21 Explain why the surface area is infinite when $y = 1/x$ is rotated around the x axis ($1 \leqslant x < \infty$). But the volume of "Gabriel's horn" is ________. It can't hold enough paint to paint its surface.

22 A disk of radius 1" can be covered by four strips of tape (width $\frac{1}{2}$"). If the strips are not parallel, prove that they can't cover the disk. **Hint**: Change to a unit sphere sliced by planes $\frac{1}{2}$" apart. Problem 14 gives surface area π for each slice.

23 A watermelon (maybe a football) is the result of rotating half of the ellipse $x = \sqrt{2}\cos t$, $y = \sin t$ (which means $x^2 + 2y^2 = 2$). Find the surface area, parametrically or not.

24 Estimate the surface area of an egg.

8.4 Probability and Calculus

Discrete probability usually involves careful counting. Not many samples are taken and not many experiments are made. There is a list of possible outcomes, and a known probability for each outcome. But probabilities go far beyond red cards and black cards. The real questions are much more practical:

1. How often will too many passengers arrive for a flight?
2. How many random errors do you make on a quiz?
3. What is the chance of exactly one winner in a big lottery?

Those are important questions and we will set up models to answer them.

There is another point. Discrete models do not involve calculus. The number of errors or bumped passengers or lottery winners is a small whole number. ***Calculus enters for continuous probability***. Instead of results that exactly equal 1 or 2 or 3, calculus deals with results that fall in a range of numbers. Continuous probability comes up in at least two ways:

(A) An experiment is repeated many times and we take *averages*.
(B) The outcome lies anywhere in an *interval* of numbers.

In the continuous case, the probability p_n of hitting a particular value $x = n$ becomes zero. Instead we have a ***probability density*** $p(x)$—which is a key idea. *The chance that a random X falls between a and b is found by integrating the density $p(x)$:*

$$\text{Prob}\{a \leqslant X \leqslant b\} = \int_a^b p(x)\,dx. \tag{1}$$

Roughly speaking, $p(x)\,dx$ is the chance of falling between x and $x + dx$. Certainly $p(x) \geqslant 0$. If a and b are the extreme limits $-\infty$ and ∞, including all possible outcomes, the probability is necessarily one:

$$\text{Prob}\{-\infty < X < +\infty\} = \int_{-\infty}^{\infty} p(x)\,dx = 1. \tag{2}$$

This is a case where infinite limits of integration are natural and unavoidable. In studying probability they create no difficulty—areas out to infinity are often easier.

Here are typical questions involving continuous probability and calculus:

4. How conclusive is a 53%–47% poll of 2500 voters?
5. Are 16 random football players safe on an elevator with capacity 3600 pounds?
6. How long before your car is in an accident?

It is not so traditional for a calculus course to study these questions. They need extra thought, beyond computing integrals (so this section is harder than average). But probability is more important than some traditional topics, and also more interesting.

Drug testing and gene identification and market research are major applications. Comparing Questions 1–3 with **4–6** brings out the relation of **discrete** to **continuous**—the differences between them, and the parallels.

It would be impossible to give here a full treatment of probability theory. I believe you will see the point (and the use of calculus) from our examples. Frank Morgan's lectures have been a valuable guide.

DISCRETE RANDOM VARIABLES

A **discrete** random variable X has a list of possible values. For two dice the outcomes are $X = 2, 3, \ldots, 12$. For coin tosses (see below), the list is infinite: $X = 1, 2, 3, \ldots$.

A ***continuous*** variable lies in an interval $a \leqslant X \leqslant b$.

EXAMPLE 1 Toss a fair coin until heads come up. The outcome X is the ***number of tosses***. The value of X is 1 or 2 or 3 or ..., and the probability is $\frac{1}{2}$ that $X = 1$ (heads on the first toss). The probability of tails then heads is $p_2 = \frac{1}{4}$. The probability that $X = n$ is $p_n = (\frac{1}{2})^n$—this is the chance of $n - 1$ tails followed by heads. ***The sum of all probabilities is necessarily*** 1:

$$p_1 + p_2 + p_3 + \cdots = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 1.$$

EXAMPLE 2 Suppose a student (not you) makes an average of 2 unforced errors per hour exam. The number of actual errors on the next exam is $X = 0$ or 1 or 2 or .... A reasonable model for the probability of n errors—when they are random and independent—is the ***Poisson model*** (pronounced Pwason):

$$p_n = \textit{probability of } n \textit{ errors} = \frac{2^n}{n!} e^{-2}.$$

The probabilities of no errors, one error, and two errors are p_0, p_1, and p_2:

$$p_0 = \frac{2^0}{0!} e^{-2} = \frac{1}{1} e^{-2} \approx .135 \qquad p_1 = \frac{2^1}{1!} e^{-2} \approx .27 \qquad p_2 = \frac{2^2}{2!} e^{-2} \approx .27.$$

The probability of more than two errors is $1 - .135 - .27 - .27 = .325$.

This Poisson model can be derived theoretically or tested experimentally. The total probability is again 1, from the infinite series (Section 6.6) for e^2:

$$p_0 + p_1 + p_2 + \cdots = \left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \cdots\right) e^{-2} = e^2 e^{-2} = 1. \tag{3}$$

EXAMPLE 3 Suppose on average 3 out of 100 passengers with reservations don't show up for a flight. If the plane holds 98 passengers, ***what is the probability that someone will be bumped?***

If the passengers come independently to the airport, use the Poisson model with 2 changed to 3. X is the number of no-shows, and $X = n$ happens with probability p_n:

$$p_n = \frac{3^n}{n!} e^{-3} \qquad p_0 = \frac{3^0}{0!} e^{-3} = e^{-3} \qquad p_1 = \frac{3^1}{1!} e^{-3} = 3e^{-3}.$$

There are 98 seats and 100 reservations. Someone is bumped if $X = 0$ or $X = 1$:

$$\text{chance of bumping} = p_0 + p_1 = e^{-3} + 3e^{-3} \approx 4/20.$$

We will soon define the ***average*** or ***expected value*** or ***mean*** of X—this model has $\mu = 3$.

CONTINUOUS RANDOM VARIABLES

If X is the lifetime of a VCR, all numbers $X \geqslant 0$ are possible. If X is a score on the SAT, then $200 \leqslant X \leqslant 800$. If X is the fraction of computer owners in a poll of 600 people, X is between 0 and 1. You may object that the SAT score is a whole number and the fraction of computer owners must be 0 or 1/600 or 2/600 or .... But it is completely impractical to work with 601 discrete possibilities. Instead we take X to be a ***continuous random variable***, falling *anywhere* in the range $X \geqslant 0$ or [200, 800] or $0 \leqslant X \leqslant 1$. Of course the various values of X are not equally probable.

EXAMPLE 4 The average lifetime of a VCR is 4 years. A reasonable model for breakdown time is an ***exponential random variable***. Its probability density is

$$p(x) = \tfrac{1}{4}e^{-x/4} \quad \text{for} \quad 0 \leqslant x < \infty.$$

The probability that the VCR will eventually break is 1:

$$\int_0^\infty \tfrac{1}{4}e^{-x/4}\,dx = \Big[-e^{-x/4}\Big]_0^\infty = 0-(-1) = 1. \tag{4}$$

The probability of breakdown within 12 years (X from 0 to 12) is .95:

$$\int_0^{12} \tfrac{1}{4}e^{-x/4}\,dx = \Big[-e^{-x/4}\Big]_0^{12} = -e^{-3}+1 \approx .95. \tag{5}$$

An exponential distribution has $p(x) = ae^{-ax}$. Its integral from 0 to x is $F(x) = 1-e^{-ax}$. Figure 8.11 is the graph for $a = 1$. It shows the area up to $x = 1$.

To repeat: ***The probability that*** $a \leqslant X \leqslant b$ ***is the integral of*** $p(x)$ ***from*** a ***to*** b.

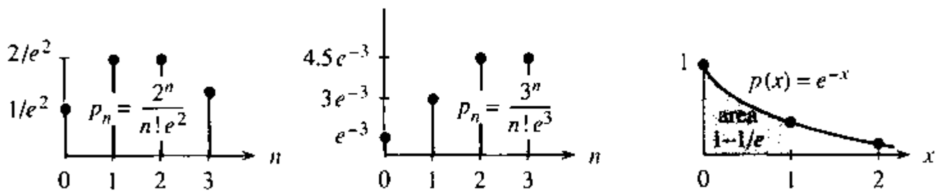


Fig. 8.11 Probabilities add to $\Sigma\, p_n = 1$. Continuous density integrates to $\int p(x)\,dx = 1$.

EXAMPLE 5 We now define the most important density function. Suppose the average SAT score is 500, and the *standard deviation* (defined below—it measures the spread around the average) is 200. Then the ***normal distribution*** of grades has

$$p(x) = \frac{1}{200\sqrt{2\pi}}e^{-(x-500)^2/2(200)^2} \quad \text{for} \quad -\infty < x < \infty.$$

This is the normal (or Gaussian) distribution with mean 500 and standard deviation 200. The graph of $p(x)$ is the famous ***bell-shaped curve*** in Figure 8.12.

A new objection is possible. The actual scores are between 200 and 800, while the density $p(x)$ extends all the way from $-\infty$ to ∞. I think the Educational Testing Service counts all scores over 800 as 800. The fraction of such scores is pretty small—in fact the normal distribution gives

$$\text{Prob}\,\{X \geqslant 800\} = \int_{800}^\infty \frac{1}{200\sqrt{2\pi}}e^{-(x-500)^2/2(200)^2}\,dx \approx .0013. \tag{6}$$

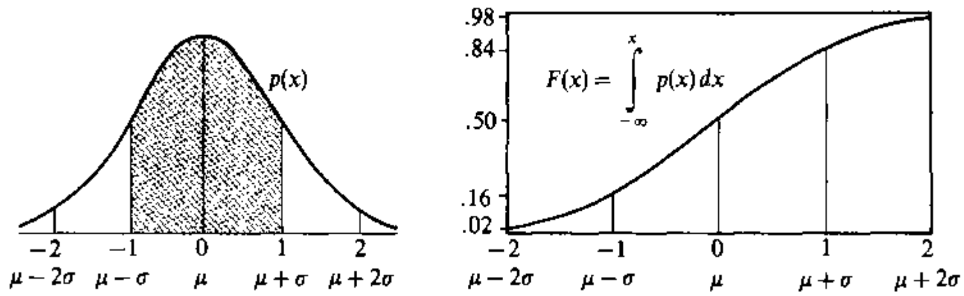


Fig. 8.12 The normal distribution (bell-shaped curve) and its cumulative density $F(x)$.

Regrettably, e^{-x^2} has no elementary antiderivative. We need numerical integration. But there is nothing the matter with that! The integral is called the "*error function*," and special tables give its value to great accuracy. The integral of $e^{-x^2/2}$ from $-\infty$ to ∞ is exactly $\sqrt{2\pi}$. Then division by $\sqrt{2\pi}$ keeps $\int p(x)\,dx = 1$.

Notice that the normal distribution involves *two parameters*. They are the mean value (in this case $\mu = 500$) and the standard deviation (in this case $\sigma = 200$). Those numbers *mu* and *sigma* are often given the "normalized" values $\mu = 0$ and $\sigma = 1$:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{becomes} \quad p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The bell-shaped graph of p is symmetric around the middle point $x = \mu$. The width of the graph is governed by the second parameter σ—which stretches the x axis and shrinks the y axis (leaving total area equal to 1). The axes are labeled to show the standard case $\mu = 0$, $\sigma = 1$ and also the graph for any other μ and σ.

We now give a name to the integral of $p(x)$. The limits will be $-\infty$ and x, so the integral $F(x)$ measures the ***probability that a random sample is below*** x:

$$\text{Prob}\{X \leqslant x\} = \int_{-\infty}^{x} p(x)\,dx = \textbf{\textit{cumulative density function}}\ F(x). \tag{7}$$

$F(x)$ accumulates the probabilities given by $p(x)$, so $dF/dx = p(x)$. The total probability is $F(\infty) = 1$. This integral from $-\infty$ to ∞ covers all outcomes.

Figure 8.12b shows the integral of the bell-shaped normal distribution. The middle point $x = \mu$ has $F = \frac{1}{2}$. By symmetry there is a 50-50 chance of an outcome below the mean. The cumulative density $F(x)$ is near .16 at $\mu - \sigma$ and near .84 at $\mu + \sigma$. The chance of falling in between is $.84 - .16 = .68$. Thus 68% of the outcomes are less than one deviation σ away from the center μ.

Moving out to $\mu - 2\sigma$ and $\mu + 2\sigma$, 95% of the area is in between. ***With 95% confidence X is less than two deviations from the mean.*** Only one sample in 20 is further out (less than one in 40 on each side).

Note that $\sigma = 200$ is not the precise value for the SAT!

MEAN, VARIANCE, AND STANDARD DEVIATION

In Example 1, X was the number of coin tosses until the appearance of heads. The probabilities were $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{8}$, ... *What is the average number of tosses?* We now find the "mean" μ of any distribution $p(x)$—not only the normal distribution, where symmetry guarantees that the built-in number μ is the mean.

To find μ, ***multiply outcomes by probabilities and add***:

$$\mu = \textbf{\textit{mean}} = \sum n p_n = 1(p_1) + 2(p_2) + 3(p_3) + \cdots. \tag{8}$$

The average number of tosses is $1(\frac{1}{2})+2(\frac{1}{4})+3(\frac{1}{8})+\cdots$. This series adds up (in Section 10.1) to $\mu=2$. Please do the experiment 10 times. I am almost certain that the average will be near 2.

When the average is $\lambda=2$ quiz errors or $\lambda=3$ no-shows, the Poisson probabilities are $p_n=\lambda^n e^{-\lambda}/n!$ Check that the formula $\mu=\Sigma\, np_n$ does give λ as the mean:

$$\left[1\frac{\lambda}{1!}+2\frac{\lambda^2}{2!}+3\frac{\lambda^3}{3!}+\cdots\right]e^{-\lambda}=\lambda\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots\right]e^{-\lambda}=\lambda e^{\lambda}e^{-\lambda}=\lambda.$$

For continuous probability, the sum $\mu=\Sigma\, np_n$ changes to $\mu=\int xp(x)\,dx$. We multiply outcome x by probability $p(x)$ and integrate. In the VCR model, integration by parts gives a mean breakdown time of $\mu=4$ years:

$$\int x\; p(x)\; dx=\int_0^\infty x(\tfrac{1}{4}e^{-x/4})\; dx=\Big[-xe^{-x/4}-4e^{-x/4}\Big]_0^\infty=4. \tag{9}$$

Together with the mean we introduce the ***variance***. It is always written σ^2, and in the normal distribution that measured the "width" of the curve. When σ^2 was 200^2, SAT scores spread out pretty far. If the testing service changed to $\sigma^2=1^2$, the scores would be a disaster. 95% of them would be within ± 2 of the mean. When a teacher announces an average grade of 72, the variance should also be announced—if it is big then those with 60 can relax. At least they have company.

> **8E** The mean μ is the expected value of X. The variance σ^2 is the expected value of $(X-\text{mean})^2=(X-\mu)^2$. Multiply outcome times probability and add:
>
> $\mu=\sum np_n \qquad \sigma^2=\sum(n-\mu)^2p_n \qquad$ (discrete)
>
> $\mu=\int_{-\infty}^{\infty} xp(x)\,dx \qquad \sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2p(x)\,dx \qquad$ (continuous)

The ***standard deviation*** (written σ) is the square root of σ^2.

EXAMPLE 6 (Yes-no poll, one person asked) The probabilities are p and $1-p$.

A fraction $p=\frac{1}{3}$ of the population thinks *yes*, the remaining fraction $1-p=\frac{2}{3}$ thinks *no*. Suppose we only ask one person. If $X=1$ for yes and $X=0$ for no, the expected value of X is $\mu=p=\frac{1}{3}$. The variance is $\sigma^2=p(1-p)=\frac{2}{9}$:

$$\mu=0\left(\frac{2}{3}\right)+1\left(\frac{1}{3}\right)=\frac{1}{3}\quad\text{and}\quad\sigma^2=\left(0-\frac{1}{3}\right)^2\left(\frac{2}{3}\right)+\left(1-\frac{1}{3}\right)^2\left(\frac{1}{3}\right)=\frac{2}{9}.$$

The standard deviation is $\sigma=\sqrt{2/9}$. When the fraction p is near one or near zero, the spread is smaller—and one person is more likely to give the right answer for everybody. The maximum of $\sigma^2=p(1-p)$ is at $p=\frac{1}{2}$, where $\sigma=\frac{1}{2}$.

The table shows μ and σ^2 for important probability distributions.

Model	*Mean*	*Variance*	*Application*
$p_1=p,\ p_0=1-p$	p	$p(1-p)$	yes-no
Poisson $p_n=\lambda^n e^{-\lambda}/n!$	λ	λ	random occurrence
Exponential $p(x)=ae^{-ax}$	$1/a$	$1/a^2$	waiting time
Normal $p(x)=\dfrac{1}{\sqrt{2\pi}\,\sigma}e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2	distribution around mean

THE LAW OF AVERAGES AND THE CENTRAL LIMIT THEOREM

We come to the center of probability theory (without intending to give proofs). The key idea is to repeat an experiment many times—poll many voters, or toss many dice, or play considerable poker. Each independent experiment produces an outcome X, and the average from N experiments is $\bar{X}$. It is called "*X bar*":

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_N}{N} = \text{average outcome.}$$

All we know about $p(x)$ is its mean μ and variance σ^2. It is amazing how much information that gives about the average $\bar{X}$:

> **8F** ***Law of Averages***: $\bar{X}$ is almost sure to approach μ as $N \to \infty$.
> ***Central Limit Theorem***: The probability density $p_N(x)$ for $\bar{X}$ approaches a normal distribution with the same mean μ and with variance σ^2/N.

No matter what the probabilities for X, ***the probabilities for*** $\bar{X}$ ***move toward the normal bell-shaped curve.*** The standard deviation is close to $\sigma/\sqrt{N}$ when the experiment is repeated N times. In the Law of Averages, "almost sure" means that the chance of $\bar{X}$ *not* approaching μ is zero. It can happen, but it won't.

Remark 1 The Boston Globe doesn't understand the Law of Averages. I quote from September 1988: "What would happen if a giant Red Sox slump arrived? What would happen if the fabled Law of Averages came into play, reversing all those can't miss decisions during the winning streak?" They think the Law of Averages evens everything up, favoring heads after a series of tails. See Problem 20.

EXAMPLE 7 *Yes-no poll of* $N = 2500$ *voters.* ***Is a*** *53%–47%* ***outcome conclusive***?

The fraction p of "yes" voters in the whole population is *not known*. That is the reason for the poll. The deviation $\sigma = \sqrt{p(1-p)}$ is also not known, but for one voter this is never more than $\frac{1}{2}$ (when $p = \frac{1}{2}$). Therefore $\sigma/\sqrt{N}$ for 2500 voters is no larger than $\frac{1}{2}/\sqrt{2500}$, which is 1%.

The result of the poll was $\bar{X} = 53\%$. With 95% confidence, this sample is within two standard deviations (here 2%) of its mean. Therefore with 95% confidence, ***the unknown mean*** $\mu = p$ ***of the whole population is between*** 51% ***and*** 55%. This poll is conclusive.

If the true mean had been $p = 50\%$, the poll would have had only a .0013 chance of reaching 53%. The error margin on each side of a poll is amazingly simple; it is always $1/\sqrt{N}$.

Remark 2 The New York Times has better mathematicians than the Globe. Two days after Bush defeated Dukakis, their poll of $N = 11{,}645$ voters was printed with the following explanation. "In theory, in 19 cases out of 20 [*there is* 95%] the results should differ by no more than one percentage point [*there is* $1/\sqrt{N}$] from what would have been obtained by seeking out all voters in the United States."

EXAMPLE 8 Football players at Caltech (if any) have average weight $\mu = 210$ pounds and standard deviation $\sigma = 30$ pounds. Are $N = 16$ players safe on an elevator with capacity 3600 pounds? 16 times 210 is 3360.

The average weight $\bar{X}$ is approximately a normal random variable with $\bar{\mu} = 210$ and $\bar{\sigma} = 30/\sqrt{N} = 30/4$. There is only a 2% chance that $\bar{X}$ is above $\bar{\mu} + 2\bar{\sigma} = 225$ (see Figure 8.12b—weights below the mean are no problem on an elevator). Since 16 times 225 is 3600, a statistician would have 98% confidence that the elevator is safe. This is an example where 98% is not good enough—I wouldn't get on.

EXAMPLE 9 (The famous Weldon Dice) Weldon threw 12 dice 26,306 times and counted the 5's and 6's. They came up in 33.77% of the 315,672 separate rolls. Thus $\bar{X} = .3377$ instead of the expected fraction $p = \frac{1}{3}$ of 5's and 6's. Were the dice fair?

The variance in each roll is $\sigma^2 = p(1-p) = 2/9$. The standard deviation of $\bar{X}$ is $\bar{\sigma} = \sigma/\sqrt{N} = \sqrt{2/9}/\sqrt{315672} \approx .00084$. For fair dice, there is a 95% chance that $\bar{X}$ will differ from $\frac{1}{3}$ by less than $2\bar{\sigma}$. (For Poisson probabilities that is false. Here $\bar{X}$ is *normal*.) But .3377 differs from .3333 by more than $5\bar{\sigma}$. The chance of falling 5 standard deviations away from the mean is only about 1 in 10,000.†

So the dice were unfair. The faces with 5 or 6 indentations were *lighter* than the others, and a little more likely to come up. Modern dice are made to compensate for that, but Weldon never tried again.

8.4 EXERCISES

Read-through questions

Discrete probability uses counting, __a__ probability uses calculus. The function $p(x)$ is the probability __b__. The chance that a random variable falls between a and b is __c__. The total probability is $\int_{-\infty}^{\infty} p(x)\,dx =$ __d__. In the discrete case $\Sigma p_n =$ __e__. The mean (or expected value) is $\mu = \int$ __f__ in the continuous case and $\mu = \Sigma np_n$ in the __g__.

The Poisson distribution with mean λ has $p_n =$ __h__. The sum $\Sigma p_n = 1$ comes from the __i__ series. The exponential distribution has $p(x) = e^{-x}$ or $2e^{-2x}$ or __j__. The standard Gaussian (or __k__) distribution has $\sqrt{2\pi}\,p(x) = e^{-x^2/2}$. Its graph is the well-known __l__ curve. The chance that the variable falls below x is $F(x) =$ __m__. F is the __n__ density function. The difference $F(x+dx) - F(x)$ is about __o__, which is the chance that X is between x and $x + dx$.

The *variance*, which measures the spread around μ, is $\sigma^2 = \int$ __p__ in the continuous case and $\sigma^2 = \Sigma$ __q__ in the discrete case. Its square root σ is the __r__. The normal distribution has $p(x) =$ __s__. If $\bar{X}$ is the __t__ of N samples from any population with mean μ and variance σ^2, the Law of Averages says that $\bar{X}$ will approach __u__. The Central Limit Theorem says that the distribution for $\bar{X}$ approaches __v__. Its mean is __w__ and its variance is __x__.

In a yes-no poll when the voters are 50-50, the mean for one voter is $\mu = 0(\frac{1}{2}) + 1(\frac{1}{2}) =$ __y__. The variance is $(0-\mu)^2 p_0 + (1-\mu)^2 p_1 =$ __z__. For a poll with $N = 100$, $\bar{\sigma}$ is __A__. There is a 95% chance that $\bar{X}$ (the fraction saying yes) will be between __B__ and __C__.

1 If $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{8}$, ..., what is the probability of an outcome $X < 4$? What are the probabilities of $X = 4$ and $X > 4$?

2 With the same $p_n = (\frac{1}{2})^n$, what is the probability that X is odd? Why is $p_n = (\frac{1}{3})^n$ an impossible set of probabilities? What multiple $c(\frac{1}{3})^n$ is possible?

3 Why is $p(x) = e^{-2x}$ not an acceptable probability density for $x \geq 0$? Why is $p(x) = 4e^{-2x} - e^{-x}$ not acceptable?

***4** If $p_n = (\frac{1}{2})^n$, show that the probability P that X is a prime number satisfies $6/16 \leq P \leq 7/16$.

5 If $p(x) = e^{-x}$ for $x \geq 0$, find the probability that $X \geq 2$ and the approximate probability that $1 \leq X \leq 1.01$.

6 If $p(x) = C/x^3$ is a probability density for $x \geq 1$, find the constant C and the probability that $X \leq 2$.

7 If you choose x completely at random between 0 and π, what is the density $p(x)$ and the cumulative density $F(x)$?

†Joe DiMaggio's 56-game hitting streak was much more improbable—I think it is statistically the most exceptional record in major sports.

In 8–13 find the mean value $\mu = \Sigma\, np_n$ or $\mu = \int xp(x)\,dx$.

8 $p_0 = 1/2,\ p_1 = 1/4,\ p_2 = 1/4$

9 $p_1 = 1/7,\ p_2 = 1/7,\ \ldots,\ p_7 = 1/7$

10 $p_n = 1/n!e \quad (p_0 = 1/e,\ p_1 = 1/e,\ p_2 = 1/2e,\ \ldots)$

11 $p(x) = 2/\pi(1+x^2), \quad x \geqslant 0$

12 $p(x) = e^{-x}$ (integrate by parts)

13 $p(x) = ae^{-ax}$ (integrate by parts)

14 Show by substitution that

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx = \sqrt{2}\,\sigma \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{2\pi}\,\sigma.$$

15 Find the cumulative probability F (the integral of p) in Problems 11, 12, 13. In terms of F, what is the chance that a random sample lies between a and b?

16 Can-Do Airlines books 100 passengers when their plane only holds 98. If the average number of no-shows is 2, what is the Poisson probability that someone will be bumped?

17 The waiting time for a bus has probability density $(1/10)e^{-x/10}$, with $\mu = 10$ minutes. What is the probability of waiting longer than 10 minutes?

18 You make a 3-minute telephone call. If the waiting time for the next incoming call has $p(x) = e^{-x}$, what is the probability that your phone will be busy?

19 Supernovas are expected about every 100 years. What is the probability that you will be alive for the next one? Use a Poisson model with $\lambda = .01$ and estimate your lifetime. (Supernovas actually occurred in 1054 (Crab Nebula), 1572, 1604, and 1987. But the future distribution doesn't depend on the date of the last one.)

20 (a) A fair coin comes up heads 10 times in a row. Will heads or tails be more likely on the next toss?

(b) The fraction of heads after N tosses is α. The expected fraction after $2N$ tosses is _______.

21 Show that the area between μ and $\mu + \sigma$ under the bell-shaped curve is a fixed number (near 1/3), by substituting $y =$ _______:

$$\int_{\mu}^{\mu+\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy.$$

What is the area between $\mu - \sigma$ and μ? The area outside $(\mu - \sigma, \mu + \sigma)$?

22 For a *yes-no* poll of two voters, explain why

$$p_0 = (1-p)^2,\ p_1 = 2p - 2p^2,\ p_2 = p^2.$$

Find μ and σ^2. N voters give the "*binomial distribution.*"

23 Explain the last step in this reorganization of the formula for σ^2:

$$\sigma^2 = \int (x-\mu)^2 p(x)\,dx = \int (x^2 - 2x\mu + \mu^2)p(x)\,dx$$
$$= \int x^2 p(x)\,dx - 2\mu \int xp(x)\,dx + \mu^2 \int p(x)\,dx$$
$$= \int x^2 p(x)\,dx - \mu^2.$$

24 Use $\int (x-\mu)^2 p(x)\,dx$ and also $\int x^2 p(x)\,dx - \mu^2$ to find σ^2 for the ***uniform distribution***: $p(x) = 1$ for $0 \leqslant x \leqslant 1$.

25 Find σ^2 if $p_0 = 1/3,\ p_1 = 1/3,\ p_2 = 1/3$. Use $\Sigma\,(n-\mu)^2 p_n$ and also $\Sigma\, n^2 p_n - \mu^2$.

26 Use Problem 23 and integration by parts (equation 7.1.10) to find σ^2 for the *exponential distribution* $p(x) = 2e^{-2x}$ for $x \geqslant 0$, which has mean $\frac{1}{2}$.

27 The waiting time to your next car accident has probability density $p(x) = \frac{1}{2}e^{-x/2}$. What is μ? What is the probability of no accident in the next four years?

28 With $p = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, find the average number μ of coin tosses by writing $p_1 + 2p_2 + 3p_3 + \cdots$ as $(p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + p_4 + \cdots) + (p_3 + p_4 + p_5 + \cdots) + \cdots$.

29 In a poll of 900 Americans, 30 are in favor of war. What range can you give with 95% confidence for the percentage of peaceful Americans?

30 Sketch rough graphs of $p(x)$ for the fraction x of heads in 4 tosses of a fair coin, and in 16 tosses. The mean value is $\frac{1}{2}$.

31 A judge tosses a coin 2500 times. How many heads does it take to prove with 95% confidence that the coin is unfair?

32 Long-life bulbs shine an average of 2000 hours with standard deviation 150 hours. You can have 95% confidence that your bulb will fail between _______ and _______ hours.

33 Grades have a normal distribution with mean 70 and standard deviation 10. If 300 students take the test and passing is 55, how many are expected to fail? (Estimate from Figure 8.12b.) What passing grade will fail 1/10 of the class?

34 The average weight of luggage is $\mu = 30$ pounds with deviation $\sigma = 8$ pounds. What is the probability that the luggage for 64 passengers exceeds 2000 pounds? How does the answer change for 256 passengers and 8000 pounds?

35 A thousand people try independently to guess a number between 1 and 1000. This is like a lottery.

(a) What is the chance that the first person fails?

(b) What is the chance P_0 that they all fail?

(c) Explain why P_0 is approximately $1/e$.

36 (a) In Problem 35, what is the chance that the first person is right and all others are wrong?

(b) Show that the probability P_1 of exactly one winner is also close to $1/e$.

(c) Guess the probability P_n of n winners (fishy question).

8.5 Masses and Moments

This chapter concludes with two sections related to engineering and physics. Each application starts with a finite number of masses or forces. Their sum is the total mass or total force. Then comes the "continuous case," in which the mass is spread out instead of lumped. Its distribution is given by a ***density function*** ρ (Greek rho), and the sum changes to an *integral.*

The first step (hardest step?) is to get the physical quantities straight. The second step is to move from sums to integrals (discrete to continuous, lumped to distributed). By now we hardly stop to think about it—although this is the key idea of integral calculus. The third step is to evaluate the integrals. For that we can use substitution or integration by parts or tables or a computer.

Figure 8.13 shows the one-dimensional case: *masses along the x axis.* The total mass is the sum of the masses. The new idea is that of ***moments***—when the mass or force is multiplied by a *distance*:

moment of mass around the y axis $= mx =$ (*mass*) times (*distance to axis*).

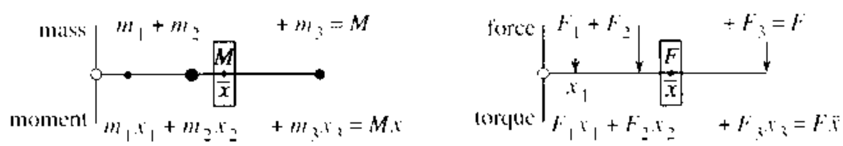


Fig. 8.13 The center of mass is at $\bar{x} =$ (total moment)/(total mass) = average distance.

The figure has masses 1, 3, 2. The total mass is 6. The "lever arms" or "moment arms" are the distances $x = 1, 3, 7$. The masses have moments 1 and 9 and 14 (since mx is 2 times 7). The total moment is $1 + 9 + 14 = 24$. Then the balance point is at $\bar{x} = M_x/M = 24/6 = 4$.

The total mass is the sum of the m's. The total moment is the sum of m_n times x_n (negative on the other side of $x = 0$). If the masses are children on a seesaw, the balance point is the center of gravity $\bar{x}$—also called the ***center of mass***:

DEFINITION $$\bar{x} = \frac{\sum m_n x_n}{\sum m_n} = \frac{\text{total moment}}{\text{total mass}}. \qquad (1)$$

If all masses are moved to $\bar{x}$, the total moment (6 times 4) is still 24. The moment equals the mass $\Sigma\, m_n$ times $\bar{x}$. ***The masses act like a single mass at*** $\bar{x}$.

Also: If we move the axis to $\bar{x}$, and leave the children where they are, the seesaw balances. The masses on the left of $\bar{x} = 4$ will offset the mass on the right. ***Reason***: The distances to the new axis are $x_n - \bar{x}$. The moments add to zero by equation (1):

$$\text{moment around new axis} = \sum m_n(x_n - \bar{x}) = \sum m_n x_n - \sum m_n \bar{x} = 0.$$

Turn now to the *continuous case*, when mass is spread out along the line. Each piece of length Δx has an average density $\rho_n =$ (mass of piece)/(length of piece) $= \Delta m/\Delta x$. As the pieces get shorter, this approaches dm/dx—the density at the point. ***The limit of*** (*small mass*)/(*small length*) ***is the density*** $\rho(x)$.

Integrating that derivative $\rho = dm/dx$, we recover the total mass: $\Sigma\, \rho_n \Delta x$ becomes

$$\textbf{\textit{total mass}}\ M = \int \rho(x)\, dx. \qquad (2)$$

When the mass is spread evenly, ρ is constant. Then $M = \rho L =$ *density times length.*

The moment formula is similar. For each piece, the moment is mass $\rho_n \Delta x$ multiplied by distance x—and we add. In the continuous limit, $\rho(x)\,dx$ is multiplied by x and we integrate:

$$\textbf{\textit{total moment around } } y \textbf{\textit{ axis}} = M_y = \int x\rho(x)\,dx. \tag{3}$$

Moment is mass times distance. Dividing by the total mass M gives "average distance":

$$\textbf{\textit{center of mass }} \bar{x} = \frac{\text{moment}}{\text{mass}} = \frac{M_y}{M} = \frac{\int x\rho(x)\,dx}{\int \rho(x)\,dx}. \tag{4}$$

Remark If you studied Section 8.4 on probability, you will notice how the formulas match up. The mass $\int \rho(x)\,dx$ is like the total probability $\int p(x)\,dx$. The moment $\int x\rho(x)\,dx$ is like the mean $\int xp(x)\,dx$. The moment of inertia $\int (x-\bar{x})^2\rho(x)\,dx$ is the variance. Mathematics keeps hammering away at the same basic ideas! The only difference is that the total probability is always 1. The mean really corresponds to the *center* of mass $\bar{x}$, but in probability we didn't notice the division by $\int p(x)\,dx = 1$.

EXAMPLE 1 With constant density ρ from 0 to L, the mass is $M = \rho L$. The moment is

$$M_y = \int_0^L x\rho\,dx = \tfrac{1}{2}\rho x^2\Big]_0^L = \tfrac{1}{2}\rho L^2.$$

The center of mass is $\bar{x} = M_y/M = L/2$. It is halfway along.

EXAMPLE 2 With density e^{-x} the mass is 1, the moment is 1, and $\bar{x}$ is 1:

$$\int_0^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_0^\infty = 1 \quad\text{and}\quad \int_0^\infty xe^{-x}\,dx = \Big[-xe^{-x} - e^{-x}\Big]_0^\infty = 1.$$

MASSES AND MOMENTS IN TWO DIMENSIONS

Instead of placing masses along the x axis, suppose m_1 is at the point (x_1, y_1) in the plane. Similarly m_n is at (x_n, y_n). Now there are *two moments* to consider. Around the y axis $M_y = \Sigma\, m_n x_n$ and around the x axis $M_x = \Sigma\, m_n y_n$. ***Please notice that the*** x***'s go into the moment*** M_y—because the x coordinate gives the distance from the y axis!

Around the x axis, the distance is y and the moment is M_x. The ***center of mass*** is the point $(\bar{x}, \bar{y})$ at which everything balances:

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_n x_n}{\sum m_n} \quad\text{and}\quad \bar{y} = \frac{M_x}{M} = \frac{\sum m_n y_n}{\sum m_n}. \tag{5}$$

In the continuous case these sums become two-dimensional integrals. The total mass is $\iint \rho(x, y)\,dx\,dy$, when the density is ρ = mass per unit area. These "double integrals" are for the future (Section 14.1). Here we consider the most important case: $\rho = $ *constant*. Think of a thin plate, made of material with constant density (say $\rho = 1$). To compute its mass and moments, the plate is cut into strips (Figure 8.14):

$$\textbf{\textit{mass }} M = \text{area of plate} \tag{6}$$

$$\textbf{\textit{moment }} M_y = \int (\text{distance } x)\,(\text{length of vertical strip})\,dx \tag{7}$$

$$\textbf{\textit{moment }} M_x = \int (\text{height } y)\,(\text{length of horizontal strip})\,dy. \tag{8}$$

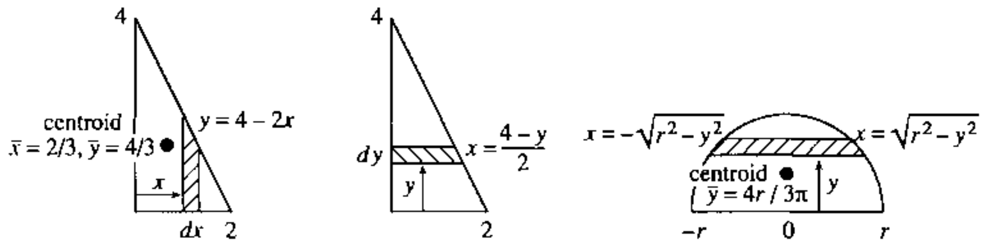


Fig. 8.14 Plates cut into strips to compute masses and moments and centroids.

The mass equals the area because $\rho = 1$. For moments, all points in a vertical strip are the same distance from the y axis. *That distance is x.* The moment is x times area, or x times length times dx—and the integral accounts for all strips.

Similarly the x-moment of a *horizontal* strip is y times strip length times dy.

EXAMPLE 3 A plate has sides $x = 0$ and $y = 0$ and $y = 4 - 2x$. Find M, M_y, M_x.

$$\text{mass } M = \text{area} = \int_0^2 y\,dx = \int_0^2 (4 - 2x)\,dx = \Big[4x - x^2\Big]_0^2 = 4.$$

The vertical strips go up to $y = 4 - 2x$, and the horizontal strips go out to $x = \frac{1}{2}(4 - y)$:

$$\text{moment } M_y = \int_0^2 x(4 - 2x)\,dx = \left[2x^2 - \frac{2}{3}x^3\right]_0^2 = \frac{8}{3}$$

$$\text{moment } M_x = \int_0^4 y\frac{1}{2}(4 - y)\,dy = \left[y^2 - \frac{1}{6}y^3\right]_0^4 = \frac{16}{3}.$$

The "center of mass" has $\bar{x} = M_y/M = 2/3$ and $\bar{y} = M_x/M = 4/3$. This is the ***centroid*** of the triangle (and also the "center of gravity"). With $\rho = 1$ these terms all refer to the same balance point $(\bar{x}, \bar{y})$. The plate will not tip over, if it rests on that point.

EXAMPLE 4 Find M_y and M_x for the half-circle below $x^2 + y^2 = r^2$.

$M_y = 0$ because the region is symmetric—Figure 8.14 balances on the y axis. In the x-moment we integrate y times the length of a horizontal strip (notice the factor 2):

$$M_x = \int_0^r y \cdot 2\sqrt{r^2 - y^2}\,dy = -\tfrac{2}{3}(r^2 - y^2)^{3/2}\Big]_0^r = \tfrac{2}{3}r^3.$$

Divide by the mass (the area $\frac{1}{2}\pi r^2$) to find the height of the centroid: $\bar{y} = M_x/M = 4r/3\pi$. This is less than $\frac{1}{2}r$ because the bottom of the semicircle is wider than the top.

MOMENT OF INERTIA

The ***moment of inertia*** comes from multiplying each mass by the ***square*** of its distance from the axis. Around the y axis, the distance is x. Around the origin, it is r:

$$I_y = \Sigma\, x_n^2 m_n \quad \text{and} \quad I_x = \Sigma\, y_n^2 m_n \quad \text{and} \quad I_0 = \Sigma\, r_n^2 m_n.$$

Notice that $I_x + I_y = I_0$ because $x_n^2 + y_n^2 = r_n^2$. In the continuous case we integrate.

The moment of inertia around the y axis is $I_y = \iint x^2 \rho(x, y)\,dx\,dy$. With a constant density $\rho = 1$, we again keep together the points on a strip. On a vertical strip they share the same x. On a horizontal strip they share y:

$$I_y = \int (x^2)\,(\text{vertical strip length})\,dx \quad \text{and} \quad I_x = \int (y^2)\,(\text{horizontal strip length})\,dy.$$

In engineering and physics, it is ***rotation*** that leads to the moment of inertia. Look at the energy of a mass m going around a circle of radius r. It has $I_0 = mr^2$.

$$\text{kinetic energy} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(r\omega)^2 = \tfrac{1}{2}I_0\omega^2. \tag{9}$$

The angular velocity is ω (radians per second). The speed is $v = r\omega$ (meters per second).

An ice skater reduces I_0 by putting her arms up instead of out. She stays close to the axis of rotation (r is small). Since her rotational energy $\frac{1}{2}I_0\omega^2$ does not change, ω increases as I_0 decreases. Then she spins faster.

Another example: It takes force to turn a revolving door. More correctly, it takes ***torque***. The force is multiplied by distance from the turning axis: $T = Fx$, so a push further out is more effective.

To see the physics, replace Newton's law $F = ma = m\,dv/dt$ by its rotational form: $T = I\,d\omega/dt$. Where F makes the mass move, the torque T makes it turn. Where m measures unwillingness to change speed, I measures unwillingness to change rotation.

EXAMPLE 5 Find the moment of inertia of a rod about (a) its end and (b) its center.

The distance x from the end of the rod goes from 0 to L. The distance from the center goes from $-L/2$ to $L/2$. Around the center, turning is easier because I is smaller:

$$I_{\text{end}} = \int_0^L x^2\,dx = \tfrac{1}{3}L^3 \qquad I_{\text{center}} = \int_{-L/2}^{L/2} x^2\,dx = \tfrac{1}{12}L^3. \tag{10}$$

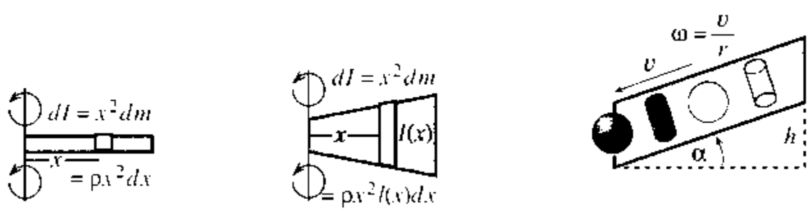


Fig. 8.15 Moment of inertia for rod and propeller. Rolling balls beat cylinders.

MOMENT OF INERTIA EXPERIMENT

Experiment: Roll a solid cylinder (a coin), a hollow cylinder (a ring), a solid ball (a marble), and a hollow ball (*not* a pingpong ball) down a slope. Galileo dropped things from the Leaning Tower—this experiment requires a Leaning Table. Objects that fall together from the tower don't roll together down the table.

Question 1 What is the order of finish? *Record your prediction first*!

Question 2 Does size make a difference if shape and density are the same?

Question 3 Does density make a difference if size and shape are the same?

Question 4 Find formulas for the velocity v and the finish time T.

To compute v, the key is that potential energy plus kinetic energy is practically constant. Energy loss from rolling friction is very small. If the mass is m and the vertical drop is h, the energy at the top (all potential) is mgh. The energy at the bottom (all kinetic) has two parts: $\frac{1}{2}mv^2$ from movement along the plane plus $\frac{1}{2}I\omega^2$ from turning. *Important fact*: $v = \omega r$ for a rolling cylinder or ball of radius r.

Equate energies and set $\omega = v/r$:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2\left(1 + \frac{I}{mr^2}\right). \tag{11}$$

***The ratio** I/mr^2 **is critical.** Call it J and solve* (11) *for* v^2:

$$v^2 = \frac{2gh}{1+J} \text{ (smaller } J \text{ means larger velocity).} \tag{12}$$

The order of J's, for different shapes and sizes, should decide the race. Apparently the density doesn't matter, because it is a factor in both I and m—so it cancels in $J = I/mr^2$. A hollow cylinder has $J = 1$, which is the largest possible—all its mass is at the full distance r from the axis. So the hollow cylinder should theoretically come in last. This experiment was developed by Daniel Drucker.

Problems 35–37 find the other three J's. Problem 40 finds the time T by integration. Your experiment will show how close this comes to the measured time.

8.5 EXERCISES

Read-through questions

If masses m_n are at distances x_n, the total mass is $M =$ __a__. The total moment around $x = 0$ is $M_y =$ __b__. The center of mass is at $\bar{x} =$ __c__. In the continuous case, the mass distribution is given by the __d__ $\rho(x)$. The total mass is $M =$ __e__ and the center of mass is at $x =$ __f__. With $\rho = x$, the integrals from 0 to L give $M =$ __g__ and $\int x\rho(x)\,dx =$ __h__ and $\bar{x} =$ __i__. The total moment is the same if the whole mass M is placed at __j__.

In a plane, with masses m_n at the points (x_n, y_n), the moment around the y axis is __k__. The center of mass has $\bar{x} =$ __l__ and $\bar{y} =$ __m__. For a plate with density $\rho = 1$, the mass M equals the __n__. If the plate is divided into vertical strips of height $y(x)$, then $M = \int y(x)\,dx$ and $M_y = \int$ __o__ dx. For a square plate $0 \leqslant x, y \leqslant L$, the mass is $M =$ __p__ and the moment around the y axis is $M_y =$ __q__. The center of mass is at $(\bar{x}, \bar{y}) =$ __r__. This point is the __s__, where the plate balances.

A mass m at a distance x from the axis has moment of inertia $I =$ __t__. A rod with $\rho = 1$ from $x = a$ to $x = b$ has $I_y =$ __u__. For a plate with $\rho = 1$ and strips of height $y(x)$, this becomes $I_y = \int$ __v__. The torque T is __w__ times __x__.

Compute the mass M along the x axis, the moment M_y around $x = 0$, and the center of mass $\bar{x} = M_y/M$.

1 $m_1 = 2$ at $x_1 = 1$, $m_2 = 4$ at $x_2 = 2$

2 $m = 3$ at $x = 0, 1, 2, 6$

3 $\rho = 1$ for $-1 \leqslant x \leqslant 3$

4 $\rho = x^2$ for $0 \leqslant x \leqslant L$

5 $\rho = 1$ for $0 \leqslant x < 1$, $\rho = 2$ for $1 \leqslant x \leqslant 2$

6 $\rho = \sin x$ for $0 \leqslant x \leqslant \pi$

Find the mass M, the moments M_y and M_x, and the center of mass $(\bar{x}, \bar{y})$.

7 Unit masses at $(x, y) = (1, 0), (0, 1)$, and $(1, 1)$

8 $m_1 = 1$ at $(1, 0)$, $m_2 = 4$ at $(0, 1)$

9 $\rho = 7$ in the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

10 $\rho = 3$ in the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$.

Find the area M and the centroid $(\bar{x}, \bar{y})$ inside curves 11–16.

11 $y = \sqrt{1 - x^2}$, $y = 0$, $x = 0$ (quarter-circle)

12 $y = x$, $y = 2 - x$, $y = 0$ (triangle)

13 $y = e^{-2x}$, $y = 0$, $x = 0$ (infinite dagger)

14 $y = x^2$, $y = x$ (lens)

15 $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ (ring)

16 $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $y = 0$ (half-ring).

Verify these engineering formulas for I_y with $\rho = 1$:

17 Rectangle bounded by $x = 0$, $x = a$, $y = 0$, $y = b$: $I_y = a^3b/3$.

18 Square bounded by $x = -\frac{1}{2}a$, $x = \frac{1}{2}a$, $y = -\frac{1}{2}a$, $y = \frac{1}{2}a$: $I_y = a^4/12$.

19 Triangle bounded by $x = 0$, $y = 0$, $x + y = a$: $I_y = a^4/12$.

20 Disk of radius a centered at $x = y = 0$: $I_y = \pi a^4/4$.

21 The moment of inertia around the point $x=t$ of a rod with density $\rho(x)$ is $I=\int (x-t)^2\rho(x)\,dx$. Expand $(x-t)^2$ and I into three terms. Show that $dI/dt=0$ when $t=\bar{x}$. The moment of inertia is smallest around the center of mass.

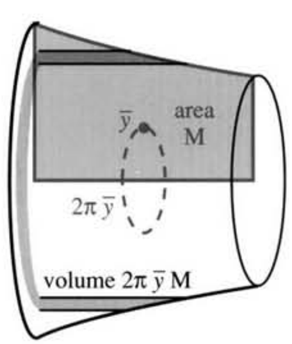
22 A region has $\bar{x}=0$ if $M_y=\int x(\text{height of strip})\,dx=0$. The moment of inertia about any other axis $x=c$ is $I=\int (x-c)^2(\text{height of strip})\,dx$. Show that $I=I_y+(\text{area})(c^2)$. This is the *parallel axis theorem*: I is smallest around the balancing axis $c=0$.

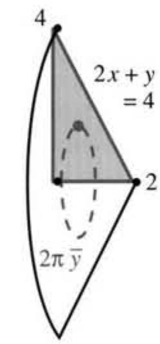
23 (With thanks to Trivial Pursuit) In what state is the center of gravity of the United States—the "geographical center" or centroid?

24 Pappus (an ancient Greek) noticed that the volume is

$$V=\int 2\pi y(\text{strip width})\,dy=2\pi M_x=2\pi\bar{y}M$$

when a region of area M is revolved around the x axis. In the first step the solid was cut into ________.



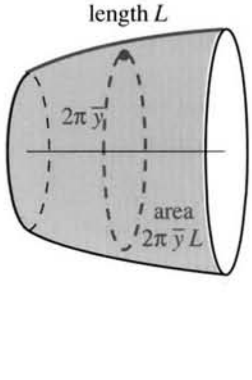


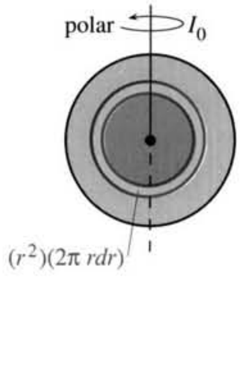
25 Use this theorem of Pappus to find the volume of a torus. Revolve a disk of radius a whose center is at height $\bar{y}=b>a$.

26 Rotate the triangle of Example 3 around the x axis and find the volume of the resulting cone—first from $V=2\pi\bar{y}M$, second from $\frac{1}{3}\pi r^2 h$.

27 Find M_x and M_y for a thin wire along the semicircle $y=\sqrt{1-x^2}$. Take $\rho=1$ so $M=\text{length}=\pi$.

28 A second theorem of Pappus gives $A=2\pi\bar{y}L$ as the surface area when a wire of length L is rotated around the x axis. Verify his formula for a horizontal wire along $y=3$ ($x=0$ to $x=L$) and a vertical wire ($y=1$ to $y=L+1$).





29 The surface area of a sphere is $A=4\pi$ when $r=1$. So $A=2\pi\bar{y}L$ leads to $\bar{y}=$ ________ for the semicircular wire in Problem 27.

30 Rotating $y=mx$ around the x axis between $x=0$ and $x=1$ produces the surface area $A=$ ________.

31 Put a mass m at the point $(x,0)$. Around the origin the torque from gravity is the force mg times the distance x. This equals g times the ________ mx.

32 If ten equal forces F are alternately down and up at $x=1,2,\ldots,10$, what is their torque?

33 The solar system has nine masses m_n at distances r_n with angular velocities ω_n. What is the moment of inertia around the sun? What is the rotational energy? What is the torque provided by the sun?

34 The disk $x^2+y^2\leqslant a^2$ has $I_0=\int_0^a r^2\,2\pi r\,dr=\frac{1}{2}\pi a^4$. Why is this different from I_y in Problem 20? Find the *radius of gyration* $\bar{r}=\sqrt{I_0/M}$. (The rotational energy $\frac{1}{2}I_0\omega^2$ equals $\frac{1}{2}M\bar{r}^2\omega^2$—when the whole mass is turning at radius $\bar{r}$.)

Questions 35–42 come from the moment of inertia experiment.

35 A solid cylinder of radius r is assembled from hollow cylinders of length l, radius x, and volume $(2\pi x)(l)(dx)$. The solid cylinder has

$$\text{mass } M=\int_0^r 2\pi xl\rho\,dx \quad\text{and}\quad I=\int_0^r x^2\,2\pi xl\rho\,dx.$$

With $\rho=7$ find M and I and $J=I/Mr^2$.

36 Problem 14.4.40 finds $J=2/5$ for a solid ball. It is less than J for a solid cylinder because the mass of the ball is more concentrated near ________.

37 Problem 14.4.39 finds $J=\frac{1}{2}\int_0^\pi \sin^3\phi\,d\phi=$ ________ for a hollow ball. The four rolling objects finish in the order ________.

38 By varying the density of the ball how could you make it roll faster than any of these shapes?

39 Answer Question 2 about the experiment.

40 For a vertical drop of y, equation (12) gives the velocity along the plane: $v^2=2gy/(1+J)$. Thus $v=cy^{1/2}$ for $c=$ ________. The vertical velocity is $dy/dt=v\sin\alpha$:

$$dy/dt=cy^{1/2}\sin\alpha \quad\text{and}\quad \int y^{-1/2}\,dy=\int c\sin\alpha\,dt.$$

Integrate to find $y(t)$. Show that the bottom is reached ($y=h$) at time $T=2\sqrt{h}/c\sin\alpha$.

41 What is the theoretical ratio of the four finishing times?

42 **True or false**:

(a) Basketballs roll downhill faster than baseballs.

(b) The center of mass is always at the centroid.

(c) By putting your arms up you reduce I_x and I_y.

(d) The center of mass of a high jumper goes over the bar (on successful jumps).

8.6 Force, Work, and Energy

Chapter 1 introduced derivatives df/dt and df/dx. The independent variable could be t or x. For velocity it was natural to use the letter t. This section is about two important physical quantities—***force*** and ***work***—for which x is the right choice.

The basic formula is $W = Fx$. ***Work equals force times distance moved*** (distance in the direction of F). With a force of 100 pounds on a car that moves 20 feet, the work is 2000 foot-pounds. If the car is rolling forward and you are pushing backward, the work is -2000 foot-pounds. If your force is only 80 pounds and the car doesn't move, the work is zero. In these examples the force is constant.

$W = Fx$ is completely parallel to $f = vt$. When v is constant, we only need multiplication. It is a *changing velocity* that requires calculus. The integral $\int v(t)\,dt$ adds up small multiplications over short times. For a changing force, we add up small pieces of work $F\,dx$ over short distances:

$$W = Fx \quad (\textit{constant force}) \qquad W = \int F(x)\,dx \quad (\textit{changing force}).$$

In the first case we lift a suitcase weighing $F = 30$ pounds up $x = 20$ feet of stairs. The work is $W = 600$ foot-pounds. The suitcase doesn't get heavier as we go up—it only seems that way. Actually it gets lighter (we study gravity below).

In the second case we stretch a spring, which needs more force as x increases. *Hooke's law says that* $F(x) = kx$. The force is proportional to the stretching distance x. Starting from $x = 0$, the work increases with the *square* of x:

$$F = kx \qquad \text{and} \qquad W = \int_0^x kx\,dx = \tfrac{1}{2}kx^2. \tag{1}$$

In metric units the force is measured in Newtons and the distance in meters. The unit of work is a Newton-meter (a joule). The 600 foot-pounds for an American suitcase would have been about 800 joules in France.

EXAMPLE 1 Suppose a force of $F = 20$ pounds stretches a spring 1 foot.

(a) *Find k.* The elastic constant is $k = F/x = 20$ pounds per foot.

(b) *Find W.* The work is $\frac{1}{2}kx^2 = \frac{1}{2} \cdot 20 \cdot 1^2 = 10$ foot-pounds.

(c) *Find x when $F = -10$ pounds.* This is compression not stretching: $x = -\frac{1}{2}$ foot.

Compressing the same spring through the same distance requires the same work. For compression x and F are negative. But the work $W = \frac{1}{2}kx^2$ is still positive. Please note that W does not equal kx times x! That is the whole point of variable force (change Fx to $\int F(x)\,dx$).

May I add another important quantity from physics? It comes from looking at the situation from the viewpoint of the spring. In its natural position, the spring rests comfortably. It feels no strain and has no energy. *Tension or compression gives it potential energy.* More stretching or more compression means more energy. ***The change in energy equals the work.*** The potential energy of the suitcase increases by 600 foot-pounds, when it is lifted 20 feet.

Write $V(x)$ for the potential energy. Here x is the height of the suitcase or the extension of the spring. In moving from $x = a$ to $x = b$, ***work = increase in potential***:

$$W = \int_a^b F(x)\,dx = V(b) - V(a). \tag{2}$$

This is absolutely beautiful. The work W is the ***definite integral.*** The potential V is the ***indefinite integral.*** If we carry the suitcase up the stairs and back down, our total work is *zero*. We may feel tired, but the trip down should have given back our energy. (It was in the suitcase.) Starting with a spring that is compressed one foot, and ending with the spring extended one foot, again we have done no work. $V = \frac{1}{2}kx^2$ is the same for $x = -1$ and $x = 1$. But an extension from $x = 1$ to $x = 3$ requires work:

$$W = \text{change in } V = \tfrac{1}{2}k(3)^2 - \tfrac{1}{2}k(1)^2.$$

Indefinite integrals like V come with a property that we know well. ***They include an arbitrary constant*** C. The correct potential is not simply $\frac{1}{2}kx^2$, it is $\frac{1}{2}kx^2 + C$. To compute a *change* in potential, we don't need C. The constant cancels. But to determine V itself, we have to choose C. By fixing $V = 0$ at one point, the potential is determined at all other points. A common choice is $V = 0$ at $x = 0$. Sometimes $V = 0$ at $x = \infty$ (for gravity). Electric fields can be "grounded" at any point.

There is another connection between the potential V and the force F. According to (2), V is the indefinite integral of F. Therefore $F(x)$ ***is the derivative of*** $V(x)$. The fundamental theorem of calculus is also fundamental to physics:

$$\text{force exerted } on \text{ spring: } F = dV/dx \tag{3a}$$

$$\text{force exerted } by \text{ spring: } F = -dV/dx \tag{3b}$$

Those lines say the same thing. One is our force pulling on the spring, the other is the "restoring force" pulling back. (3a) and (3b) are a warning that the sign of F depends on the point of view. Electrical engineers and physicists use the minus sign. In mechanics the plus sign is more common. It is one of the ironies of fate that $F = V'$, while distance and velocity have those letters reversed: $v = f'$. Note the change to capital letters and the change to x.

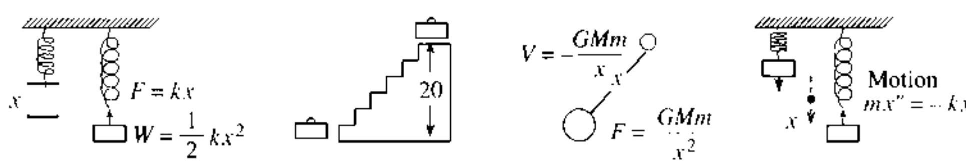


Fig. 8.16 Stretched spring; suitcase 20 feet up; moon of mass m; oscillating spring.

EXAMPLE 2 ***Newton's law of gravitation*** (inverse square law):

force to overcome gravity $= GMm/x^2$ force exerted by gravity $= -GMm/x^2$

An engine pushes a rocket forward. Gravity pulls it back. The gravitational constant is G and the Earth's mass is M. The mass of the rocket or satellite or suitcase is m, and the potential is the indefinite integral:

$$V(x) = \int F(x)\,dx = -GMm/x + C. \tag{4}$$

Usually $C = 0$, which makes the potential zero at $x = \infty$.

Remark When carrying the suitcase upstairs, x changed by 20 feet. The weight was regarded as constant—which it nearly is. But an exact calculation of work uses the integral of $F(x)$, not just the multiplication 30 times 20. The serious difference comes when the suitcase is carried to $x = \infty$. With constant force that requires infinite work. With the correct (decreasing) force, the work equals V at infinity (which is zero) minus V at the pickup point x_0. The change in V is $W = GMm/x_0$.

KINETIC ENERGY

This optional paragraph carries the physics one step further. Suppose you release the spring or drop the suitcase. The external force changes to $F = 0$. But the internal force still acts on the spring, and gravity still acts on the suitcase. They both start moving. The potential energy of the suitcase is converted to *kinetic energy*, until it hits the bottom of the stairs.

Time enters the problem, either through Newton's law or Einstein's:

$$(\textbf{\textit{Newton}})\quad F = ma = m\frac{dv}{dt} \qquad (\textbf{\textit{Einstein}})\quad F = \frac{d}{dt}(mv). \tag{5}$$

Here we stay with Newton, and pretend the mass is constant. Exercise 21 follows Einstein; the mass increases with velocity. There $m = m_0/\sqrt{1 - v^2/c^2}$ goes to infinity as v approaches c, the speed of light. That correction comes from the theory of relativity, and is not needed for suitcases.

What happens as the suitcase falls? From $x = a$ at the top of the stairs to $x = b$ at the bottom, potential energy is lost. But kinetic energy $\frac{1}{2}mv^2$ is gained, as we see from integrating Newton's law:

$$\text{force } F = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}$$

$$\text{work } \int_a^b F\,dx = \int_a^b mv\frac{dv}{dx}\,dx = \frac{1}{2}mv^2(b) - \frac{1}{2}mv^2(a). \tag{6}$$

This same force F is given by $-dV/dx$. So the work is also the change in V:

$$\text{work } \int_a^b F\,dx = \int_a^b \left(-\frac{dV}{dx}\right)dx = -V(b) + V(a). \tag{7}$$

Since (6) = (7), the total energy $\frac{1}{2}mv^2 + V$ (*kinetic plus potential*) is constant:

$$\tfrac{1}{2}mv^2(b) + V(b) = \tfrac{1}{2}mv^2(a) + V(a). \tag{8}$$

This is the law of ***conservation of energy***. The total energy is conserved.

EXAMPLE 3 Attach a mass m to the end of a stretched spring and let go. The spring's energy $V = \frac{1}{2}kx^2$ is gradually converted to kinetic energy of the mass. At $x = 0$ the change to kinetic energy is complete: the original $\frac{1}{2}kx^2$ has become $\frac{1}{2}mv^2$. Beyond $x = 0$ the potential energy increases, the force reverses sign and pulls back, and kinetic energy is lost. Eventually all energy is potential—when the mass reaches the other extreme. It is *simple harmonic motion*, exactly as in Chapter 1 (where the mass was the shadow of a circling ball). The equation of motion is the statement that ***the rate of change of energy is zero*** (and we cancel $v = dx/dt$):

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0 \quad \text{or} \quad m\frac{d^2x}{dt^2} + kx = 0. \tag{9}$$

That is $F = ma$ in disguise. For a spring, the solution $x = \cos\sqrt{k/m}\,t$ will be found in this book. For more complicated structures, engineers spend a billion dollars a year computing the solution.

PRESSURE AND HYDROSTATIC FORCE

Our forces have been concentrated at a single points. That is not the case for ***pressure***. A fluid exerts a force all over the base and sides of its container. Suppose a water tank or swimming pool has constant depth h (in meters or feet). The water has weight-density $w \approx 9800\ \text{N/m}^3 \approx 62\ \text{lb/ft}^3$. On the base, the pressure is w times h. The force is wh times the base area A:

$$F = whA \text{ (pounds or Newtons)} \qquad p = F/A = wh \text{ (lb/ft}^2 \text{ or N/m}^2\text{)}. \tag{10}$$

Thus *pressure is force per unit area*. Here p and F are computed by multiplication, because the depth h is constant. Pressure is proportional to depth (as divers know). Down the side wall, h varies and we need calculus.

The pressure on the side is still wh—*the same in all directions*. We divide the side into horizontal strips of thickness Δh. Geometry gives the length $l(h)$ at depth h (Figure 8.17). The area of a strip is $l(h)\,\Delta h$. The pressure wh is nearly constant on the strip—the depth only changes by Δh. ***The force on the strip is*** $\Delta F = whl\Delta h$. Adding those forces, and narrowing the strips so that $\Delta h \to 0$, the total force approaches an integral:

$$\textbf{\textit{total force}}\ F = \int whl(h)\,dh \tag{11}$$

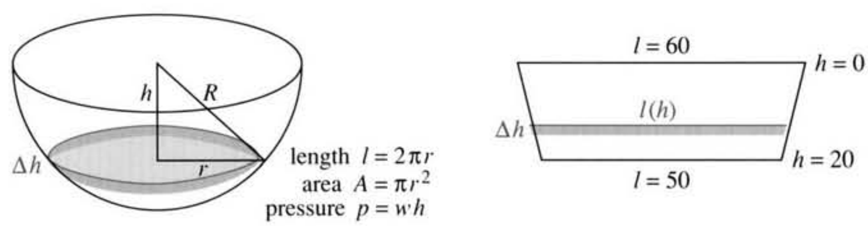


Fig. 8.17 Water tank and dam: length of side strip $= l$, area of layer $= A$.

EXAMPLE 4 Find the total force on the trapezoidal dam in Figure 8.17.

The side length is $l = 60$ when $h = 0$. The depth h increases from 0 to 20. The main problem is to find l at an in-between depth h. With straight sides the relation is linear: $l = 60 + ch$. We choose c to give $l = 50$ when $h = 20$. Then $50 = 60 + c(20)$ yields $c = -\frac{1}{2}$.

The total force is the integral of whl. So substitute $l = 60 - \frac{1}{2}h$:

$$F = \int_0^{20} wh(60 - \tfrac{1}{2}h)\,dh = \Big[30wh^2 - \tfrac{1}{6}wh^3\Big]_0^{20} = 12000w - \tfrac{1}{6}(8000w).$$

With distance in feet and $w = 62\ \text{lb/ft}^3$, F is in pounds. With distance in meters and $w = 9800\ \text{N/m}^3$, the force is in Newtons.

Note that (weight-density w) = (mass-density ρ) times $(g) = (1000)(9.8)$. These SI units were chosen to make the density of water at 0°C exactly $\rho = 1000\ \text{kg/m}^3$.

EXAMPLE 5 ***Find the work to pump water out of a tank. The area at depth*** h ***is*** $A(h)$.

Imagine lifting out *one layer of water at a time*. The layer weighs $wA(h)\,\Delta h$. The work to lift it to the top is its weight times the distance h, or $whA(h)\,\Delta h$. The work to empty the whole tank is the integral:

$$W = \int whA(h)\,dh. \tag{12}$$

Suppose the tank is the bottom half of a sphere of radius R. The cross-sectional area at depth h is $A = \pi(R^2 - h^2)$. Then the work is the integral (12) from 0 to R. It equals $W = \pi w R^4/4$.

Units: w = force/(distance)^3 times R^4 = (distance)^4 gives work W = (force)(distance).

8.6 EXERCISES

Read-through questions

Work equals __a__ times __b__. For a spring the force is $F =$ __c__, proportional to the extension x (this is __d__ law). With this variable force, the work in stretching from 0 to x is $W = \int$ __e__ = __f__. This equals the increase in the __g__ energy V. Thus W is a __h__ integral and V is the corresponding __i__ integral, which includes an arbitrary __j__. The derivative dV/dx equals __k__. The force of gravity is $F =$ __l__ and the potential is $V =$ __m__.

In falling, V is converted to __n__ energy $K =$ __o__. The total energy $K + V$ is __p__ (this is the law of __q__ when there is no external force).

Pressure is force per unit __r__. Water of density w in a pool of depth h and area A exerts a downward force $F =$ __s__ on the base. The pressure is $p =$ __t__. On the sides the __u__ is still wh at depth h, so the total force is $\int whl\,dh$, where l is __v__. In a cubic pool of side s, the force on the base is $F =$ __w__, the length around the sides is $l =$ __x__, and the total force on the four sides is $F =$ __y__. The work to pump the water out of the pool is $W = \int whA\,dh =$ __z__.

1 (a) Find the work W when a constant force $F = 12$ pounds moves an object from $x = .9$ feet to $x = 1.1$ feet.

(b) Compute W by integration when the force $F = 12/x^2$ varies with x.

2 A 12-inch spring is stretched to 15 inches by a force of 75 pounds.

(a) What is the spring constant k in *pounds per foot*?

(b) Find the work done in stretching the spring.

(c) Find the work to stretch it 3 more inches.

3 A shock-absorber is compressed 1 inch by a weight of 1 ton. Find its spring constant k in pounds per foot. What potential energy is stored in the shock-absorber?

4 A force $F = 20x - x^3$ stretches a nonlinear spring by x.

(a) What work is required to stretch it from $x = 0$ to $x = 2$?

(b) What is its potential energy V at $x = 2$, if $V(0) = 5$?

(c) What is $k = dF/dx$ for a small additional stretch at $x = 2$?

5 (a) A 120-lb person makes a scale go down x inches. How much work is done?

(b) If the same person goes x inches down the stairs, how much potential energy is lost?

6 A rocket burns its 100 kg of fuel at a steady rate to reach a height of 25 km.

(a) Find the weight of fuel left at height h.

(b) How much work is done lifting fuel?

7 Integrate to find the work in winding up a hanging cable of length 100 feet and weight density 5 lb/ft. How much additional work is caused by a 200-pound weight hanging at the end of the cable?

8 The great pyramid (height 500′—you can see it from Cairo) has a square base 800′ by 800′. Find the area A at height h. If the rock weighs $w = 100$ lb/ft^3, approximately how much work did it take to lift all the rock?

9 The force of gravity on a mass m is $F = -GMm/x^2$. With $G = 6 \cdot 10^{-17}$ and Earth mass $M = 6 \cdot 10^{24}$ and rocket mass $m = 1000$, compute the work to lift the rocket from $x = 6400$ to $x = 6500$. (The units are kgs and kms and Newtons, giving work in Newton-kms.)

10 The approximate work to lift a 30-pound suitcase 20 feet is 600 foot-pounds. The exact work is the change in the potential $V = -GmM/x$. Show that ΔV is 600 times a correction factor $R^2/(R^2 - 10^2)$, when x changes from $R - 10$ to $R + 10$. (This factor is practically 1, when R = radius of the Earth.)

11 Find the work to lift the rocket in Problem 9 from $x = 6400$ out to $x = \infty$. If this work equals the original kinetic energy $\frac{1}{2}mv^2$, what was the original v (*the escape velocity*)?

12 The kinetic energy $\frac{1}{2}mv^2$ of a rocket is converted into potential energy $-GMm/x$. Starting from the Earth's radius $x = R$, what x does the rocket reach? If it reaches $x = \infty$ show that $v = \sqrt{2GM/R}$. This escape velocity is 25,000 miles per hour.

13 It takes 20 foot-pounds of work to stretch a spring 2 feet. How much work to stretch it one more foot?

14 A barrel full of beer is 4 feet high with a 1 foot radius and an opening at the bottom. How much potential energy is lost by the beer as it comes out of the barrel?

15 A rectangular dam is 40 feet high and 60 feet wide. Compute the total side force F on the dam when (a) the water is at the top (b) the water level is halfway up.

16 A triangular dam has an 80-meter base at a depth of 30 meters. If water covers the triangle, find

(a) the pressure at depth h
(b) the length l of the dam at depth h
(c) the total force on the dam.

17 A cylinder of depth H and cross-sectional area A stands full of water (density w). (a) Compute the work $W = \int wAh\,dh$ to lift all the water to the top. (b) Check the units of W. (c) What is the work W if the cylinder is only half full?

18 In Problem 17, compute W in both cases if $H = 20$ feet, $w = 62\ \text{lb/ft}^3$, and the base is a circle of radius $r = 5$ feet.

19 How much work is required to pump out a swimming pool, if the area of the base is 800 square feet, the water is 4 feet deep, and the top is one foot above the water level?

20 For a cone-shaped tank the cross-sectional area increases with depth: $A = \pi r^2 h^2/H^2$. Show that the work to empty it is half the work for a cylinder with the same height and base. What is the ratio of volumes of water?

21 In relativity the mass is $m = m_0/\sqrt{1 - v^2/c^2}$. Find the correction factor in Newton's equation $F = m_0 a$ to give Einstein's equation $F = d(mv)/dt = (d(mv)/dv)(dv/dt) =$ ________ $m_0 a$.

22 Estimate the depth of the *Titanic*, the pressure at that depth, and the force on a cabin door. Why doesn't every door collapse at the bottom of the Atlantic Ocean?

23 A swimming pool is 4 meters wide, 10 meters long, and 2 meters deep. Find the weight of the water and the total force on the bottom.

24 If the pool in Problem 23 has a shallow end only one meter deep, what fraction of the water is saved? Draw a cross-section (a trapezoid) and show the direction of force on the sides and the sloping bottom.

25 In what ways is work like a definite integral and energy like an indefinite integral? Their derivative is the ________.