CHAPTER 5 INTEGRALS

5.1 The Idea of the Integral (page 181)

Problems 1–3 review sums and differences from Section 1.2. This chapter goes forward to integrals and derivatives.

1. If $f_0, f_1, f_2, f_3, f_4 = 0, 2, 6, 12, 20$, find the differences $v_j = f_j - f_{j-1}$ and the sum of the v's.

   - The differences are $v_1, v_2, v_3, v_4 = 2, 4, 6, 8$. The sum is $2 + 4 + 6 + 8 = 20$. This equals $f_4 - f_0$.

2. If $v_1, v_2, v_3, v_4 = 3, 3, 3, 3$ and $f_0 = 5$, find the f's. Show that $f_4 - f_0$ is the sum of the v's.

   - Each new f_j is $f_{j-1} + v_j$. So $f_1 = f_0 + v_1 = 5 + 3 = 8$. Similarly $f_2 = f_1 + v_2 = 8 + 3 = 11$. Then $f_3 = 14$ and $f_4 = 17$. The sum of the v's is 12. The difference between f_{last} and f_{first} is also $17 - 5 = 12$.

3. (This is Problem 5.1.5) Show that $f_j = \frac{r^j}{r-1}$ has differences $v_j = f_j - f_{j-1} = r^{j-1}$.

   - The formula gives $f_0 = \frac{r^0}{r-1} = \frac{1}{r-1}$. Then $f_1 = \frac{r}{r-1}$ and $f_2 = \frac{r^2}{r-1}$. Now find the differences:

   $$v_1 = \frac{r}{r-1} - \frac{1}{r-1} = \frac{r-1}{r-1} = 1 \text{ and } v_2 = \frac{r^2}{r-1} - \frac{r}{r-1} = \frac{r^2-r}{r-1} = r.$$

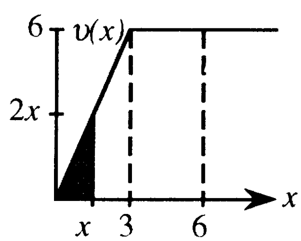
   In general $f_j - f_{j-1} = r^{j-1}$. This is v_j. Adding the v's gives the ***geometric series*** $1 + r + r^2 + \cdots + r^{n-1}$. Its sum is $f_n - f_0 = \frac{r^n}{r-1} - \frac{1}{r-1} = \frac{r^n-1}{r-1}$.

4. Suppose $v(x) = 2x$ for $0 < x < 3$ and $v(x) = 6$ for $x > 3$. Sketch and find the area from 0 to x under the graph of $v(x)$.

   - There are really two cases to think about. If $x < 3$, the shaded triangle with base x and height $2x$ has area $= \frac{1}{2}$(base) (height) $= \frac{1}{2}x(2x) = x^2$. If $x > 3$ the area is that of a triangle plus a rectangle. The triangle has base 3 and height 6 and area 9. The rectangle has base $(x - 3)$ and height 6. Total area $= 9 + 6(x - 3) = 6x - 9$. The area $f(x)$ has a two-part formula:

   $$f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 6x - 9, & x > 3. \end{cases}$$

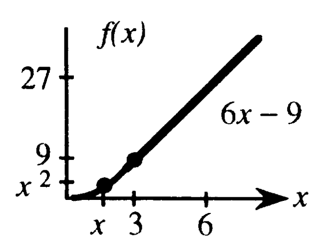
Areas under $v(x)$ give $f(x)$



small triangle $\frac{1}{2}x(2x) = x^2$

complete triangle $\frac{1}{2}3(6) = 9$

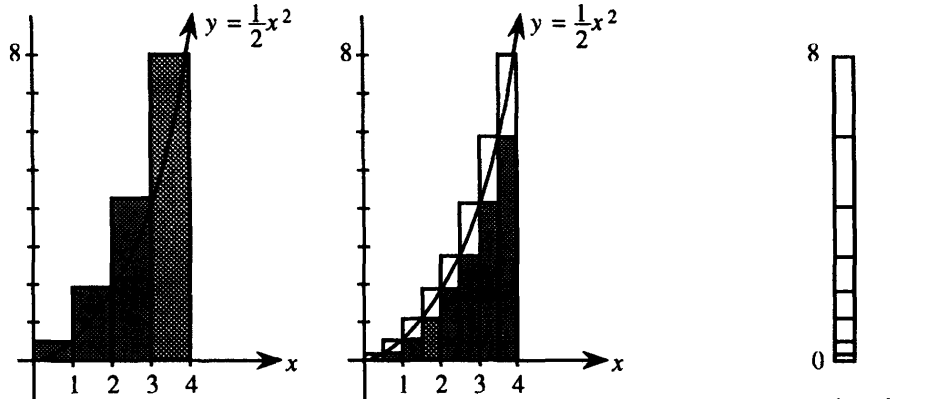
add rectangle $6(x-3) = 6x - 18$



5. Use four rectangles to approximate the area under the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 4$. Then do the same using eight rectangles.

   - The heights of the four rectangles are $f(1) = \frac{1}{2}$, $f(2) = 2$, $f(3) = \frac{9}{2}$, $f(4) = 8$. The width of each rectangle is one. The sum of the four areas is $1 \cdot \frac{1}{2} + 1 \cdot 2 + 1 \cdot \frac{9}{2} + 1 \cdot 8 = 15$. The sketch shows that the actual area under the curve is less than 15.

The second figure shows eight rectangles. Their total area is still greater than the curved area, but less than 15. (Can you see why?) Each rectangle has width $\frac{1}{2}$. Their heights are determined by $y = \frac{1}{2}x^2$, where $x = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$. The total area of the rectangles is $\frac{1}{2}(\frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 + \frac{25}{8} + \frac{9}{2} + \frac{49}{8} + 8) =$ 12.75. If we took 16 rectangles we would get an even closer estimate. ***The actual area is*** $10\frac{2}{3}$.



four outer rectangles — eight outer, eight inner — outer minus inner

6. Use the same curve $y = \frac{1}{2}x^2$ as in Problem 5. This time *inscribe* eight rectangles – the rectangles should touch the curve but remain inside it. What is the area approximation using this method? Give bounds on the same true area A under the curve.

   - The eight inscribed rectangles are shown, counting the first one with height zero. The total area is $\frac{1}{2}[f(0) + f(\frac{1}{2}) + \cdots + f(\frac{7}{2})] = \frac{1}{2}[0 + \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 + \frac{25}{8} + \frac{9}{2} + \frac{49}{8}] = 8.75$. The true area A is greater than 8.75 and less than 12.75.

   The differences between the outer and the inner rectangles add to a **single rectangle** with base $\frac{1}{2}$ and height $f(4) = 8$. Difference in areas $= \frac{1}{2}[f(4) - f(0)] = 4$. This is $12.75 - 8.75$.

   Hint for Exercises 5.1.11 – 14: These refer to an optimist and a pessimist. The pessimist would *under*estimate income, using inner rectangles. The optimist *over*estimates. She prefers *outer* rectangles that contain the graph.

Read-throughs and selected even-numbered solutions :

The problem of summation is to add $v_1 + \cdots + v_n$. It is solved if we find f's such that $v_j = \mathbf{f_j} - \mathbf{f_{j-1}}$. Then $v_1 + \cdots + v_n$ equals $\mathbf{f_n} - \mathbf{f_0}$. The cancellation in $(f_1 - f_0) + (f_2 - f_1) + \cdots + (f_n - f_{n-1})$ leaves only $\mathbf{f_n}$ and $-\mathbf{f_0}$. Taking sums is the **reverse (or inverse)** of taking differences.

The differences between 0, 1, 4, 9 are $v_1, v_2, v_3 = \mathbf{1, 3, 5}$. For $f_j = j^2$ the difference between f_{10} and f_9 is $v_{10} = \mathbf{19}$. From this pattern $1 + 3 + 5 + \cdots + 19$ equals **100**.

For functions, finding the integral is the reverse of **finding the derivative.** If the derivative of $f(x)$ is $v(x)$, then the **integral** of $v(x)$ is $f(x)$. If $v(x) = 10x$ then $f(x) = \mathbf{5x^2}$. This is the **area** of a triangle with base x and height $10x$.

Integrals begin with sums. The triangle under $v = 10x$ out to $x = 4$ has area **80**. It is approximated by four rectangles of heights 10, 20, 30, 40 and area **100**. It is better approximated by eight rectangles of heights $\mathbf{5, 10, \cdots, 40}$ and area **90**. For n rectangles covering the triangle the area is the sum of $\mathbf{\frac{4}{n}}(\mathbf{\frac{40}{n}} + \mathbf{\frac{80}{n}} + \cdots + \mathbf{40}) = \mathbf{80} + \mathbf{\frac{80}{n}}$. As $n \to \infty$ this sum should approach the number **80**. *That is the integral of $v = 10x$ from 0 to 4.*

6 $f_0 = \frac{1-1}{r-1} = 0; 1 + r + \cdots + r^n = f_n = \mathbf{\frac{r^n - 1}{r - 1}}$.

8 The f's are $\mathbf{0, 1, -1, 2, -2}, \cdots$ Here $v_j = (-1)^{j+1}j$ or $v_j = \begin{cases} j & j \text{ odd} \\ -j & j \text{ even} \end{cases}$ and $f_j = \begin{cases} \frac{j+1}{2} & j \text{ odd} \\ \frac{-j}{2} & j \text{ even} \end{cases}$

12 The last rectangle for the pessimist has height $\sqrt{\frac{15}{4}}$. Since the optimist's last rectangle of area $\frac{1}{4}\sqrt{\frac{16}{4}} = \frac{1}{2}$ is missed, the total area is reduced by $\frac{1}{2}$.

16 Under the $\sqrt{x}$ curve, the first triangle has base 1, height 1, area $\frac{1}{2}$. To its right is a rectangle of area 3. Above the rectangle is a triangle of base 3, height 1, area $\frac{3}{2}$. The total area $\frac{1}{2} + 3 + \frac{3}{2} = 5$ is below the curve.

5.2 Antiderivatives (page 186)

The text on page 186 explains the difference between antiderivative, indefinite integral, and definite integral. If you're given a function $y = v(x)$ and asked for an ***antiderivative***, you give any function whose derivative is $v(x)$. If you need the ***indefinite integral***, you take an antiderivative and add $+C$ for any arbitrary constant. If you need a definite integral you will be given two endpoints $x = a$ and $x = b$. Find an antiderivative $y = f(x)$ and then compute $f(b) - f(a)$. In this problem your answer is a number, not a function.

The definite integral gives the exact area under the curve $v(x)$ from $x = a$ to $x = b$. (This has not been proved in the text yet, but this is the direction we are headed.) Finding antiderivatives is like solving puzzles–it takes a bit of insight as well as experience. Questions 1–5 are examples of this process. In each case, find the indefinite integral $f(x)$ of $v(x)$.

1. $v(x) = 9x^3 - 8$

   - An antiderivative of x^n is $\frac{1}{n+1}x^{n+1}$. For x^3 the power is $n = 3$. An antiderivative of x^3 is $\frac{1}{4}x^4$. Multiplying by 9 gives $\frac{9}{4}x^4$ as an antiderivative of $9x^3$. For the term -8, an antiderivative is $-8x$. Put these together to get the indefinite integral $f(x) = \frac{9}{4}x^4 - 8x + C$. ***Check*** $f'(x) = x^3 - 8 + 0 = v(x)$.

2. $v(x) = \sqrt{2x+5}$

   - The square root is the "$\frac{1}{2}$ power." Adding 1 gives $y = (2x+5)^{3/2}$. But now $\frac{dy}{dx}$ is $\frac{3}{2}(2x+5)^{1/2} \cdot 2 = 3\sqrt{2x+5}$. The first guess was off by a factor of 3. Correct this by $f(x) = \frac{1}{3}(2x+5)^{3/2} + C$. Check by taking $f'(x)$.

3. $v(x) = 3\sec 4x \tan 4x$.

   - We know that $y = \sec u$ has $\frac{dy}{du} = \sec u \tan u$. So the first guess at an antiderivative is $y = \sec 4x$. This gives $\frac{dy}{dx} = 4\sec 4x \tan 4x$. (The 4 comes from the chain rule: $u = 4x$ and $\frac{du}{dx} = 4$.) Everything is right but the leading 4, where we want 3. Adjust to get $f(x) = \frac{3}{4}\sec 4x + C$.

4. (This is 5.3.11) $v(x) = \sin x \cos x$.

   - There are two approaches, each illuminating in its own way. The first is to recognize $v(x)$ as $\frac{1}{2}\sin 2x$. Since the derivative of the cosine is minus the sine, $y = -\cos 2x$ is a first guess. That gives $\frac{dy}{dx} = 2\sin 2x$. To get $\frac{1}{2}$ instead of 2, divide by 4. The correct answer is $f(x) = -\frac{1}{4}\cos 2x + C$.

     A second approach is to recognize $v(x) = \sin x \cos x$ as $u\frac{du}{dx}$, where $u = \sin x$. An antiderivative for $u\frac{du}{dx}$ is $\frac{1}{2}u^2$. So $f(x) = \frac{1}{2}(\sin x)^2 + C$. (Find $f'(x)$ to check.)

Wait a minute. How can $f(x) = -\frac{1}{4}\cos 2x + C$ and $f(x) = \frac{1}{2}\sin^2 x + C$ both be correct? The answer to this reasonable question lies in the identity $\cos 2x = 1 - 2\sin^2 x$. The first answer $-\frac{1}{4}\cos 2x$ is $-\frac{1}{4} + \frac{1}{2}\sin^2 x$. The two answers are **the same**, except for the added constant $-\frac{1}{4}$. *For any $v(x)$ all antiderivatives are the same except for constants!*

5. $v(x) = \frac{1}{1+9x^2}$.

   - This seems impossible unless you remember that $\frac{1}{1+u^2}$ is the derivative of $\tan^{-1} u$. Our problem has $u^2 = 9x^2$ and $u = 3x$. If we let $y = \tan^{-1} 3x$, we get $\frac{dy}{dx} = \frac{1}{1+9x^2} \cdot 3$. (The 3 comes from the chain rule.) The coefficient should be adjusted to give $f(x) = \frac{1}{3}\tan^{-1} 3x + C$.

   Questions 6 and 7 use definite integrals to find area under curves.

6. Find the area under the parabola $v = \frac{1}{2}x^2$ from $x = 0$ to $x = 4$. (Section 5.1 of this Guide outlined ways to estimate this area. Now you can find it exactly.)

   - Area $= \int_a^b v(x)dx = \int_0^4 \frac{1}{2}x^2 dx$. We need an antiderivative of $\frac{1}{2}x^2$. It is $\frac{1}{2}$ times $\frac{1}{3}x^3$, or $\frac{x^3}{6}$:

$$\text{Area } = \int_0^4 \frac{1}{2}x^2 dx = \frac{4^3}{6} - \frac{0^3}{6} = \frac{64}{6} = 10\frac{2}{3}.$$

7. (This is 5.2.23) For the curve $y = \frac{1}{\sqrt{x}}$ the area between $x = 0$ and $x = 1$ is 2. Verify that $f(1) - f(0) = 2$.

   - Since $\frac{1}{\sqrt{x}} = x^{-1/2}$, an antiderivative is $f(x) = 2x^{1/2}$. Therefore $\int_0^1 \frac{1}{\sqrt{x}}dx = 2(1)^{1/2} - 2(0)^{1/2} = 2 - 0 = 2$. The unusual part of this problem is that the curve goes up to $y = \infty$ when $x = 0$. But the area underneath is still 2.

8. (This is 5.2.26) Draw $y = v(x)$ so that the area $f(x)$ from 0 to x increases until $x = 1$, stays constant to $x = 2$, and decreases to $f(3) = 1$.

   - Where the area increases, $v(x)$ is positive (from 0 to 1). There is no new area from $x = 1$ to $x = 2$, so $v(x) = 0$ on that interval. The area decreases after $x = 2$, so $v(x)$ must go below the x axis. The total area from 0 to 3 is $f(3) = 1$, so the area from 2 to 3 is one unit less than the area from 0 to 1. One solution has $v(x) = 2$ then 0 then -1 in the three intervals. Can you find a *continuous* $v(x)$?

Read-throughs and selected even-numbered solutions :

Integration yields the **area** under a curve $y = v(x)$. It starts from rectangles with the base $\mathbf{\Delta x}$ and heights $v(x)$ and areas $\mathbf{v(x)\Delta x}$. As $\Delta x \to 0$ the area $v_1\Delta x + \cdots + v_n\Delta x$ becomes the **integral** of $v(x)$. The symbol for the indefinite integral of $v(x)$ is $\int \mathbf{v(x)dx}$.

The problem of integration is solved if we find $f(x)$ such that $\mathbf{\frac{df}{dx}} = \mathbf{v(x)}$. Then f is the **antiderivative** of v, and $\int_2^6 v(x)dx$ equals $\mathbf{f(6)}$ minus $\mathbf{f(2)}$. The limits of integration are **2 and 6.** This is a **definite** integral, which is a **number** and not a function $f(x)$.

The example $v(x) = x$ has $f(x) = \mathbf{\frac{1}{2}x^2}$. It also has $f(x) = \mathbf{\frac{1}{2}x^2 + 1}$. The area under $v(x)$ from 2 to 6 is **16.** The constant is canceled in computing the difference $\mathbf{f(6)}$ minus $\mathbf{f(2)}$. If $v(x) = x^8$ then $f(x) = \mathbf{\frac{1}{9}x^9}$.

The sum $v_1 + \cdots + v_n = f_n - f_0$ leads to the Fundamental Theorem $\int_a^b v(x)dx = \mathbf{f(b) - f(a)}$. The **indefinite** integral is $f(x)$ and the **definite** integral is $f(b) - f(a)$. Finding the **area** under the v-graph is the opposite of finding the **slope** of the f-graph.

6 $\frac{x^{1/3}}{x^{2/3}} = x^{-1/3}$ which has antiderivative $f(x) = \frac{3}{2}x^{2/3}$; $f(1) - f(0) = \mathbf{\frac{3}{2}}$.
10 $f(x) = \sin x - x\cos x$; $f(1) - f(0) = \mathbf{\sin 1 - \cos 1}$
18 Areas 0, 1, 2, 3 add to $A_4 = \mathbf{6}$. Each rectangle misses a triangle of base $\frac{4}{N}$ and height $\frac{4}{N}$. There are N triangles of total area $N \cdot \frac{1}{2}(\frac{4}{N})^2 = \frac{8}{N}$. So the N rectangles have area $\mathbf{8 - \frac{8}{N}}$.
22 Two rectangles have base $\frac{1}{2}$ and heights 2 and 1, with area $\mathbf{\frac{3}{2}}$. Four rectangles have base $\frac{1}{4}$ and heights 4, 3, 2, 1 with area $\frac{10}{4} = \mathbf{\frac{5}{2}}$. Eight rectangles have area $\frac{7}{2}$. The limiting area under $y = \frac{1}{x}$ is **infinite.**
28 The area $f(4) - f(3)$ is $-\frac{1}{2}$, and $f(3) - f(2)$ is -1, and $f(2) - f(1)$ is $\frac{1}{2}(\frac{2}{3})(2) - \frac{1}{2}(\frac{1}{3})(1)$. Total $\mathbf{-1}$. The graph of f_4 is x^2 to $x = 1$.

5.3 Summation Versus Integration (page 194)

Problems 1–8 offer practice with sigma notation. The summation limits $\sum_{k=1}^{5}$ are like the integration limits $\int_{x=a}^{b}$.

1. Write out the terms of $\sum_{k=1}^{5}(x + k) = S$.

   - Do not make any substitutions for x. Just let k go from 1 to 5 and add: $S = (x+1) + (x+2) + (x+3) + (x+4) + (x+5) = 5x + 15$.

2. Write out the first three terms and the last term of $S = \sum_{i=0}^{n}(-1)^i 2^{i+1}$.

   - $S = (-1)^0 2^{0+1} + (-1)^1 2^{1+1} + (-1)^2 2^{2+1} + \cdots + (-1)^n 2^{n+1} = 2 - 2^2 + 2^3 - \cdots + (-1)^n 2^{n+1}$.

3. Write this sum using sigma notation: $243 - 81 + 27 - 9 + 3 - 1$.

   - The first thing to notice is that the terms are alternating positive and negative. That means a factor of either $(-1)^i$ or $(-1)^{i-1}$. The first term 243 is positive. If we start the sum with $i = 1$, then $(-1)^{i-1}$ is the right choice. This gives a positive sign for $i = 1, 3, 5, \cdots$.

     Now look at the numbers 243, 81, 27, 9, 3, and 1. Notice the pattern $81 = \frac{243}{3}$ and $27 = \frac{243}{3^2}$. In general the "ith" term is $\frac{243}{3^{i-1}}$. So we can write $\sum_{i=1}^{6}(-1)^{i-1}\frac{243}{3^{i-1}} = \sum_{i=1}^{6} 243(-\frac{1}{3})^{i-1}$.
   - Second solution: Start the sum at $i = 0$. We'll use a new dummy variable j, to avoid confusion between the "old i" and the "new i." We want $j = 0$ when $i = 1$. *This means* $j = i - 1$. Since i goes from 1 to 6, j goes from 0 to 5. This gives $\sum_{j=0}^{5} 243(-\frac{1}{3})^j$.

5. Rewrite $\sum_{i=3}^{2n} a^i b^{i-3}$ so that the indexing starts at $j = 0$ instead of $i = 3$.

   - Changing the dummy variable will require changes in four places: $\sum_{[]}^{[]} a^{[]} b^{[]}$. Replace $i = 3$ with $j = 0$. This means $j = i - 3$ and $i = j + 3$. The indexing ends at $i = 2n$, which means $j = 2n - 3$. Also a^i becomes a^{j+3}. The factor b^{i-3} becomes b^j. The answer is $\sum_{j=0}^{2n-3} a^{j+3} b^j$.

6. Compute the sum $3 + 6 + 9 + \cdots + 3000 = S$.

   - Don't use your calculator! Use your head! Each term is a multiple of 3, so $S = 3(1 + 2 + \cdots + 1000)$. The part in parentheses is the sum of the first 1000 integers. This is $\frac{1}{2}(1000)(1001)$ by equation 2 on page 189. Then $S = 3[\frac{1}{2}(1000)(1001)] = 1,501,500$.

7. Compute the sum $S = \sum_{i=1}^{100}(i+50)^2$. Write out the first and last terms to get a feel for what's happening.

- The sum is $51^2 + 52^2 + \cdots + 149^2 + 150^2$. Equation 7 on page 191 gives the sum of the first n squares. Then

$$\begin{aligned} S &= [\text{sum of the first 150 squares}] - [\text{sum of the first 50 squares}] \\ &= [\tfrac{1}{3}(150)^3 + \tfrac{1}{2}(150)^2 + \tfrac{1}{6}(150)] - [\tfrac{1}{3}(50)^3 + \tfrac{1}{2}(50)^2 + \tfrac{1}{6}(50)] \\ &= 1136275 - 42925 = 1093350. \end{aligned}$$

8. Compute $S = \sum_{i=1}^{n}(\frac{1}{2i} - \frac{1}{2i+2})$. The answer is a function of n. What happens as $n \to \infty$?

- Writing the first few terms and the last term almost always makes the sum clearer. Take $i = 1, 2, 3$ to find $(\frac{1}{2} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{6}) + (\frac{1}{6} - \frac{1}{8})$. Regroup those terms so that $\frac{1}{4}$ cancels $-\frac{1}{4}$ and $\frac{1}{6}$ cancels $-\frac{1}{6}$:

$$S = \frac{1}{2} + (-\frac{1}{4} + \frac{1}{4}) + (-\frac{1}{6} + \frac{1}{6}) + \cdots + (-\frac{1}{2n} + \frac{1}{2n}) - \frac{1}{2n-2} = \frac{1}{2} - \frac{1}{2n-2}.$$

  This kind of series is called *"telescoping". It is like our sums of differences.* The middle terms collapse to leave $S = \frac{1}{2} - \frac{1}{2n-2}$. As $n \to \infty$ this approaches $\frac{1}{2}$. The sum of the infinite series is $\frac{1}{2}$.

9. (This is 5.4.19) Prove by induction that $f_n = 1 + 3 + \cdots + (2n-1) = n^2$.

In words, the sum of the first n odd integers is n^2. ***A proof by induction has two parts.*** First part: Check the formula for the first value of n. Since the first odd integer is $1 = 1^2$, this part is easy and f_1 is correct. The second part of the proof is to check that $f_n = f_{n-1} + v_n$. Then a formula that is correct for f_{n-1} remains correct at the next step. *When v_n is added, f_n is still correct:*

$$f_{n-1} + v_n = (n-1)^2 + (2n-1) = n^2 - 2n + 1 + 2n - 1 = n^2.$$

The induction proof is complete. The correctness for $n = 1$ leads to $n = 2$, $n = 3$, $\cdots$, and every n.

10. Find q in the formula $1^6 + 2^6 + 3^6 + \cdots + n^6 = qn^7 +$ correction.

- The correction will involve powers below 7. We are concerned here with the coefficient of n^7. The sum $1^6 + 2^6 + 3^6 + \cdots + n^6$ is like adding n rectangles of width 1 under the curve $y = x^6$. The sum is like the area under $y = x^6$ from 0 to n. This is given by the definite integral $\int_0^n x^6 dx$. Since $\frac{1}{7}x^7$ is an antiderivative for $v(x) = x^6$, the definite integral is $\frac{1}{7}(n)^7 - \frac{1}{7}(0)^7 = \frac{1}{7}n^7$. The coefficient q is $\frac{1}{7}$. (This is informal reasoning, not proof.)

11. (This is 5.4.23) Add $n = 400$ to the table for $S_n = 1 + \cdots + n$ and find the relative error E_n. Guess a formula for E_n.

- The question refers to the table on page 193. The sum of the first n integers, $S_n = 1 + 2 + \cdots + n$, is being approximated by the area $I_n = \int_0^n x\,dx$, which is under the line $y = x$ from $x = 0$ to $x = n$. The error D_n is the difference between the true sum and the integral. "Relative error" is the proportional error $E_n = D_n/I_n$.

  Now for specifics: The first 400 integers add to $\frac{1}{2}n(n+1) = \frac{1}{2}(400)(401) = 80,200$. The integral is

$$\begin{aligned} I_{400} &= \int_0^{400} x\,dx = \tfrac{1}{2}(400)^2 - \tfrac{1}{2}0^2 = 80,000 \\ D_{400} &= 80,200 - 80,000 = 200 \text{ and } E_{400} = \tfrac{200}{80,000} = \tfrac{1}{400}. \end{aligned}$$

The table shows $E_{100} = \frac{1}{100}$ and $E_{200} = \frac{1}{200}$. It seems that $E_n = \frac{1}{n}$.

Read-throughs and selected even-numbered solutions :

The Greek letter $\sum$ indicates summation. In $\sum_1^n v_j$ the dummy variable is **j**. The limits are **j = 1 and j = n**, so the first term is $\mathbf{v_1}$ and the last term is $\mathbf{v_n}$. When $v_j = j$ this sum equals $\mathbf{\frac{1}{2}n(n+1)}$. For $n = 100$ the leading term is $\mathbf{\frac{1}{2}100^2 = 5000}$. The correction term is $\mathbf{\frac{1}{2}n = 50}$. The leading term equals the integral of $v = x$ from 0 to 100, which is written $\int_0^{100} \mathbf{x\,dx}$. The sum is the total **area** of 100 rectangles. The correction term is the area between the **sloping line** and the **rectangles.**

The sum $\sum_{i=3}^6 i^2$ is the same as $\sum_{j=1}^4 \mathbf{(j+2)^2}$ and equals **86.** The sum $\sum_{i=4}^5 v_i$ is the same as $\sum_{\mathbf{i=0}}^{\mathbf{1}} v_{i+4}$ and equals $\mathbf{v_4 + v_5}$. For $f_n = \sum_{j=1}^n v_j$ the difference $f_n - f_{n-1}$ equals $\mathbf{v_n}$.

The formula for $1^2 + 2^2 + \cdots + n^2$ is $f_n = \mathbf{\frac{1}{6}n(n+1)(2n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n}$. To prove it by mathematical induction, check $f_1 = \mathbf{1}$ and check $f_n - f_{n-1} = \mathbf{n^2}$. The area under the parabola $v = x^2$ from $x = 0$ to $x = 9$ is $\mathbf{\frac{1}{3}9^3}$. This is close to the area of $\mathbf{9/\Delta x}$ rectangles of base Δx. The correction terms approach zero very **slowly.**

16 $\sum_{i=1}^n v_i = \sum_{j=0}^{n-1} \mathbf{v_{j+1}}$ and $\sum_{i=0}^6 i^2 = \sum_{i=2}^8 \mathbf{(i-2)^2}$.
20 $f_1 = \frac{1}{4}(1)^2(2)^2 = 1; f_n - f_{n-1} = \frac{1}{4}n^2(n+1)^2 - \frac{1}{4}(n-1)^2n^2 = \frac{1}{4}n^2(4n) = n^3$.
24 $S_{50} = 42925; I_{50} = 41666\frac{2}{3}; D_{50} = 1258\frac{1}{3}; E_{50} = 0.0302; E_n$ is approximately $\frac{1.5}{n}$ and exactly $\frac{1.5}{n} + \frac{1}{2n^2}$.
28 $xS = x + x^2 + x^3 + \cdots$ equals $S - 1$. Then $S = \frac{1}{1-x}$. If $x = 2$ the sums are $S = \infty$.
36 The rectangular area is $\Delta x \sum_{j=1}^{1/\Delta x} v((j-1)\Delta x)$ or $\Delta x \sum_{i=0}^{(1/\Delta x)-1} v(i\Delta x)$.

5.4 Indefinite Integrals and Substitutions (page 200)

All the integrals in this section are either direct applications of the known forms on page 196 or have substitutions which lead to those forms. The trick, which takes experience, is choosing the right u. As you gain skill, you will do many of these substitutions in your head. To start, write them out.

1. Find the indefinite integral $\int 2x\sqrt{x^2-5}\,dx$.

   - Take $u = x^2 - 5$. Then $\frac{du}{dx} = 2x$, or $du = 2x\,dx$. This is great because we can take out "$2x\,dx$" in the problem and substitute "du". The problem is now $\int u^{1/2}du$ which equals $\frac{2}{3}u^{3/2} + C$. The final step is to put x back. The answer is $\frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^2-5)^{3/2} + C$.

2. Find $\int \frac{\cos x}{\sin^4 x}dx$.

   - Choose $u = \sin x$ because then $\frac{du}{dx} = \cos x$. We can replace "$\cos x\,dx$" by "du". The problem is now $\int \frac{du}{u^4} = \int u^{-4}du = -\frac{1}{3}u^{-3} + C = -\frac{1}{3}(\sin x)^{-3} + C$.

3. (This is 5.4.14) Find $\int t^3\sqrt{1-t^2}dt$.

   - It's hard to know what substitution to make, but the expression under the radical is a good candidate. Let $u = 1-t^2$. Then $du = -2t\,dt$. Decompose the problem this way: $\int(t^2)(\sqrt{1-t^2})(t\,dt)$. Substitute

$t^2 = 1 - u, \sqrt{1-t^2} = u^{1/2}$ and $t\ dt = -\frac{1}{2}du$. The problem is now

$$-\frac{1}{2}\int (1-u)u^{1/2}du = -\frac{1}{2}\int (u^{1/2} - u^{3/2})du = -\frac{1}{2}\int u^{1/2}du + \frac{1}{2}\int u^{3/2}du.$$

- The last step used the linearity property of integrals.
- The solution is $-\frac{1}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C = -\frac{1}{3}(1-t^2)^{3/2} + \frac{1}{5}(1-t^2)^{5/2} + C$.

4. Find $\int \frac{dx}{\sqrt{1-4x^2}}$. Choosing $u = 1 - 4x^2$ fails because $du = -8x\ dx$ is not present in the problem.

   - Mentally sifting through the "known" integrals, we remember $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$. The original problem $\int \frac{dx}{\sqrt{1-4x^2}}$ has this form if $u = 2x$. Next step: $du = 2\ dx$. Replace "dx" with "$\frac{1}{2}du$":

$$\int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{\frac{1}{2}du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} 2x + C.$$

5. Find a function $y(x)$ that solves the differential equation $\frac{dy}{dx} = \sqrt{xy}$. (This is 5.4.26)

   - Here are two methods. The first is "guess and check," the second is "separation of variables."

   "Guess and check" We guess $y = cx^n$ and try to figure out c and n. Since $\frac{dy}{dx} = ncx^{n-1}$ and we are given $\frac{dy}{dx} = \sqrt{xy}$, we must have $ncx^{n-1} = x^{1/2}(cx^n)^{1/2}$. The power of x on the left is $n-1$ and the power on the right is $\frac{1}{2} + \frac{n}{2}$. They are equal for $n = 3$. The left side is now $3cx^2$ and the right is $c^{1/2}x^2$. This means $3c = c^{1/2}$ and $c = \frac{1}{9}$. The method gives $y = \frac{1}{9}x^3$ and we check $\frac{dy}{dx} = \sqrt{xy}$.

   "Separation of variables" Starting with $\frac{dy}{dx} = x^{1/2}y^{1/2}$, first get x and dx on one side of the equal sign and y and dy on the other: $\boldsymbol{y^{-1/2}\ dy = x^{1/2}dx}$. Now integrate both sides: $2y^{1/2} + C_1 = \frac{2}{3}x^{3/2} + C_2$. Divide by 2 and let "$C$" replace the awkward constant $\frac{C_2}{2} - \frac{C_1}{2}$. Then $y^{1/2} = \frac{1}{3}x^{3/2} + C$, and $y = (\frac{1}{3}x^{3/2} + C)^2$. (Again, check $\frac{dy}{dx} = \sqrt{xy}$.)

   Let $C = 0$ and you get the answer $\frac{1}{9}x^3$ obtained by "guess and check." The former is just one solution; the latter is a whole class of solutions, one for every C.

6. Solve the differential equation $\frac{d^2y}{dx^2} = x$.

   The second derivative is x, so the first derivative must be $\frac{1}{2}x^2 + C_1$. The reason for writing C_1 instead of C becomes clear at the next step: Integrate $\frac{dy}{dx} = \frac{1}{2}x^2 + C_1$ to get $y = \frac{1}{6}x^3 + C_1x + C_2$. Each integration gives another "constant of integration."

   The functions $y = \frac{1}{6}x^3, y = \frac{1}{6}x^3 - x + 27$, and $y = \frac{1}{6}x^3 - 8$ are all solutions to $y'' = x$.

Read-throughs and selected even-numbered solutions :

Finding integrals by substitution is the reverse of the **chain** rule. The derivative of $(\sin x)^3$ is $\mathbf{3(\sin x)^2\cos x}$. Therefore the antiderivative of $\mathbf{3(\sin x)^2\cos x}$ is $\mathbf{(\sin x)^3}$. To compute $\int(1+\sin x)^2\cos x\ dx$, substitute $u = \mathbf{1 + \sin x}$. Then $du/dx = \mathbf{\cos x}$ so substitute $du = \mathbf{\cos x\ dx}$. In terms of u the integral is $\int \mathbf{u^2\ du} = \mathbf{\frac{1}{3}u^3}$. Returning to x gives the final answer.

The best substitutions for $\int \tan(x+3)\sec^2(x+3)dx$ and $\int(x^2+1)^{10}x\ dx$ are $u = \mathbf{\tan(x+3)}$ and $u = \mathbf{x^2+1}$. Then $du = \mathbf{\sec^2(x+3)dx}$ and $\mathbf{2x\ dx}$. The answers are $\mathbf{\frac{1}{2}\tan^2(x+3)}$ **and** $\mathbf{\frac{1}{22}(x^2+1)^{11}}$. The antiderivative of $v\, dv/dx$ is $\mathbf{\frac{1}{2}v^2}$. $\int 2x\, dx/(1+x^2)$ leads to $\int \frac{\mathbf{du}}{\mathbf{u}}$, which we don't yet know. The integral $\int dx/(1+x^2)$ is known immediately as $\mathbf{\tan^{-1}x}$.

6 $\frac{-2}{9}(1-3x)^{3/2}+C$ **10** $\cos^3 x \sin 2x$ equals $2\cos^4 x \sin x$ and its integral is $\frac{-2}{5}\cos^5 x + C$
16 The integral of $x^{1/2}+x^2$ is $\frac{2}{3}x^{3/2}+\frac{1}{3}x^3+C$.
20 Write $\sin^3 x$ as $(1-\cos^2 x)\sin x$. The integrals of $-\cos^2 x \sin x$ and $\sin x$ give $\frac{1}{3}\cos^3 x - \cos x + C$.
26 $dy/dx = x/y$ gives $y\, dy = x\, dx$ or $y^2 = x^2 + C$ or $y = \sqrt{x^2+C}$.
28 $y = \frac{1}{120}x^5 + C_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5$
34 (a) **False:** The derivative of $\frac{1}{2}f^2(x)$ is $f(x)\frac{df}{dx}$ (b) **True:** The chain rule gives $\frac{d}{dx}f(v(x)) = \frac{df}{dx}(v(x))$ times $\frac{dv}{dx}$ (c) **False:** These are inverse *operations* not inverse functions and (d) is **True.**

5.5 The Definite Integral (page 205)

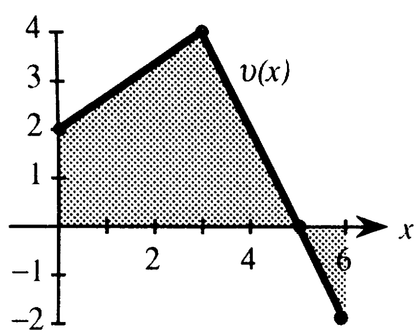
1. Make a sketch to show the meaning of $\int_1^3 \sqrt{x}\, dx$. Then find the value.

   - $\int_1^3 \sqrt{x}\, dx$ is the area under the curve $y=\sqrt{x}$ from $x=1$ to $x=3$. Since $f(x)=\frac{2}{3}x^{3/2}$ is an antiderivative for $v(x)=\sqrt{x}$, the definite integral is

$$\int_1^3 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}]_1^3 = \frac{2}{3}(3)^{3/2} - \frac{2}{3}(1)^{3/2} = 2\sqrt{3} - \frac{2}{3} \approx 2.797.$$

2. Use integral notation to describe the area between $y=\sqrt{x}$ and $y=0$ and $x=2$ and $x=4$.

   - The area under $y=\sqrt{x}$ from $x=2$ to $x=4$ is $\int_2^4 \sqrt{x}\, dx$. Don't leave off the dx!



Quick question
Which x gives the maximum area $f(x)$?
$f(x)$ is a maximum when $v(x) = \frac{df}{dx} =$ ____.

3. The function $f(x) = \int_0^x v(t)dt$ gives the area under the curve $y=v(t)$ from $t=0$ to $t=x$. From the graph of v, find $f(3), f(5), f(6)$, and $f(0)$. Note: This is not a graph of $f(x)$! The function $f(x)$ is the *area* under this graph.

   - You could write a formula for $f(x)$ but it's just as easy to read off the areas. $f(3)$ is the trapezoid area with base 3: $f(3) = \frac{1}{2}(2+4)3 = 9$. To find $f(5)$, add the area of the triangle with base 2: $f(5) = 9+\frac{1}{2}(4)(2) = 13$. The next triangle has area 1 but it is below the x axis. This area is negative so that $f(6) = f(5) - 1 = 12$. Finally $f(0) = 0$ since the area from $t=0$ to $t=0$ is zero!

   Problems 4 and 5 will give practice with definite integrals – especially changes of limits.

4. $\int_0^{\pi/4} \sec^3 x \tan x \, dx = \int_?^? u^2 \, du$ (with $u = \sec x, du = \sec x \tan x \, dx$).

   - The lower limit is the value of u when $x = 0$ (the original lower limit). Thus $u(0) = \sec\ 0 = 1$. The upper limit is the value of u when $x = \frac{\pi}{4}$ (the original upper limit). This is $\sec \frac{\pi}{4} = \sqrt{2}$.
   - Now $\int_1^{\sqrt{2}} u^2 du = \frac{1}{3}u^3]_1^{\sqrt{2}} = \frac{2\sqrt{2}}{3} - \frac{1}{3} \approx 0.609$.

5. $\int_{-2}^{0} x^5\sqrt{1-x^3} \, dx = \int_?^? (1-u)\sqrt{u} \, \frac{du}{-3}$ (with $u = 1 - x^3$).

   - We replaced x^3 with $1-u$ and x^2dx with $-\frac{1}{3}du$. The lower limit is $u(-2) = 1-(-2)^3 = 9$ and the upper limit is $u(0) = 1$. The integral is

$$-\frac{1}{3}\int_9^1 (u^{1/2} - u^{3/2})du = -\frac{1}{3}(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2})]_9^1 = -\frac{1}{3}((\frac{2}{3}-\frac{2}{5}) - (\frac{2}{3}\cdot 27 - \frac{2}{5}\cdot 243)) \approx -26.5.$$

Problems 6-7 help flesh out the idea of the maximum M_k, minimum m_k, and sums S and s.

6. Find the maximum and minimum of $y = \sin \pi x$ in the intervals from 0 to $\frac{1}{4}$, $\frac{1}{4}$ to $\frac{1}{2}$, $\frac{1}{2}$ to $\frac{3}{4}$, $\frac{3}{4}$ to 1. Use these M's and m's in the upper sum S and lower sum s enclosing $\int_0^1 \sin \pi x \, dx$.

   - The maximum values M in the four intervals are $\frac{\sqrt{2}}{2}$, 1, 1, $\frac{\sqrt{2}}{2}$. The upper sum with $\Delta x = \frac{1}{4}$ is $S = \frac{1}{4}(\frac{\sqrt{2}}{2} + 1 + 1 + \frac{\sqrt{2}}{2}) \approx .85$. The minimum values m_k are $0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$. The lower sum is $s = \frac{1}{4}(0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0) \approx .35$. The integral is between .35 and .85. (It is exactly $[-\frac{\cos \pi x}{\pi}]_0^1 = \frac{2}{\pi} \approx .64$.)

7. Repeat Problem 6 with $\Delta x = \frac{1}{8}$ and eight intervals. Compare the answers.

   - Adding $\frac{1}{8}$ times the maximum values M_k gives $S \approx .75$. Adding $\frac{1}{8}$ times the minimum values m_k gives $s \approx .50$. As expected, s has grown and S is smaller. The correct value .64 is still in between.

Read-throughs and selected even-numbered solutions :

If $\int_a^x v(x)dx = f(x) + C$, the constant C equals $-\mathbf{f(a)}$. Then at $x = a$ the integral is **zero**. At $x = b$ the integral becomes $\mathbf{f(b) - f(a)}$. The notation $f(x)]_a^b$ means $\mathbf{f(b) - f(a)}$. Thus $\cos x]_0^\pi$ equals $\mathbf{-2}$. Also $[\cos x + 3]_0^\pi$ equals $\mathbf{-2}$, which shows why the antiderivative includes an arbitrary **constant**. Substituting $u = 2x-1$ changes $\int_1^3 \sqrt{2x-1} \, dx$ into $\int_\mathbf{1}^\mathbf{5} \frac{\mathbf{1}}{\mathbf{2}}\sqrt{\mathbf{u}} \, \mathbf{du}$ (with limits on u).

The integral $\int_a^b v(x)dx$ can be defined for any **continuous** function $v(x)$, even if we can't find a simple **antiderivative.** First the meshpoints $x_1, x_2, \cdots$ divide $[a, b]$ into subintervals of length $\Delta x_k = \mathbf{x_k - x_{k-1}}$. The upper rectangle with base Δx_k has height M_k = **maximum of v(x) in interval k.**
The upper sum S is equal to $\mathbf{\Delta x_1 M_1 + \Delta x_2 M_2} + \cdots$. The lower sum s is $\mathbf{\Delta x_1 m_1 + \Delta x_2 m_2} + \cdots$. The **area** is between s and S. As more meshpoints are added, S **decreases** and s **increases.** If S and s approach the same **limit,** that defines the integral. The intermediate sums S^*, named after **Riemann,** use rectangles of height $v(x_k^*)$. Here x_k^* is any point between $\mathbf{x_{k-1}}$ **and** $\mathbf{x_k}$, and $S^* = \sum \mathbf{\Delta x_k v(x_k^*)}$ approaches the area.

4 $C = -f(\sin \frac{\pi}{2}) = -\mathbf{f(1)}$ so that $\int v(u)du = f(u) + C$
6 $C = \mathbf{0}$. No constant in the derivative!

10 Set $x = 2t$ and $dx = 2dt$. Then $\int_{x=0}^{2} v(x)dx = \int_{t=0}^{1} v(2t)(2dt)$ so $C = \mathbf{2}$.

16 Choose $u = x^2$ with $du = 2x\,dx$ and $u = 0$ at $x = 0$ and $u = 1$ at $x = 1$. Then $\int_0^1 \frac{du}{2\sqrt{1-u}} = -\sqrt{1-u}]_0^1 = \mathbf{+1}$. (Could also choose $u = 1 - x^2$.)

22 Maximum of x in the four intervals is: $M_k = -\frac{1}{2}, 0, \frac{1}{2}, 1$. Minimum is $m_k = -1, -\frac{1}{2}, 0, \frac{1}{2}$. Then $S = \frac{1}{2}(-\frac{1}{2} + 0 + \frac{1}{2} + 1) = \mathbf{\frac{1}{2}}$ and $s = \frac{1}{2}(-1 - \frac{1}{2} + 0 + \frac{1}{2}) = -\mathbf{\frac{1}{2}}$.

5.6 Properties of the Integral and Average Value (page 212)

Properties 1–7 are elegantly explained by the accompanying figures (5.11 and 5.12). Focus on them.

1. Explain how property 4 applies to $\int_{-\pi/3}^{\pi/3} \sec x\, dx$. Note that $\sec(-x) = \sec x$.

2. Explain how property 4 applies to $\int_{-\sqrt{3}}^{\sqrt{3}} \tan^{-1} x\, dx$. Note that $\tan^{-1}(-x) = -\tan^{-1} x$.

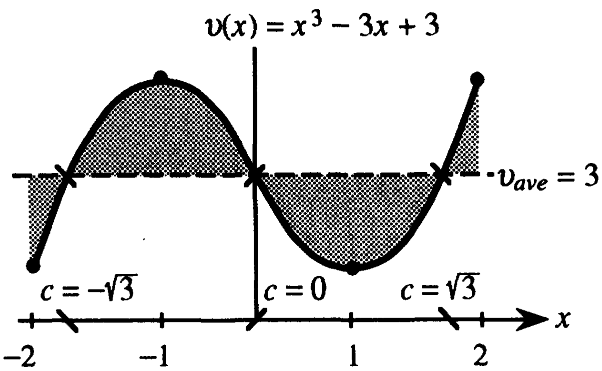
   - Solution to Problem 1: Since $y = \sec x$ is an ***even*** function, the area on the left of the y axis equals the area on the right. Property 4 states that $\int_{-\pi/3}^{\pi/3} \sec x\, dx = 2\int_0^{\pi/3} \sec x\, dx$.
   - Solution to Problem 2: Since $y = \tan^{-1} x$ is an ***odd*** function, the area on the left is equal in absolute value to the area on the right. One area is negative while the other is positive. Property 4 says that $\int_{-\sqrt{3}}^{\sqrt{3}} \tan^{-1} x\, dx = 0$.

3. For the function $y = x^3 - 3x + 3$, compute the average value on $[-2, 2]$. Then draw a sketch like those in Figure 5.13 showing the area divided into two equal portions by a horizontal line at the average value.

   - Equation 3 on page 208 gives average value as $\frac{1}{b-a}\int_a^b v(x)dx$. Since the interval is $[-2, 2]$ we have $a = -2$ and $b = 2$. The function we're averaging is $v(x) = x^3 - 3x + 3$. The average value is

$$\frac{1}{2-(-2)}\int_{-2}^{2}(x^3 - 3x + 3)dx = \frac{1}{4}[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 3x]_{-2}^{2} = \frac{1}{4}(4+8) = 3.$$

   - ***Quick reason*** : x^3 and $3x$ are *odd functions* (average zero). The average of 3 is 3.



In this sketch it is easy to see that the two shaded areas above the dotted line fit into the shaded areas below the line. The area under the curve equals that of a rectangle whose height is the average value.

4. (Refer to Problem 3) The Mean Value Theorem says that the function $v(x) = x^3 - 3x + 3$ actually takes on its average value somewhere in the interval $(-2, 2)$. Find the value(s) c for which $v(c) = v_{ave}$.

   - Question 3 found $v_{ave} = 3$. Question 4 is asking where the graph crosses the dotted line. Solve $v(x) = v_{ave}$. There are three solutions (three c's):
   - $x^3 - 3x + 3 = 3$ or $x^3 - 3x = 0$ or $x(x^2 - 3) = 0$. Then $c = 0$ or $c = \pm\sqrt{3}$.

5. (This is 5.6.25) Find the average distance from $x = a$ to points in the interval $0 \le x \le 2$.

   The full solution depends on where a is. First assume $a > 2$. The distance to x is $a - x$. The *average* distance is $\frac{1}{2-0}\int_0^2 (a - x)dx = a - 1$. This is the distance to the middle point $x = 1$.

   If $a < 0$, the distance to x is $x - a$ and the average distance is $\frac{1}{2-0}\int_0^2 (x - a)dx = 1 - a$.

   Now suppose a lies in the interval $[0, 2]$. The distance $|x - a|$ is never negative. This is best considered in two sections, $x < a$ and $x > a$. The average of $(x - a)$ is

$$\frac{1}{2-0}\int_0^2 |x-a|dx = \frac{1}{2}\left(\int_0^a (a-x)dx + \int_a^2 (x-a)dx\right) = 1 - a + \frac{a^2}{2}.$$

6. A point P is chosen randomly along a semicircle (equal probability for equal arcs). What is the average distance y from the x axis? The radius is 1. (This is 5.6.27, drawn on page 212)

   - Since the problem states "equal probability for equal arcs" we imagine the semicircle divided by evenly spaced radii (like equal slices of pie). The central angle of each arc is $\Delta\theta$, and the total angle is π. The probability of a particular wedge being chosen is $\frac{\Delta\theta}{\pi}$. A point P on the unit circle has height $y = \sin\theta$.

     For a finite number of arcs, the average distance would be $\sum \sin\theta \frac{\Delta\theta}{\pi}$. As we let $\Delta\theta \to 0$, the sum becomes an integral ($\Delta\theta$ becomes $d\theta$). The average distance is $\int_0^\pi \sin\theta \frac{d\theta}{\pi} = \frac{1}{\pi}(-\cos\theta)]_0^\pi = \frac{2}{\pi}$.

Read-throughs and selected even-numbered solutions :

The integrals $\int_0^b v(x)dx$ and $\int_b^5 v(x)dx$ add to $\mathbf{\int_0^5 v(x)dx}$. The integral $\int_3^1 v(x)\,dx$ equals $-\mathbf{\int_1^3 v(x)dx}$. The reason is **that the steps Δx are negative**. If $v(x) \le x$ then $\int_0^1 v(x)\,dx \le \mathbf{\frac{1}{2}}$. The average value of $v(x)$ on the interval $1 \le x \le 9$ is defined by $\mathbf{\frac{1}{8}\int_1^9 v(x)\,dx}$. It is equal to $v(c)$ at a point $x = c$ which is **between 1 and 9.** The rectangle across the interval with height $v(c)$ has the same area as **the region under v(x).** The average value of $v(x) = x + 1$ on the interval $1 \le x \le 9$ is **6.**

If x is chosen from 1,3,5,7 with equal probabilities $\frac{1}{4}$, its expected value (average) is **4.** The expected value of x^2 is **21.** If x is chosen from $1, 2, \cdots, 8$ with probabilities $\frac{1}{8}$, its expected value is **4.5.** If x is chosen from $1 \le x \le 9$, the chance of hitting an integer is **zero.** The chance of falling between x and $x + dx$ is $p(x)dx = \mathbf{\frac{1}{8}dx}$. The expected value $E(x)$ is the integral $\mathbf{\int_1^9 \frac{x}{8}\,dx}$. It equals **5.**

6 $v_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\sin x)^9 dx = \mathbf{0}$ (odd function over symmetric interval $-\pi$ to π). This equals $(\sin c)^9$ at $c = -\pi$ and **0** and π.

8 $2\int_1^5 x\,dx = x^2]_1^5 = \mathbf{24}$. Remember to reverse sign in the integral from 5 to 1.

20 Property 6 proves both (a) and (b) because $v(x) \le |v(x)|$ and also $-v(x) \le |v(x)|$. So their integrals maintain these inequalities.

32 The square has area 1. The area under $y = \sqrt{x}$ is $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}$.

38 Average size is $\frac{G}{n}$. The chance of an individual belonging to group 1 is $\frac{x_1}{G}$. The expected size is sum of size times probability: $E(x) = \sum \frac{x_i^2}{G}$. This exceeds $\frac{G}{n}$ by the Schwarz inequality: $(1x_1 + \cdots + 1x_n)^2 \le (1^2 + \cdots + 1^2)(x_1^2 + \cdots + x_n^2)$ is the same as $G^2 \le n \sum x_i^2$.

40 This formula for $f(x)$ jumps from 9 to -9. The correct formula (with continuous f) is x^2 then $18 - x^2$. Then $f(4) - f(0) = \mathbf{2}$, which is $\int_0^4 v(x)dx$.

5.7 The Fundamental Theorem and Its Applications (page 219)

The Fundamental Theorem is almost anticlimactic at this point! It has all been built into the theory since page 1. The applications are where most of the challenge and interest lie. You must see the difference between Parts 1 and 2 – derivative of integral and then integral of derivative. In Problems 1-5, find $F'(x)$.

1. $F(x) = \int_{\pi/4}^{x} \sin 4t\,dt$. This is a direct application of the Fundamental Theorem (Part 1 on page 214). Here $v(t) = \sin 4t$. The theorem says that $F'(x) = \sin 4x$. As a check, pretend you don't believe the Fundamental Theorem. Since $-\frac{1}{4}\cos 4t$ is an antiderivative of $\sin 4t$, the integral of v is $F(x) = -\frac{1}{4}\cos 4x + \frac{1}{4}\cos \pi$. The derivative of F is $F'(x) = \sin 4x$. *The integral of $v(t)$ from a to x has derivative $v(x)$.*

2. $F(x) = \int_x^5 \frac{1}{t}dt$. Now x is the lower limit rather than the upper limit. We expect a minus sign.

   - To "fix" this write $F(x) = -\int_5^x \frac{1}{t}dt$. The Fundamental Theorem gives $F'(x) = -\frac{1}{x}$. Notice that *we do not have to know this integral $F(x)$, in order to know that its derivative is $-\frac{1}{x}$.*

3. $F(x) = \int_2^{6x} t^3 dt$. The derivative of $F(x)$ is ***not*** $(6x)^3$.

   - When the limit of integration is a function of x, we need the chain rule. Set $u = 6x$ and $F(x) = \int_2^u t^3 dt$. $\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = u^3 \cdot 6 = (6x)^3 \cdot 6$. *There is an extra 6 from the derivative of the limit!*

4. $F(x) = \int_{2x}^{8x} \sin^{-1} u\,du$. Both limits $a(x)$ and $b(x)$ depend on x.

   - Equation (6) on page 215 is $F'(x) = v(b(x))\frac{db}{dx} - v(a(x))\frac{da}{dx}$. The chain rule applies at both of the limits $b(x) = 8x$ and $a(x) = 2x$. Don't let it bother you that the dummy variable is u and not t. The rule gives $F'(x) = 8\sin^{-1} 8x - 2\sin^{-1} 2x$. The extra 8 and 2 are from derivatives of limits.

5. $F(x) = \frac{1}{x}\int_0^x \tan t\,dt$.

   - $F(x)$ is a product, so this calls for the product rule. (In fact $F(x)$ is the average value of $\tan x$.) Its derivative is $F'(x) = \frac{1}{x}\tan x - \frac{1}{x^2}\int_0^x \tan t\,dt$.

Problems 6 and 7 ask you to set up integrals involving regions other than vertical rectangles.

6. Find the area A bounded by the curve $x = 4 - y^2$, and the two axes.

- Think of the area being approximated by thin *horizontal* rectangles. Each rectangle has *height* Δy and length x. The last rectangle is at $y = 2$ (where the region ends).

  The area is $A \approx \sum x\ \Delta y$, where the sum is taken over the $\frac{2}{\Delta y}$ rectangles. As we let the number of rectangles become infinite, the sum becomes an integral. $A \approx \sum x\ \Delta y$ becomes $A = \int_0^2 x\ dy$. The $\approx$ has become $=$, the $\sum$ has become $\int$, and Δy has become dy. The limits of integration, 0 and 2, are the lowest and highest values of y in the area.

The antiderivative is not $\frac{1}{2}x^2$ because we have "dy" and not "dx." Before integrating we must replace x by $4 - y^2$. Then $A = \int_0^2 (4 - y^2)dy = [4y - \frac{1}{3}y^3]_0^2 = \frac{16}{3}$.

7. Find the volume V of a cone of radius 3 feet and height 6 feet.

- Think of the cone being approximated by a stack of thin disks. Each disk (a very short cylinder) has volume $\pi r^2 \Delta h$. The disk radius decreases with h to approximate the cone. The total volume of the disks is $\sum \pi r^2 \Delta h \approx V$.

  Increasing the number of disks to infinity gives $V = \int_0^6 \pi r^2 dh$. The height variable h goes from 0 to 6. Before integrating we must replace r^2 with an expression using h. By similar triangles, $\frac{r}{h} = \frac{3}{6}$ and $r = \frac{1}{2}h$. Then $V = \int_0^6 \pi(\frac{h}{2})^2 dh = \frac{\pi}{4}[\frac{1}{3}h^3]_0^6 = 18\pi$ cubic feet.

Read-throughs and selected even-numbered solutions :

The area $f(x) = \int_a^x v(t)\ dt$ is a function of **x**. By Part **1** of the Fundamental Theorem, its derivative is $\mathbf{v(x)}$. In the proof, a small change Δx produces the area of a thin **rectangle.** This area Δf is approximately $\mathbf{\Delta x}$ times $\mathbf{v(x)}$. So the derivative of $\int_a^x t^2\ dt$ is $\mathbf{x^2}$.

The integral $\int_x^b t^2\ dt$ has derivative $\mathbf{-x^2}$. The minus sign is because **x is the lower limit.** When both limits $a(x)$ and $b(x)$ depend on x, the formula for df/dx becomes $\mathbf{v(b(x))\frac{db}{dx}}$ minus $\mathbf{v(a(x))\frac{da}{dx}}$. In the example $\int_2^{3x} t\ dt$, the derivative is $\mathbf{9x}$.

By Part **2** of the Fundamental Theorem, the integral of df/dx is $\mathbf{f(x) + C}$. In the special case when $df/dx = 0$, this says that **the integral is constant.** From this special case we conclude: If $dA/dx = dB/dx$ then $A(x) = \mathbf{B(x) + C}$. If an antiderivative of $1/x$ is $\ln x$ (whatever that is), then automatically $\int_a^b dx/x = \mathbf{\ln b - \ln a}$.

The square $0 \le x \le s, 0 \le y \le s$ has area $A = \mathbf{s^2}$. If s is increased by Δs, the extra area has the shape of **an L.** That area ΔA is approximately $\mathbf{2s\ \Delta s}$. So $dA/ds = \mathbf{2s}$.

6 $\frac{d}{dx}\int_{-x}^{x/2} v(u)du = \frac{1}{2}v(\frac{x}{2}) - (-1)v(-x) = \mathbf{\frac{1}{2}v(\frac{x}{2}) + v(-x)}$

10 $\frac{d}{dx}(\frac{1}{2}\int_x^{x+2} x^3 dx) = \mathbf{\frac{1}{2}(x+2)^3 - \frac{1}{2}x^3}$ **18** $\frac{d}{dx}\int_{a(x)}^{b(x)} 5dt = 5\frac{db}{dx} - 5\frac{da}{dx}$.

30 When the side s is increased, only *two* strips are added to the square (on the right side and top). So $dA = 2s\ ds$ which agrees with $A = s^2$.

34 $\int x\ dy = \int_0^1 \sqrt{y}\ dy = \frac{2}{3}y^{3/2}]_0^1 = \frac{2}{3}$.

38 The triangle ends at the line $x + y = 1$ or $r\cos\theta + r\sin\theta = 1$. The area is $\frac{1}{2}$, by geometry. So the area integral $\int_{\theta=0}^{\pi/2} \frac{1}{2}r^2 d\theta = \frac{1}{2}$: Substitute $r = \frac{1}{\cos\theta + \sin\theta}$.

40 Rings have area $2\pi r\ dr$, and $\int_2^3 2\pi r\ dr = \pi r^2]_2^3 = 5\pi$. Strips are difficult because they go in and out of the ring (see Figure 14.5b on page 528).

5.8 Numerical Integration (page 226)

1. Estimate $\int_0^1 \sin\frac{\pi x}{2}dx$ using each method: **L, R, M, T, S**. Take $n = 4$ intervals of length $\Delta x = \frac{1}{4}$.

- The intervals end at $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The values of $y(x) = \sin\frac{\pi x}{2}$ at those endpoints are $y_0 = \sin 0 = 0, y_1 = \sin\frac{\pi}{8} = .3827, y_2 = .7071, y_3 = .9239$, and $y_4 = \sin\frac{\pi}{2} = 1$. Now for the five methods:

$$\begin{aligned}\text{Left rectangular area} \quad L_4 &= \Delta x(y_0 + \cdots + y_3) = \tfrac{1}{4}(0 + .3827 + .7071 + .9239) = .5034.\\ \text{Right rectangular area} \quad R_4 &= \tfrac{1}{4}(.3827 + .7071 + .9239 + 1) = .7534.\\ \text{Trapezoidal rule} \quad T_4 &= \tfrac{1}{2}R_4 + \tfrac{1}{2}L_4 = \tfrac{1}{4}(\tfrac{1}{2}\cdot 0 + .3827 + .7071 + 0.9239 + \tfrac{1}{2}\cdot 1) = 0.6284.\end{aligned}$$

To apply the midpoint rule, we need the values of $y = \sin\frac{\pi x}{2}$ at the midpoints $x = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}$, and $\frac{7}{8}$. The first midpoint $\frac{1}{8}$ is halfway between 0 and $\frac{1}{4}$, and the calculator gives $y_{1/2} = \sin(\frac{\pi}{2}(\frac{1}{8})) = 0.1951$. The other midpoint values are $y_{3/2} = \sin(\frac{\pi}{2}(\frac{3}{8})) = 0.5556, y_{5/2} = 0.8315, y_{7/2} = 0.9808$. Now we're ready to find M_4:

$$M_4 = \Delta x(y_{1/2} + y_{3/2} + y_{5/2} + y_{7/2}) = \frac{1}{4}(0.1951 + 0.5556 + 0.8315 + 0.9808) = 0.6408.$$

Simpson's rule gives S_4 as $\frac{1}{3}T_4 + \frac{2}{3}M_4$ or directly from the $1-4-2-4-2-4-2-4-1$ pattern:

$$\begin{aligned}S_4 &= \tfrac{1}{6}\Delta x[y_0 + 4y_{1/2} + 2y_1 + 4y_{3/2} + 2y_2 + 4y_{5/2} + 2y_3 + 4y_{7/2} + y_4]\\ &= \tfrac{1}{6}(\tfrac{1}{4})[0 + 4(0.1951) + 2(0.3827) + 4(0.5556) + 2(0.7071)\\ &\quad +4(0.8315) + 2(0.9239) + 4(0.9808) + 1] = 0.6366.\end{aligned}$$

2. Find $\int_0^1 \sin\frac{\pi x}{2}dx$ by integration. Give the predicted and actual errors for each method in Problem 1.

- The integral is $\int_0^1 \sin\frac{\pi x}{2}dx = \frac{2}{\pi} \approx 0.6366$ (accurate to 4 places). The predicted error for R_4 is $\frac{1}{2}\Delta x(y_n - y_0) = \frac{1}{2}(\frac{1}{4})(1-0) = 0.125$. The actual error is $0.7534 - 0.6366 = 0.1168$. The predicted error for L_4 is -0.125 (*just change sign!*) and the actual error is -0.1322.

  For T_4, the predicted error is $\frac{1}{12}(\Delta x)^2(y'(b) - y'(a)) = \frac{1}{12}(\frac{1}{4})^2(0 - \frac{\pi}{2}) \approx -0.0082$. The factor $\frac{\pi}{2}$ is $y' = \frac{\pi}{2}\cos\frac{\pi x}{2}$ at $x = 0$. *This prediction is excellent:* $0.6284 - 0.6366 = -0.0082$.

  The predicted error for M_4 is $-\frac{1}{24}(\Delta x)^2[y'(b) - y'(a)]$. This is half the predicted error for T_4, or 0.0041. The actual error is $0.6408 - 0.6366 = 0.0042$.

For Simpson's rule S_4, the predicted error in Figure 5.20 is $\frac{1}{2880}(\Delta x)^4(y'''(b) - y'''(a)) = \frac{1}{2880}(\frac{1}{4})^4(-\frac{\pi^3}{2^3}) = -0.000021$. That is to say, Simpson's error is predicted to be 0 to four places. The actual error to four places is $0.6366 - 0.6366 = 0.0000$.

3. (This is 5.8.6) Compute π to 6 places as $4\int_0^1 \frac{dx}{1+x^2}$ using any rule for numerical integration.

- Check first that $4\int_0^1 \frac{dx}{1+x^2} = 4\tan^{-1}x]_0^1 = 4(\frac{\pi}{4}) = \pi$. To obtain 6-place accuracy, we want error $< 5\times 10^{-7}$. Examine the error predictions for R_n, T_n, and S_n to see which Δx will be necessary.

  The predicted error for R_n is $\frac{1}{2}\Delta x[y(b) - y(a)] = \frac{1}{2}(\Delta x)[\frac{4}{2} - \frac{4}{1}] = -\Delta x$. This is $-\frac{1}{n}$, since $\Delta x = \frac{1}{n}$ for an interval of length 1 cut into n pieces. Ignore the negative sign since the size of the error is what interests us. To obtain $\frac{1}{n} < 5\cdot 10^{-7}$ we need $n > \frac{1}{5}\cdot 10^7$. Over 2 million rectangles will be necessary!

Go up an order of accuracy to T_n. Its error is estimated by $\frac{1}{12}(\Delta x)^2(y'(1) - y'(0)) = \frac{1}{12}(\frac{1}{n})^2(-2)$. We want $\frac{1}{6n^2} < 5 \cdot 10^{-7}$ or $n^2 > \frac{1}{30} \cdot 10^7$. This means that $n > 577$ trapezoids will be necessary.

Simpson offers a better hope for reducing the work. Simpson's error is about $\frac{1}{2880}(\Delta x)^4(y'''(1) - y'''(0))$. Differentiate $\frac{4}{1+x^2}$ three times to find $y'''(1) = 12$ and $y'''(0) = 0$. Simpson's error is about $\frac{1}{240n^4}$. To make the calculation accurate to 6 places, we need $\frac{1}{240n^4} < 5 \times 10^{-7}$ or $n^4 > \frac{1}{1200} \times 10^7$ or $n > 9$. Much better than 2 million!

Read-throughs and selected even-numbered solutions :

To integrate $y(x)$, divide $[a, b]$ into n pieces of length $\Delta x = \mathbf{(b - a)/n}$. R_n and L_n place a **rectangle** over each piece, using the height at the right or **left** endpoint: $R_n = \Delta x(y_1 + \cdots + y_n)$ and $L_n = \mathbf{\Delta x(y_0 + \cdots + y_{n-1})}$. These are **first**-order methods, because they are incorrect for $y = \mathbf{x}$. The total error on $[0, 1]$ is approximately $\mathbf{\frac{\Delta x}{2}(y(1) - y(0))}$. For $y = \cos \pi x$ this leading term is $\mathbf{-\Delta x}$. For $y = \cos 2\pi x$ the error is very small because $[0, 1]$ is a complete **period.**

A much better method is $T_n = \frac{1}{2}R_n + \mathbf{\frac{1}{2}L_n} = \Delta x[\frac{1}{2}y_0 + \mathbf{1}y_1 + \cdots + \mathbf{\frac{1}{2}}y_n]$. This **trapezoidal** rule is **second**-order because the error for $y = x$ is **zero.** The error for $y = x^2$ from a to b is $\mathbf{\frac{1}{6}(\Delta x)^2(b - a)}$. The **midpoint** rule is twice as accurate, using $M_n = \Delta x[\mathbf{y_{\frac{1}{2}}} + \cdots + \mathbf{y_{n-\frac{1}{2}}}]$.

Simpson's method is $S_n = \frac{2}{3}M_n + \mathbf{\frac{1}{3}T_n}$. It is **fourth**-order, because the powers $\mathbf{1, x, x^2, x^3}$ are integrated correctly. The coefficients of $y_0, y_{1/2}, y_1$ are $\mathbf{\frac{1}{6}, \frac{4}{6}, \frac{1}{6}}$ times Δx. Over three intervals the weights are $\Delta x/6$ times $1 - \mathbf{4} - \mathbf{2} - \mathbf{4} - \mathbf{2} - \mathbf{4} - \mathbf{1}$. Gauss uses **two** points in each interval, separated by $\Delta x/\sqrt{3}$. For a method of order p the error is nearly proportional to $\mathbf{(\Delta x)^p}$.

2 The trapezoidal error has a factor $(\Delta x)^2$. It is reduced by 4 when Δx is cut in half. The error in Simpson's rule is proportional to $(\Delta x)^4$ and is reduced by 16.

8 The trapezoidal rule for $\int_0^{2\pi} \frac{dx}{3+\sin x} = \frac{\pi}{\sqrt{2}} = 2.221441$ gives $\frac{2\pi}{3} \approx 2.09$ (two intervals), $\frac{7\pi}{9} \approx 2.221$ (three intervals), $\frac{17\pi}{24} \approx 2.225$ (four intervals is worse??), and 7 digits for T_5. Curious that $M_n = T_n$ for odd n.

10 The midpoint rule is exact for 1 and x. For $y = x^2$ the integral from 0 to Δx is $\frac{1}{3}(\Delta x)^3$ and the rule gives $(\Delta x)(\frac{\Delta x}{2})^2$. This error $\frac{1}{4}(\Delta x)^3 - \frac{1}{3}(\Delta x)^3 = -\frac{1}{12}(\Delta x)^3$ does equal $-\frac{(\Delta x)^2}{24}(y'(\Delta x) - y'(0))$.

14 Correct answer $\frac{2}{3}$. $T_1 = .5, T_{10} \approx .66051, T_{100} \approx .66646$. $M_1 \approx .707, M_{10} \approx .66838, M_{100} \approx .66673$. What is the rate of decrease of the error?

18 The trapezoidal rule $T_4 = \frac{\pi}{8}(\frac{1}{2} + \cos^2 \frac{\pi}{8} + \cos^2 \frac{\pi}{4} + \cos^2 \frac{3\pi}{8} + 0)$ gives the correct answer $\frac{\pi}{4}$.

22 Any of these stopping points should give the integral as 0.886227 $\cdots$ Extra correct digits depend on the computer design.

5 Chapter Review Problems

Review Problems

R1 Draw any up and down curve between $x = 0$ and $x = 3$, starting at $v(0) = -1$. Aligned below it draw the integral $f(x)$ that gives the area from 0 to x.

R2 Find the derivatives of $f(x) = \int_x^{2x} v(t)dt$ and $f(x) = \int_{1/x}^{1+x} v(t)dt$.

R3 Use sketches to illustrate these approximations to $\int_0^1 x^4\, dx$ with $\Delta x = 1$: Left rectangle, Right rectangle, Trapezoidal, Midpoint, and Simpson. Find the errors.

R4 What is the difference between a *definite* integral and an *indefinite* integral?

R5 Sketch a function $v(x)$ on $[a, b]$. Show graphically how the average value relates to $\int_a^b v(x)dx$.

R6 Show how a step function has no antiderivative but has a definite integral.

R7 Compute $\int_{-1}^{4}(5 + |x|)dx$ from a graph (and from the area of a triangle).

R8 1000 lottery tickets are sold. One person will win \$250, ten will win free \$1 tickets to the next lottery. The others lose. Is the expected value of a ticket equal to 26¢?

Drill Problems (Find an antiderivative by substitution or otherwise)

D1 $\int \cos^3 x \sin x\, dx$

D2 $\int x\sqrt{3x^2 - 7}\, dx$

D3 $\int \frac{(x+2)dx}{(x^2+4x+3)^2}$

D4 $\int (x^4 - 20x)^7(x^3 - 5)dx$

D5 $\int \frac{\cos x}{\sqrt{1-\sin x}}dx$

D6 $\int \frac{x\, dx}{x^4+1}$ (try $u = x^2$)

D7 $\int \frac{dx}{x^2\sqrt{x^4-1}}$ (try $u = x^2$)

D8 $\int \frac{x^3 dx}{\sqrt{x^2+25}}$

D9 $\int \frac{dx}{1+9x^2}$

D10 $\int x^2 \cos(x^3)dx$

D11 $\int (x - 4)\sqrt[3]{5 - 2x}\, dx$

D12 $\int \frac{\sin^{-1} x\, dx}{\sqrt{1-x^2}}$

Questions **D13 – D18** are definite integrals. Use the Fundamental Theorem of Calculus to check answers.

D13 $\int_0^1 x\sqrt{3x}\, dx$ Ans $\frac{2\sqrt{3}}{5}$

D14 $\int_{-\pi/2}^{\pi/2} \sec^2(\frac{x}{2})dx$ Ans 4

D15 $\int_{-1}^{1} \frac{dx}{1+x^2}$ Ans $\pi/2$

D16 $\int_0^{2\pi} \sin^3 2x\, \cos 2x\, dx$ Ans 0

D17 $\int_2^1 x\sqrt{2 - x}\, dx$ Ans $-\frac{14}{15}$

D18 $\int_0^{\pi/4} \sec^2 x\, \tan x\, dx$ Ans $\frac{1}{2}$

Use numerical methods to estimate **D19 – D22**. You should get more accurate answers than these:

D19 $\int_1^{10} \frac{dx}{x}$ (close to 2.3)

D20 $\int_{1/2}^{2} \sin \frac{1}{t} dt$ (close to 1.1)

D21 $\int_0^{\pi/3} \sec^3 x\, dx$ (close to 2.4)

D22 $\int_0^5 (x^2 + 4x + 3)dx$ (close to 107)

D23 Find the average value of $f(x) = \sec^2 x$ on the interval $[0, \frac{\pi}{4}]$. Ans $\frac{4}{\pi}$

D24 If the average value of $f(x) = x^3 + kx - 2$ on [0,2] is 6, find k. Ans 6