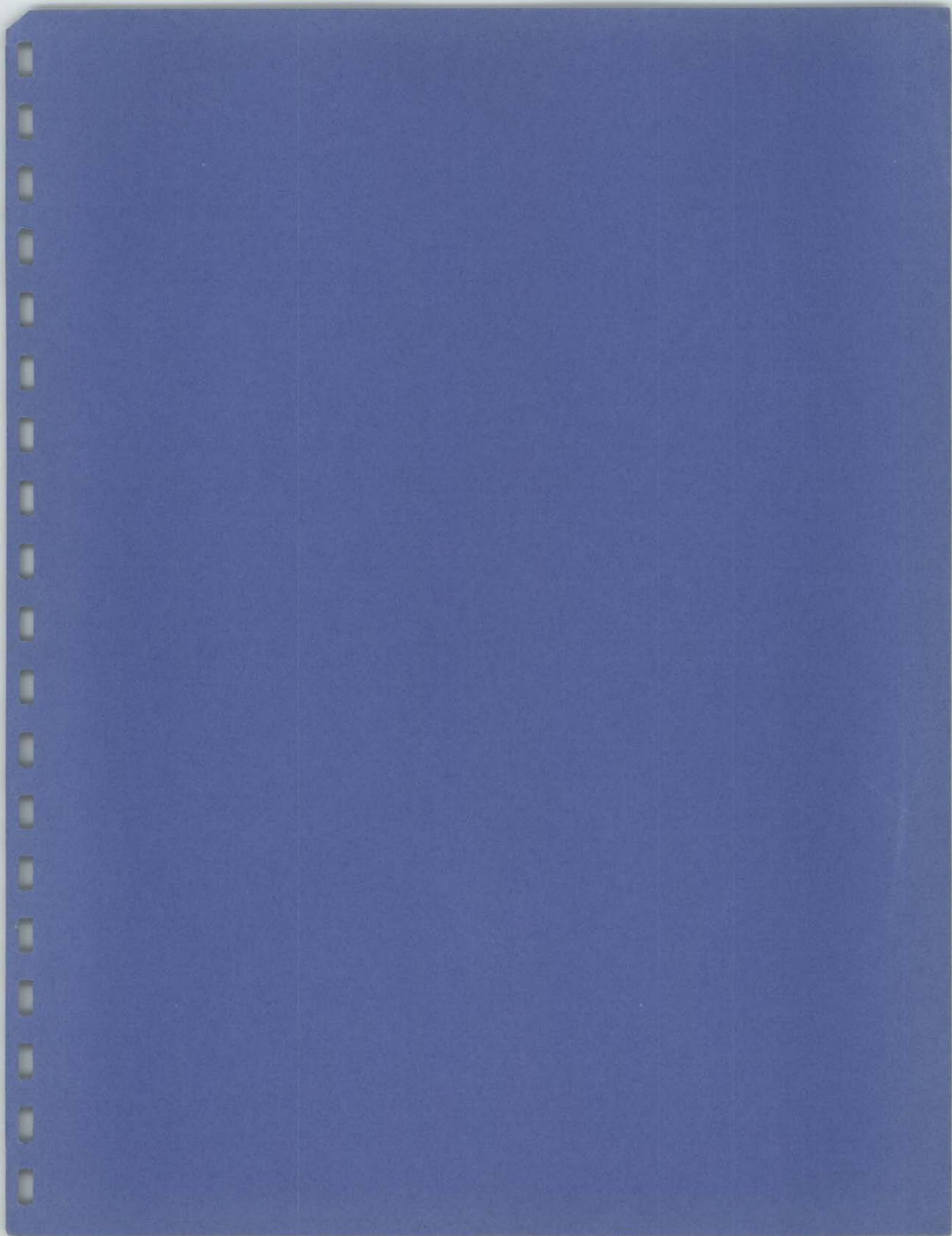
(b) In particular if  $M = 1000$ ,  $M^2 = 1,000,000$ ; hence

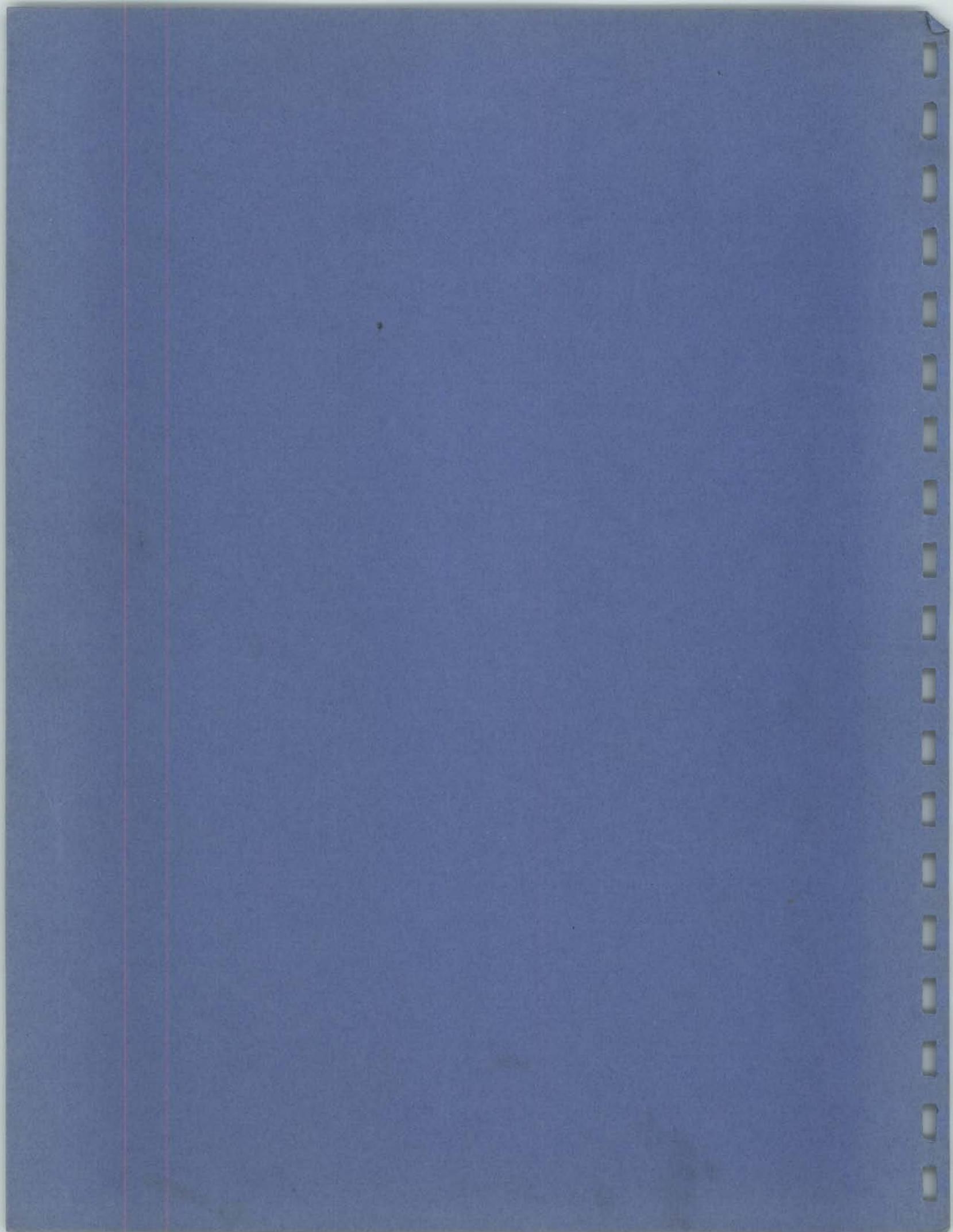
$$x-1 < \frac{1}{1,000,000} = 10^{-6}$$

That is  $x$  must be in the open interval  $(1, 1+10^{-6})$ . (As a quick check,  $x = 1+10^{-6} \rightarrow x-1 = 10^{-6} \rightarrow \sqrt{x-1} = 10^{-3} \rightarrow \frac{1}{\sqrt{x-1}} = 10^3 = 1,000$ .)









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