CHAPTER 12 MOTION ALONG A CURVE

12.1 The Position Vector (page 452)

This section explains the key vectors that describe motion. They are functions of t. In other words, the time is a "parameter." We know more than the path that the point travels along. We also know *where the point is at every time t*. So we can find the *velocity* (by taking a derivative).

In Chapter 1 the velocity $v(t)$ was the derivative of the distance $f(t)$. That was in one direction, along a line. Now $f(t)$ is replaced by the ***position vector*** $\mathbf{R}(t)$. Its derivative is the ***velocity vector*** $\mathbf{v}(t)$:

$$\textbf{position } \mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \textbf{velocity } \mathbf{v}(t) = \frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

The next derivative is the ***acceleration vector*** $\mathbf{a} = d\mathbf{v}/dt$.

The direction of $\mathbf{v}$ is a unit vector $\frac{\mathbf{v}}{|\mathbf{v}|}$. This is the ***tangent vector*** $\mathbf{T}$. It is tangent to the path because $\mathbf{v}$ is tangent to the path. But it does not tell us the speed because we divide by $|\mathbf{v}|$; the length is always $|\mathbf{T}| = 1$. The length $|\mathbf{v}|$ is the speed ds/dt. Integrate to find the distance traveled. Remember: $\mathbf{R}$, $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$ are vectors.

Read-throughs and selected even-numbered solutions :

The position vector $\mathbf{R}(t)$ along the curve changes with the parameter t. The velocity is $\mathbf{dR/dt}$. The acceleration is $\mathbf{d^2R/dt^2}$. If the position is $\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, then $\mathbf{v} = \mathbf{j} + \mathbf{2t\,k}$ and $\mathbf{a} = \mathbf{2k}$. In that example the speed is $|\mathbf{v}| = \sqrt{\mathbf{1+4t^2}}$. This equals $\mathbf{ds/dt}$, where s measures **the distance along the curve.** Then $s = \int (\mathbf{ds/dt})\mathbf{dt}$.

The tangent vector is in the same direction as the **velocity,** but $\mathbf{T}$ is a **unit vector.** In general $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ and in the example $\mathbf{T} = (\mathbf{j} + \mathbf{2t\,k})/\sqrt{\mathbf{1+4t^2}}$.

Steady motion along a line has $\mathbf{a} =$ **zero.** If the line is $x = y = z$, the unit tangent vector is $\mathbf{T} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{\mathbf{3}}$. If the speed is $|\mathbf{v}| = \sqrt{3}$, the velocity vector is $\mathbf{v} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. If the initial position is (1,0,0), the position vector is $\mathbf{R}(t) = (\mathbf{1+t})\mathbf{i} + \mathbf{t\,j} + \mathbf{t\,k}$. The general equation of a line is $x = x_0 + tv_1$, $y = \mathbf{y_0} + \mathbf{tv_2}$, $z = \mathbf{z_0} + \mathbf{t\,v_3}$. In vector notation this is $\mathbf{R}(t) = \mathbf{R_0} + \mathbf{t\,v}$. Eliminating t leaves the equations $(x-x_0)/v_1 = (y-y_0)/v_2 = (\mathbf{z}-\mathbf{z_0})/\mathbf{v_3}$. A line in space needs **two** equations where a plane needs **one.** A line has one parameter where a plane has **two.** The line from $\mathbf{R}_0 = (1,0,0)$ to (2,2,2) with $|\mathbf{v}| = 3$ is $\mathbf{R}(t) = (\mathbf{1+t})\mathbf{i} + \mathbf{2t\,j} + \mathbf{2t\,k}$.

Steady motion around a circle (radius r, angular velocity ω) has $x = \mathbf{r\cos\omega t}$, $y = \mathbf{r\sin\omega t}$, $z = 0$. The velocity is $\mathbf{v} = -\mathbf{r}\omega\,\mathbf{\sin}\,\omega\mathbf{t\,i} + \mathbf{r}\,\omega\,\mathbf{\cos}\,\omega\mathbf{t\,j}$. The speed is $|\mathbf{v}| = \mathbf{r}\omega$. The acceleration is $\mathbf{a} = -\mathbf{r}\omega^2(\mathbf{\cos}\,\omega\mathbf{t\,i} + \mathbf{\sin}\,\omega\mathbf{t\,j})$, which has magnitude $\mathbf{r}\omega^2$ and direction **toward (0,0).** Combining upward motion $\mathbf{R} = t\mathbf{k}$ with this circular motion produces motion around a **helix.** Then $\mathbf{v} = -\mathbf{r}\omega\,\mathbf{\sin}\,\omega\mathbf{t\,i} + \mathbf{r}\omega\,\mathbf{\cos}\,\omega\mathbf{t\,j} + \mathbf{k}$ and $|\mathbf{v}| = \sqrt{\mathbf{1+r^2}\omega^2}$.

2 The path is the line $x + y = 2$. The speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{\mathbf{2}}$.

10 The line is $(x,y,z) = (3,1,-2) + \mathbf{t}(-\mathbf{1}, -\frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{2}}{\mathbf{3}})$. Then at $t = 3$ this gives (0, 0, 0). The speed is $\frac{\text{distance}}{\text{time}} = \frac{\sqrt{9+1+4}}{3} = \frac{\sqrt{\mathbf{14}}}{\mathbf{3}}$. For speed e^t choose $(x,y,z) = (\mathbf{3},\mathbf{1},-\mathbf{2}) + \frac{\mathbf{e^t}}{\sqrt{\mathbf{14}}}(-\mathbf{3},-\mathbf{1},\mathbf{2})$.

14 $x^2 + y^2 = (1+t)^2 + (2-t)^2$ is a minimum when $2(1+t) - 2(2-t) = 0$ or $4t = 2$ or $\mathbf{t} = \frac{\mathbf{1}}{\mathbf{2}}$. The path crosses $y = x$ when $1 + t = 2 - t$ or $t = \frac{1}{2}$ (again) at $\mathbf{x} = \mathbf{y} = \frac{\mathbf{3}}{\mathbf{2}}$. The line never crosses a **parallel line like** $\mathbf{x = 2 + t, y = 2 - t}$.

20 If $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ along a path then $\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{x^2} + \mathbf{y^2}) = \mathbf{0}$ and $x^2 + y^2 =$ constant.

32 Given only the path $y = f(x)$, it is impossible to find the **velocity** but still possible to find the **tangent vector** (or the *slope*).

40 The first particle has speed 1 and arrives at $t = \frac{\pi}{2}$. The second particle arrives when $v_2 t = 1$ and $-v_1 t = 1$, so $t = \frac{1}{v_2}$ and $v_1 = -v_2$. Its speed is $\sqrt{v_1^2 + v_2^2} = \sqrt{2}v_2$. So it should have $\sqrt{\mathbf{2}}\mathbf{v_2} < \mathbf{1}$ (to go slower) and $\frac{\mathbf{1}}{\mathbf{v_2}} < \frac{\pi}{2}$ (to win). OK to take $\mathbf{v_2} = \frac{\mathbf{2}}{\mathbf{3}}$.

42 $\mathbf{v} \times \mathbf{w}$ is perpendicular to both lines, so the distance between lines is the length of the projection of $\mathbf{u} = Q - P$ onto $\mathbf{v} \times \mathbf{w}$. The formula for the distance is $\frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|}$.

12.2 Plane Motion: Projectiles and Cycloids (page 457)

This section discusses two particularly important motions. One is from a ball in free flight (a projectile). The other is from a point on a rolling wheel. The first travels on a parabola and the second on a cycloid. In both cases we have to figure out the position vector $\mathbf{R}(t)$ from the situation:

$$\textbf{projectile motion} \qquad x(t) = (\mathrm{v}_0 \cos\alpha)t \quad \text{and} \quad y(t) = (\mathrm{v}_0 \sin\alpha)t - \frac{1}{2}gt^2.$$

Actually this came from two integrations. In free flight the only acceleration is gravity: $\mathbf{a} = -g\mathbf{k}$ (simple but important). One integration gives the velocity $\mathbf{v} = -gt\mathbf{k} + \mathbf{v}(0)$. Another integration gives the position $\mathbf{R}(t) = -\frac{1}{2}gt^2\,\mathbf{k} + \mathbf{v}(0)t + \mathbf{R}(0)$. Assume that the starting point $\mathbf{R}(0)$ is the origin. Split $\mathbf{v}(0)$ into its x and y components $\mathrm{v}_0 \cos x$ and $\mathrm{v}_0 \sin\alpha$ (where α is the *launch angle*). The component $-\frac{1}{2}gt^2$ is all in the y direction. Now you have the components $x(t)$ and $y(t)$ given above. The constant g is 9.8 meters/sec^2.

1. If a ball is launched at $\alpha = 60°$ with speed 50 meters/second, how high does it go and how far does it go?

   - To find how *far* it goes before hitting the ground, solve $y = 0$ to find the flight time $\mathbf{T} = 2\,\mathrm{v}_0(\sin\alpha)/g = 8.8$ seconds. Compute $x(\mathbf{T}) = (50)(\frac{1}{2})(8.8) = 220$ meters for the distance (the range). The highest point is at the half-time $t = 4.4$. Then $y_{\max} = (50)(\frac{\sqrt{3}}{2})(4.4) - \frac{1}{2}(9.8)(4.4)^2 = 96$ meters.

   The **cycloid** is in Figure 12.6. The parameter is not t but the rotation angle θ:

$$\textbf{position} \quad x(\theta) = a(\theta - \sin\theta) \text{ and } y(\theta) = a(1 - \cos\theta) \text{ and slope} \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)}.$$

The text computes the area $3\pi a^2$ under the cycloid and the length $8a$ of the curve.

2. Where is the cycloid's slope zero? Where is the slope infinite?

   - The slope is zero at $\theta = \pi$, because then $\sin\theta = 0$. Top of cycloid is flat.
   - The slope is infinite at $\theta = 0$, even though $\sin\theta = 0$ again. *Reason*: $1 - \cos\theta$ is also zero. We have $\frac{0}{0}$ and go to l'Hôpital. The ratio of θ derivatives is $\frac{a\cos\theta}{a\sin\theta} \to \infty$ at $\theta = 0$. Bottom of cycloid is a cusp.

Read-throughs and selected even-numbered solutions :

A projectile starts with speed v_0 and angle α. At time t its velocity is $dx/dt = \mathbf{v_0} \cos\alpha, dy/dt = \mathbf{v_0} \sin\alpha - \mathbf{gt}$

(the downward acceleration is g). Starting from (0,0), the position is $x = \mathbf{v_0 \cos \alpha\, t}, y = \mathbf{v_0 \sin \alpha\, t - \frac{1}{2}gt^2}$. The flight time back to $y = 0$ is $T = \mathbf{2v_0(\sin \alpha)/g}$. At that time the horizontal range is $R = \mathbf{(v_0^2 \sin 2\alpha)/g}$. The flight path is a **parabola**.

The three quantities $v_0, \alpha, \mathbf{t}$ determine the projectile's motion. Knowing v_0 and the position of the target, we **cannot** solve for α. Knowing α and the position of the target, we **can** solve for v_0.

A **cycloid** is traced out by a point on a rolling circle. If the radius is a and the turning angle is θ, the center of the circle is at $x = \mathbf{a\theta}, y = \mathbf{a}$. The point is at $x = \mathbf{a(\theta - \sin\theta)}, y = \mathbf{a(1 - \cos\theta)}$, starting from (0,0). It travels a distance **8a** in a full turn of the circle. The curve has a **cusp** at the end of every turn. An upside-down cycloid gives the **fastest** slide between two points.

6 If the maximum height is $\frac{(v_0 \sin\alpha)^2}{2g} = 6$ meters, then $\sin^2\alpha = \frac{12(9.8)}{30^2} \approx .13$ gives $\alpha \approx .37$ or $21°$.

10 Substitute into $(gx/v_0)^2 + 2gy = g^2t^2\cos^2\alpha + 2gv_0 t\sin\alpha - t^2 = 2gv_0 t\sin\alpha - g^2t^2\sin^2\alpha$. This is less than v_0^2 because $\mathbf{(v_0 - g\,t\,\sin\alpha)^2 \geq 0}$. For $y = H$ the largest x is when equality holds: $v_0^2 = (gx/v_0)^2 + 2gH$ or $\mathbf{x} = \sqrt{\mathbf{v_0^2 - 2gH}}(\frac{\mathbf{v_0}}{\mathbf{g}})$. If $2gH$ is larger than v_0, the height H can't be reached.

20 $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$ becomes $\frac{0}{0}$ at $\theta = 0$, so use **l'Hôpital's Rule**: The ratio of derivatives is $\frac{\cos\theta}{\sin\theta}$ which becomes infinite. $\frac{\sin\theta}{1-\cos\theta} \approx \frac{\theta}{\theta^2/2} = \frac{2}{\theta}$ equals **20** at $\theta = \frac{1}{10}$ and **−20** at $\theta = -\frac{1}{10}$. The slope is 1 when $\sin\theta = 1 - \cos\theta$ which happens at $\theta = \frac{\pi}{2}$.

38 On the line $x = \frac{\pi}{2}y$ the distance is $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\pi/2)^2 + 1}\,dy$. The last step in equation (5) integrates $\frac{\text{constant}}{\sqrt{y}}$ to give $\frac{\sqrt{\pi^2+4}}{2\sqrt{2g}}[2\sqrt{y}]_0^{2a} = \sqrt{\pi^2+4}\,\frac{2\sqrt{2a}}{2\sqrt{2g}} = \sqrt{\pi^2+4}\,\sqrt{\frac{a}{g}}$.

40 I have read (but don't believe) that the rolling circle jumps as the weight descends.

12.3 Curvature and Normal Vector (page 463)

The tangent vector $\mathbf{T}$ is a unit vector along the path. When the path is a straight line, $\mathbf{T}$ is constant. When the path curves, $\mathbf{T}$ changes. *The rate of change gives the curvature* κ(kappa). This comes from the shape of the path but not the speed. The curvature is $|\frac{d\mathbf{T}}{ds}|$ not $|\frac{d\mathbf{T}}{dt}|$. Since we know the position $\mathbf{R}$ and velocity $\mathbf{v}$ and tangent $\mathbf{T}$ and distance s as functions of t, the available derivatives are time derivatives. So use the chain rule:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|d\mathbf{T}/dt|}{|ds/dt|} = \frac{|d\mathbf{T}/dt|}{|\mathbf{v}|}.$$

1. Find the curvature $\kappa(t)$ if $x = 3\cos 2t$ and $y = 3\sin 2t$ (a *circle*). Thus $\mathbf{R} = 3\cos 2t\,\mathbf{i} + 3\sin 2t\,\mathbf{j}$.

   - The velocity is $\mathbf{v} = \frac{d\mathbf{R}}{dt} = -6\sin 2t\,\mathbf{i} + 6\cos 2t\,\mathbf{j}$. The speed is $|\mathbf{v}| = 6$. The tangent vector is $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\sin 2t\,\mathbf{i} + \cos 2t\,\mathbf{j}$. $\mathbf{T}$ is a unit vector but it changes direction. Its time derivative is

$$\frac{d\mathbf{T}}{dt} = -2\cos 2t\,\mathbf{i} - 2\sin 2t\,\mathbf{j} \text{ and } \left|\frac{d\mathbf{T}}{dt}\right| = 2 \text{ and } \kappa = \frac{2}{6} = \frac{1}{3}.$$

   Since the path is a circle of radius 3, we were expecting its curvature to be $\frac{1}{3}$.

There is also a new vector involved as $\mathbf{T}$ changes direction. It is the ***normal vector*** $\mathbf{N}$, which tells which way $\mathbf{T}$ is changing. The rule is that $\mathbf{N}$ is always a unit vector perpendicular to $\mathbf{T}$. In the xy plane this leaves practically no choice (once we know $\mathbf{T}$). In xyz space we need a formula:

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \quad \text{or} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

2. Find the normal vector $\mathbf{N}$ to the circle in Problem 1. The derivative $d\mathbf{T}/dt$ is already computed above.

   - The length of $|\frac{d\mathbf{T}}{dt}|$ is 2. Divide by 2 to get $\mathbf{N} = \frac{1}{2}\frac{d\mathbf{T}}{dt} = -\cos 2t\ \mathbf{i} - \sin 2t\ \mathbf{j}$. Check that $\mathbf{N}$ is perpendicular to $\mathbf{T} = -\sin 2t\ \mathbf{i} + \cos 2t\ \mathbf{j}$.

3. From the fact that $\mathbf{T}\cdot\mathbf{T} = 1$ show that $\frac{d\mathbf{T}}{dt}$ is perpendicular to $\mathbf{T}$.

   - The derivative of the dot product $\mathbf{T}\cdot\mathbf{T}$ is like the ordinary product rule (first vector dot derivative of second vector plus second vector dot derivative of first vector). Here that gives $2\mathbf{T}\cdot\frac{d\mathbf{T}}{dt}$ = derivative of $1 = 0$. Since $\mathbf{N}$ is in the direction of $\frac{d\mathbf{T}}{dt}$, this means $\mathbf{T}\cdot\mathbf{N} = 0$.

The third unit vector, perpendicular to $\mathbf{T}$ and $\mathbf{N}$, is $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ in Problem 25.

Read-throughs and selected even-numbered solutions :

The curvature tells how fast the curve **turns.** For a circle of radius a, the direction changes by 2π in a distance $\mathbf{2\pi a}$, so $\kappa = \mathbf{1/a}$. For a plane curve $y = f(x)$ the formula is $\kappa = |y''|/(\mathbf{1} + (\mathbf{y}')^{\mathbf{2}})^{\mathbf{3/2}}$. The curvature of $y = \sin x$ is $|\sin \mathbf{x}|/(\mathbf{1} + \cos^{\mathbf{2}}\mathbf{x})^{\mathbf{3/2}}$. At a point where $y'' = 0$ (an **inflection** point) the curve is momentarily straight and $\kappa =$ **zero**. For a space curve $\kappa = |\mathbf{v}\times\mathbf{a}|/|\mathbf{v}|^{\mathbf{3}}$.

The normal vector $\mathbf{N}$ is perpendicular to **the curve (and therefore to v and T).** It is a **unit** vector along the derivative of $\mathbf{T}$, so $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$. For motion around a circle $\mathbf{N}$ points **inward.** Up a helix $\mathbf{N}$ also points **inward.** Moving at unit speed on any curve, the time t is the same as the **distance** s. Then $|\mathbf{v}| = \mathbf{1}$ and $d^2s/dt^2 = \mathbf{0}$ and $\mathbf{a}$ is in the direction of $\mathbf{N}$. Acceleration equals $\mathbf{d^2s/dt^2\ T} + \kappa|\mathbf{v}|^{\mathbf{2}}\ \mathbf{N}$. At unit speed around a unit circle, those components are **zero and one.** An astronaut who spins once a second in a radius of one meter has $|\mathbf{a}| = \omega^{\mathbf{2}} = (\mathbf{2\pi})^{\mathbf{2}}$ meters/sec^2, which is about **4g.**

2 $y = \ln x$ has $\kappa = \frac{|y''|}{(1+y'^2)^{3/2}} = \frac{1/x^2}{(1+\frac{1}{x^2})^{3/2}} = \frac{\mathbf{x}}{(\mathbf{x^2+1})^{\mathbf{3/2}}}$. Maximum of κ when its derivative is zero: $(x^2+1)^{3/2} = x\frac{3}{2}(x^2+1)^{1/2}(2x)$ or $x^2+1 = 3x^2$ or $\mathbf{x^2 = \frac{1}{2}}$.

14 When $\kappa = 0$ the path is a **straight line.** This happens when $\mathbf{v}$ and $\mathbf{a}$ are **parallel.** Then $\mathbf{v}\times\mathbf{a} = \mathbf{0}$.

18 Using equation (8), $\mathbf{v}\times\mathbf{a} = |\mathbf{v}|\mathbf{T}\times(\frac{d^2s}{dt^2}\mathbf{T} + \kappa(\frac{ds}{dt})^2\mathbf{N}) = \kappa|\mathbf{v}|^3\mathbf{T}\times\mathbf{N}$ because $\mathbf{T}\times\mathbf{T} = \mathbf{0}$ and $|\mathbf{v}|$ is the same as $|\frac{ds}{dt}|$. Since $|\mathbf{T}\times\mathbf{N}| = 1$ this gives $|\mathbf{v}\times\mathbf{a}| = \kappa|\mathbf{v}|^3$ or $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$.

22 The parabola through the three points is $y = x^2 - 2x$ which has a constant second derivative $\frac{d^2y}{dx^2} = 2$. The circle through the three points has radius $= 1$ and $\kappa = \frac{1}{\textbf{radius}} = 1$. These are the smallest possible (Proof?)

32 $\mathbf{T} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}$ gives $\frac{d\mathbf{T}}{d\theta} = -\sin\theta\ \mathbf{i} + \cos\theta\ \mathbf{j}$ so $|\frac{d\mathbf{T}}{d\theta}| = 1$. Then $\kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}}{d\theta}||\frac{d\theta}{ds}| = |\frac{d\theta}{ds}|$. Curvature is **rate of change of slope of path.**

12.4 Polar Coordinates and Planetary Motion (page 468)

This section has two main ideas, one from mathematics and the other from physics and astronomy. The mathematics idea is to switch from **i** and **j** (unit vectors across and up) to a polar coordinate system $\mathbf{u}_r$ and $\mathbf{u}_\theta$ (unit vectors out from and around the origin). Very suitable for travel in circles – the tangent vector **T** is just $\mathbf{u}_\theta$. Very suitable for central forces – the force vector is a multiple of $\mathbf{u}_r$.

Just one difficulty: $\mathbf{u}_r$ and $\mathbf{u}_\theta$ change direction as you move, which didn't happen for **i** and **j**:

$$\mathbf{u}_r = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j} \qquad \text{and} \qquad \mathbf{u}_\theta = \frac{d\mathbf{u}_r}{d\theta} = -\sin\theta\ \mathbf{i} + \cos\theta\ \mathbf{j}.$$

The idea from physics is ***gravity***. This is a central force from the sun. (We ignore planets that are smaller and stars that are farther away.) By using polar coordinates we derive the path of the planets around the sun. Kepler found ellipses in the measurements. Newton found ellipses by using $\mathbf{F} = m\mathbf{a}$ and $\mathbf{F} = \text{constant}/r^2$. We find ellipses by the chain rule (page 467).

An important step in mathematical physics is to write the velocity and acceleration in polar coordinates:

$$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \qquad \text{and} \qquad \mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{u}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{u}_\theta$$

Physicists tend to like those formulas. Other people tend to hate them. It was not until writing this book that I understood what Newton did in Calculus III. He solved Problems 12.4.34 and 14.4.26 and so can you.

Read-throughs and selected even-numbered solutions :

A central force points toward **the origin**. Then $\mathbf{R} \times d^2\mathbf{R}/dt^2 = \mathbf{0}$ because **these vectors are parallel.** Therefore $\mathbf{R} \times d\mathbf{R}/dt$ is a **constant** (called **H**).

In polar coordinates, the outward unit vector is $\mathbf{u}_r = \cos\theta\,\mathbf{i} + \mathbf{sin}\,\theta\,\mathbf{j}$. Rotated by 90° this becomes $\mathbf{u}_\theta = -\mathbf{sin}\,\theta\,\mathbf{i} + \mathbf{cos}\,\theta\,\mathbf{j}$. The position vector **R** is the distance r times $\mathbf{u_r}$. The velocity $\mathbf{v} = d\mathbf{R}/dt$ is $\mathbf{(dr/dt)u}_r + \mathbf{(r\, d\theta/dt)u}_\theta$. For steady motion around the circle $r = 5$ with $\theta = 4t$, **v** is $-\mathbf{20\,sin\,4t\,i} + \mathbf{20\,cos\,4t\,j}$ and $|\mathbf{v}|$ is **20** and **a** is $-\mathbf{80\,cos\,4t\,i} - \mathbf{80\,sin\,4t\,j}$.

For motion under a circular force, r^2 times $\mathbf{d\theta/dt}$ is constant. Dividing by 2 gives Kepler's second law $dA/dt = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{r^2 d\theta/dt} =$ **constant**. The first law says that the orbit is an **ellipse** with the sun at **a focus**. The polar equation for a conic section is $\mathbf{1/r} = C - D\cos\theta$. Using $\mathbf{F} = m\mathbf{a}$ we found $q_{\theta\theta} + \mathbf{q} = C$. So the path is a conic section; it must be an ellipse because **planets come around again**. The properties of an ellipse lead to the period $T = \mathbf{2\pi a^{3/2}/\sqrt{GM}}$, which is Kepler's third law.

8 The distance $r\theta$ around the circle is the integral of the speed $8t$: thus $4\theta = 4t^2$ and $\theta = t^2$. The circle is complete at $\mathbf{t} = \sqrt{\mathbf{2\pi}}$. At that time $\mathbf{v} = r\frac{d\theta}{dt}\mathbf{u}_\theta = 4(2\sqrt{2\pi})\mathbf{j}$ and $\mathbf{a} = -\mathbf{4(8\pi)i} + \mathbf{4(2)j}$.

12 Since $\mathbf{u_r}$ has constant length, its derivatives are **perpendicular** to itself. In fact $\frac{d\mathbf{u}_r}{dr} = 0$ and $\frac{\mathbf{du_r}}{\mathbf{d\theta}} = \mathbf{u}_\theta$.

14 $R = re^{i\theta}$ has $\frac{d^2R}{dt^2} = \frac{d^2r}{dt^2}e^{i\theta} + 2\frac{dr}{dt}(ie^{i\theta}\frac{d\theta}{dt}) + ir\frac{d^2\theta}{dt^2}e^{i\theta} + i^2r(\frac{d\theta}{dt})^2e^{i\theta}$. (Note repeated term gives factor 2.) The coefficient of $e^{i\theta}$ is $\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2$. The coefficient of $ie^{i\theta}$ is $2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$. These are the $\mathbf{u_r}$ and $\mathbf{u}_\theta$ components of **a**.

20 (a) **False:** The paths are conics but they could be hyperbolas and possibly parabolas.

(b) **True:** A circle has $r =$ constant and $r^2 \frac{d\theta}{dt} =$ constant so $\frac{d\theta}{dt} =$ constant.

(c) **False:** The central force might not be proportional to $\frac{1}{r^2}$.

32 $T = \frac{2\pi}{\sqrt{GM}}(1.6 \cdot 10^9)^{3/2} \approx 71$ years. So the comet will return in the year $1986 + 71 = 2057$.

34 First derivative: $\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{C - D\cos\theta}\right) = \frac{-D\sin\theta \frac{d\theta}{dt}}{(C - D\cos\theta)^2} = -D\sin\theta\, r^2 \frac{d\theta}{dt} = -Dh\sin\theta$.

Next derivative: $\frac{d^2r}{dt^2} = -Dh\cos\theta \frac{d\theta}{dt} = \frac{-Dh^2\cos\theta}{r^2}$. But $C - D\cos\theta = \frac{1}{r}$ so $-D\cos\theta = (\frac{1}{r} - C)$.

The acceleration terms $\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2$ combine into $(\frac{1}{r} - C)\frac{h^2}{r^2} - \frac{h^2}{r^3} = -C\frac{h^2}{r^2}$. Conclusion by Newton:

The elliptical orbit $r = \frac{1}{C - D\cos\theta}$ requires acceleration $= \frac{\text{constant}}{r^2}$: *the inverse square law.*

12 Chapter Review Problems

Graph Problems

G1 Draw these three curves as t goes from 0 to 2π. Mark distances on the x and y axes.

$$(a)\quad x = 2t, y = \sin t \qquad (b)\quad x = 2\sin t, y = \sin t \qquad (c)\quad x = 2\cos t, y = \sin t$$

G2 Draw a helix $x = \cos t,\ y = \sin t,\ z = t$. Also draw its projections on the xy and yz planes.

Review Problems

R1 Find $x(t)$ and $y(t)$ for travel around the unit circle at speed 2. Start at $(x, y) = (-1, 0)$.

R2 Find $x(t)$ and $y(t)$ along the circle $(x-3)^2 + (y-4)^2 = 5^2$.

R3 Find $x(t)$ and $y(t)$ to produce travel along a cycloid. At what time do you repeat?

R4 Find the tangent vector $\mathbf{T}$ and the speed $|\mathbf{v}|$ for your cycloid travel.

R5 Find the velocity vector and the speed along the curve $x = t^2, y = \sqrt{t}$. What curve is it?

R6 Explain why the velocity vector $\mathbf{v}(t) = d\mathbf{R}/dt$ is tangent to the curve. $\mathbf{R}(t)$ gives the position.

R7 Is the acceleration vector $\mathbf{a}(t) = d\mathbf{v}/dt = d^2\mathbf{R}/dt^2$ always perpendicular to $\mathbf{v}$ and the curve?

R8 Choose $x(t), y(t)$, and $z(t)$ for travel from (2,0,0) at $t = 0$ to (8,3,6) at $t = 3$ with no acceleration.

R9 Figure 12.5a shows flights at 30° and 60°. The range R is the same. One flight goes ___ times as high as the other.

R10 Why do flights at $\alpha = 10°$ and 80° have the same range? Are the flight times T the same?

R11 If $\mathbf{T}(t)$ is always a unit vector, why isn't $d\mathbf{T}/dt = 0$? Prove that $\mathbf{T} \cdot d\mathbf{T}/dt = 0$.

R12 Define the curvature $\kappa(t)$. Compute it for $x = y = z = \sin t$.

R13 If a satellite stays above Hawaii, why does that determine its distance from Earth?

Drill Problems

D1 With position vector $\mathbf{R} = 3t\,\mathbf{i} + 4t\,\mathbf{j}$ find the velocity $\mathbf{v}$ and speed $|\mathbf{v}|$ and unit tangent $\mathbf{T}$.

D2 True or false: A projectile that goes from (0,0) to (1,1) has starting angle $\alpha = 45°$.

D3 State the r, θ equation for an ellipse. Where is its center point? At the sun?

D4 Give the equations (using t) for the line through (1,4,5) in the direction of $\mathbf{i} + 2\mathbf{j}$.

D5 Give the equations for the same line without the parameter t. (Eliminate t.).

D6 State the equations $x =$ ___ and $y =$ ___ for a ball thrown down at a 30° angle with speed $\mathbf{v}_0$.

D7 What are the rules for the derivatives of $\mathbf{v}(t) \cdot \mathbf{w}(t)$ and $\mathbf{v}(t) \times \mathbf{w}(t)$?

D8 Give the formula for the curvature κ of $y = f(x)$. Compute κ for $y = \sqrt{1 - x^2}$ and explain.

D9 At $(x, y) = (3, 4)$ what are the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$? Why are they the same at (6,8)?

D10 Motion in a central force field always stays in a (circle, ellipse, plane). Why is $\mathbf{R} \times \mathbf{a} = \mathbf{0}$?

D11 What integral gives the distance traveled back to $y = 0$ on the curved flight $x = t, y = t - \frac{1}{2}gt^2$?

D12 What is the velocity $\mathbf{v} =$ ___$\mathbf{u}_r +$ ___$\mathbf{u}_\theta$ along the spiral $r = t, \theta = t$?