CHAPTER 11 VECTORS AND MATRICES

11.1 Vectors and Dot Products (page 405)

One side of the brain thinks of a vector as a pair of numbers (x, y) or a triple of numbers (x, y, z) or a string of n numbers $(x_1, x_2, \cdots, x_n)$. This is the algebra side. The other hemisphere of the brain deals with geometry. It thinks of a vector as a point (or maybe an arrow.) For (x, y) the point is in a plane and for (x, y, z) the point is in 3-dimensional space. The picture gets fuzzy in n dimensions but our intuition is backed up by algebra.

The big operation on vectors is to take ***linear combinations*** $a\mathbf{u} + b\mathbf{v}$. The second biggest operation is to take *dot products* $\mathbf{u} \cdot \mathbf{v}$. Those are the key steps in matrix multiplication and they are best seen by example!

$$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 14.$$

Linear combinations produce vectors. Dot products yield numbers. The dot product of a vector with itself is the length squared. The dot product of a vector with another vector reveals the angle between them:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ has } \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = 5. \text{ The length is } |\mathbf{v}| = \sqrt{5}.$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ have } \cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{14}{\sqrt{5}\sqrt{41}} = .9778. \text{ The angle is } \theta = \cos^{-1} .9778 = .21$$

Since cosines are never above 1, dot products $\mathbf{v} \cdot \mathbf{w}$ are never above $|\mathbf{v}|$ times $|\mathbf{w}|$. This is the *Cauchy-Schwarz inequality*.

1. Find the lengths of $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$. What is the angle between them?

   - Since $\mathbf{v} \cdot \mathbf{v} = 9 + 16 = 25$, the length is $|\mathbf{v}| = \sqrt{25} = 5$. Similarly $\mathbf{w} \cdot \mathbf{w} = 25$ and the length is $|\mathbf{w}| = 5$. The dot product is $\mathbf{v} \cdot \mathbf{w} = 12 - 12 = 0$. *The cosine of θ is zero*! Therefore $\theta = 90°$ and these two vectors are ***perpendicular***.

   The arrow from the origin to the point $\mathbf{v} = (3, 4)$ has slope $\frac{4}{3}$. The arrow to the other point $\mathbf{w} = (4, -3)$ has slope $-\frac{3}{4}$. The slopes multiply to give -1. This is our old test for perpendicular lines. Dot products give a new and better test $\mathbf{v} \cdot \mathbf{w} = 0$. This applies in three dimensions or in n dimensions.

Read-throughs and selected even-numbered solutions :

A vector has length and **direction**. If $\mathbf{v}$ has components 6 and $-\mathbf{8}$, its length is $|\mathbf{v}| = \mathbf{10}$ and its direction vector is $\mathbf{u} = .\mathbf{6i} - .\mathbf{8j}$. The product of $|\mathbf{v}|$ with $\mathbf{u}$ is $\mathbf{v}$. This vector goes from (0,0) to the point $x = \mathbf{6}, y = -\mathbf{8}$. A combination of the coordinate vectors $\mathbf{i} = (\mathbf{1}, \mathbf{0})$ and $\mathbf{j} = (\mathbf{0}, \mathbf{1})$ produces $\mathbf{v} = \mathbf{x}\,\mathbf{i} + \mathbf{y}\,\mathbf{j}$.

To add vectors we add their **components**. The sum of $(6, -8)$ and (1,0) is $(\mathbf{7}, -\mathbf{8})$. To see $\mathbf{v} + \mathbf{i}$ geometrically, put the **tail** of $\mathbf{i}$ at the **head** of $\mathbf{v}$. The vectors form a **parallelogram with diagonal $\mathbf{v} + \mathbf{i}$**. (The other diagonal is $\mathbf{v} - \mathbf{i}$). The vectors $2\mathbf{v}$ and $-\mathbf{v}$ are $(\mathbf{12}, -\mathbf{16})$ and $(-\mathbf{6}, \mathbf{8})$. Their lengths are **20** and **10**.

In a space without axes and coordinates, the tail of $\mathbf{V}$ can be placed **anywhere**. Two vectors with the same **components or the same length and direction** are the same. If a triangle starts with $\mathbf{V}$ and continues with $\mathbf{W}$, the third side is $\mathbf{V} + \mathbf{W}$. The vector connecting the midpoint of $\mathbf{V}$ to the midpoint of $\mathbf{W}$ is $\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{V} + \mathbf{W})$. That vector is **half of** the third side. In this coordinate-free form the dot product is $\mathbf{V} \cdot \mathbf{W} = |\mathbf{V}||\mathbf{W}|\,\mathbf{cos}\,\theta$.

Using components, $\mathbf{V} \cdot \mathbf{W} = \mathbf{V_1W_1} + \mathbf{V_2W_2} + \mathbf{V_3W_3}$ and $(1, 2, 1) \cdot (2, -3, 7) = \mathbf{3}$. The vectors are perpendicular if $\mathbf{V} \cdot \mathbf{W} = \mathbf{0}$. The vectors are parallel if **$\mathbf{V}$ is a multiple of $\mathbf{W}$**. $\mathbf{V} \cdot \mathbf{V}$ is the same as $|\mathbf{V}|^{\mathbf{2}}$. The dot product of $\mathbf{U}+\mathbf{V}$ with $\mathbf{W}$ equals $\mathbf{U}\cdot\mathbf{W}+\mathbf{V}\cdot\mathbf{W}$. The angle between $\mathbf{V}$ and $\mathbf{W}$ has $\cos\theta = \mathbf{V}\cdot\mathbf{W}/|\mathbf{V}||\mathbf{W}|$. When $\mathbf{V}\cdot\mathbf{W}$ is negative then θ is **greater than 90°**. The angle between $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$ is $\boldsymbol{\pi/3}$ **with cosine** $\frac{1}{2}$. The Cauchy-Schwarz inequality is $|\mathbf{V}\cdot\mathbf{W}| \le |\mathbf{V}||\mathbf{W}|$, and for $\mathbf{V}=\mathbf{i}+\mathbf{j}$ and $\mathbf{W}=\mathbf{i}+\mathbf{k}$ it becomes $\mathbf{1}\le\mathbf{2}$.

12 We want $\mathbf{V}\cdot(\mathbf{W}-c\mathbf{V})=0$ or $\mathbf{V}\cdot\mathbf{W}=c\mathbf{V}\cdot\mathbf{V}$. Then $c=\frac{6}{3}=2$ and $\mathbf{W}-c\mathbf{V}=(\mathbf{-1},\mathbf{0},\mathbf{1})$.

14 (a) Try two possibilities: keep clock vectors 1 through 5 or 1 through 6. The five add to $1 + 2\cos 30° + 2\cos 60° = 2\sqrt{3} = 3.73$ (in the direction of 3:00). The six add to $2\cos 15° + 2\cos 45° + 2\cos 75° = \mathbf{3.86}$ which is longer (in the direction of 3:30). (b) The 12 o'clock vector (call it $\mathbf{j}$ because it is vertical) is subtracted from all twelve clock vectors. So the sum changes from $\mathbf{V}=0$ to $\mathbf{V}^* = -12\mathbf{j}$.

18 (a) The points $\mathbf{tB}$ form a **line from the origin in the direction of B.** (b) $\mathbf{A}+\mathbf{tB}$ forms a **line from A in the direction of B.** (c) $\mathbf{sA}+\mathbf{tB}$ forms a **plane containing A and B.** (d) $\mathbf{v}\cdot\mathbf{A}=\mathbf{v}\cdot\mathbf{B}$ means $\frac{\cos\theta_1}{\cos\theta_2}$ = fixed number $\frac{|\mathbf{B}|}{|\mathbf{A}|}$ where θ_1 and θ_2 are the angles from $\mathbf{v}$ to $\mathbf{A}$ and $\mathbf{B}$. Then $\mathbf{v}$ is on the plane through the origin that gives this fixed number. (If $|\mathbf{A}|=|\mathbf{B}|$ the plane bisects the angle between those vectors.)

28 $\mathbf{I}\cdot\mathbf{J}=\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\cdot\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}=\frac{1-1}{2}=0$. Add $\mathbf{i}+\mathbf{j}=\sqrt{2}\mathbf{I}$ to $\mathbf{i}-\mathbf{j}=\sqrt{2}\mathbf{J}$ to find $\mathbf{i}=\frac{\sqrt{2}}{2}(\mathbf{I}+\mathbf{J})$. Substitute back to find $\mathbf{j}=\frac{\sqrt{2}}{2}(\mathbf{I}-\mathbf{J})$. Then $\mathbf{A}=2\mathbf{i}+3\mathbf{j}=\sqrt{2}(\mathbf{I}+\mathbf{J})+\frac{3\sqrt{2}}{2}(\mathbf{I}-\mathbf{J})=a\mathbf{I}+b\mathbf{J}$ with $a=\sqrt{2}+\frac{3\sqrt{2}}{2}$ and $b=\sqrt{2}-\frac{3\sqrt{2}}{2}$.

34 The diagonals are $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}-\mathbf{A}$. Suppose $|\mathbf{A}+\mathbf{B}|^2=\mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}$ equals $|\mathbf{B}-\mathbf{A}|^2 = \mathbf{B}\cdot\mathbf{B}-\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}$. After cancelling this is $4\mathbf{A}\cdot\mathbf{B}=0$ (note that $\mathbf{A}\cdot\mathbf{B}$ is the same as $\mathbf{B}\cdot\mathbf{A}$). The region is a **rectangle.**

40 Choose $\mathbf{W}=(\mathbf{1},\mathbf{1},\mathbf{1})$. Then $\mathbf{V}\cdot\mathbf{W}=V_1+V_2+V_3$. The Schwarz inequality $|\mathbf{V}\cdot\mathbf{W}|^2\le|\mathbf{V}|^2|\mathbf{W}|^2$ is $(V_1+V_2+V_3)^2\le 3(V_1^2+V_2^2+V_3^2)$.

42 $|\mathbf{A}+\mathbf{B}|\le|\mathbf{A}|+|\mathbf{B}|$ or $|\mathbf{C}|\le|\mathbf{A}|+|\mathbf{B}|$ says that any side length is less than **the sum of the other two side lengths.** Proof: $|\mathbf{A}+\mathbf{B}|^2\le$ (using Schwarz for $\mathbf{A}\cdot\mathbf{B}$)$|\mathbf{A}|^2+2|\mathbf{A}||\mathbf{B}|+|\mathbf{B}|^2=(|\mathbf{A}|+|\mathbf{B}|)^2$.

11.2 Planes and Projections (page 414)

The main point is to understand the equation of a plane (in three dimensions). In general it is $ax+by+cz=d$. Specific examples are $2x+4y+z=14$ and $2x+4y+z=0$. *Those planes are parallel. They have the same normal vector* $\mathbf{N}=(2,4,1)$. The equations of the planes can be written as $\mathbf{N}\cdot\mathbf{P}=d$. A typical point in $\mathbf{N}\cdot\mathbf{P}=14$ is $(x,y,z)=(2,2,2)$ because then $2x+4y+z=14$. A typical point in $\mathbf{N}\cdot\mathbf{P}=0$ is $(2,-1,0)$ because then $2x+4y+z=0$.

From $\mathbf{N}\cdot\mathbf{P}=0$ we see that $\mathbf{N}$ is perpendicular (= *normal*) to every vector $\mathbf{P}$ in the plane. The components of $\mathbf{P}$ are (x,y,z) and the vector is $\mathbf{P}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. This plane goes through the origin, because $x=0$, $y=0$, $z=0$ satisfies the equation $2x+4y+z=0$. The other plane $\mathbf{N}\cdot\mathbf{P}=14$ does not go through the origin. Its equation can be written as $\mathbf{N}\cdot\mathbf{P}=\mathbf{N}\cdot\mathbf{P}_0$ where $\mathbf{P}_0$ is any particular point on the plane (since every one of those points has $\mathbf{N}\cdot\mathbf{P}_0=14$). Then the plane equation is $\mathbf{N}\cdot(\mathbf{P}-\mathbf{P}_0)=0$, which shows that the normal vector $\mathbf{N}=(2,4,1)$ is perpendicular to every vector $\mathbf{P}-\mathbf{P}_0$ lying in the plane.

To repeat: $\mathbf{P}$ and $\mathbf{P}_0$ are vectors *to* the plane, starting at $(0,0,0)$. $\mathbf{P}-\mathbf{P}_0$ is a vector *in* the plane.

The distance *to* the plane is the shortest vector **P** with $\mathbf{N}\cdot\mathbf{P} = 14$. The shortest **P** is in the same direction as **N**. Call it $\mathbf{P} = t\,\mathbf{N}$. Then $\mathbf{N}\cdot\mathbf{P} = 14$ becomes $t\,\mathbf{N}\cdot\mathbf{N} = 14$. In this example $\mathbf{N}\cdot\mathbf{N} = 2^2 + 4^2 + 1^2 = 21$. Therefore $21t = 14$ and $t = \frac{2}{3}$. The shortest **P** is $\frac{2}{3}\,\mathbf{N}$.

The general formula for distance from (0,0,0) ***to*** $\mathbf{N}\cdot\mathbf{P} = d$ ***is*** $|d|$ ***divided by*** $|\mathbf{N}|$. The distance to $\mathbf{N}\cdot\mathbf{P} = 0$ is 0.

1. Find the plane through $\mathbf{P}_0 = (1,2,2)$ perpendicular to $\mathbf{N} = (3,1,4)$.

   - Because of **N** the equation must be $3x+y+4z = d$. Because of $\mathbf{P}_0$ the number d must be $3\cdot 1+2+4\cdot 2 = 13$. The plane is $3x + y + 4z = 13$ or $\mathbf{N}\cdot\mathbf{P} = \mathbf{N}\cdot\mathbf{P}_0$.

2. Find the plane parallel to $x + y + z = 4$ but going through (1,4,5).

   - To be parallel the equation must be $x + y + z = d$. To go through (1,4,5) we need $d = 10$. The plane $x + y + z = 10$ has the desired normal vector $\mathbf{N} = (1,1,1)$. The distance from the origin is $\frac{|d|}{|\mathbf{N}|} = \frac{10}{\sqrt{3}}$.

Projection onto a line We look for the projection of a vector **B** in the direction of **A**. The projection is an unknown number x times **A**. Geometry says that the vector from **B** down to $x\mathbf{A}$ should be perpendicular to **A**:

$$\mathbf{A}\cdot(\mathbf{B} - x\mathbf{A}) = 0 \text{ or } \mathbf{A}\cdot\mathbf{B} = x\mathbf{A}\cdot\mathbf{A}. \text{ Therefore } x = \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}} \text{ and } \mathbf{P} = x\mathbf{A} = \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{B}}\mathbf{A}.$$

The projection is **P** (*a new use for this letter*). Its length is $|\mathbf{B}||\cos\theta|$. This is the length of **B** "*along* **A**."

3. In terms of **A** and **B**, find the length of the projection **P**.

   - The length is $|\mathbf{B}||\cos\theta| = |\mathbf{B}|\frac{|\mathbf{A}\cdot\mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = \frac{|\mathbf{A}\cdot\mathbf{B}|}{|\mathbf{A}|}$.
   - From the formula $\mathbf{P} = \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\mathbf{A}$ the length is again $\frac{|\mathbf{A}\cdot\mathbf{B}|}{|\mathbf{A}|}$.

4. Project the vector $\mathbf{B} = (1,3,1)$ onto the vector $\mathbf{A} = (2,1,2)$. Find the component **P** along **A** and also find the component $\mathbf{B}-\mathbf{P}$ perpendicular to **A**.

   - The dot product is $\mathbf{A}\cdot\mathbf{B} = 7$. Also $\mathbf{A}\cdot\mathbf{A} = 9$. Therefore $\mathbf{P} = \frac{7}{9}\mathbf{A} = (\frac{14}{9}, \frac{7}{9}, \frac{14}{9})$.

   The perpendicular component is $\mathbf{B}-\mathbf{P} = (1,3,1)-\mathbf{P} = (-\frac{5}{9}, \frac{20}{9}, -\frac{5}{9})$. Check that it really is perpendicular: $\mathbf{A}\cdot(\mathbf{B}-\mathbf{P}) = -\frac{10}{9} + \frac{20}{9} - \frac{10}{9} = 0$.

Read-throughs and selected even-numbered solutions :

A plane in space is determined by a point $P_0 = (x_0, y_0, z_0)$ and a **normal** vector **N** with components (a, b, c). The point $P = (x, y, z)$ is on the plane if the dot product of **N** with $\mathbf{P} - \mathbf{P_0}$ is zero. (*That answer was not P!*) The equation of this plane is $a(\mathbf{x} - \mathbf{x_0}) + b(\mathbf{y} - \mathbf{y_0}) + c(\mathbf{z} - \mathbf{z_0}) = 0$. The equation is also written as $ax + by + cz = d$, where d equals $\mathbf{ax_0} + \mathbf{by_0} + \mathbf{cz_0}$ or $\mathbf{N}\cdot\mathbf{P_0}$. A parallel plane has the same **N** and a different **d**. A plane through the origin has $d = \mathbf{0}$.

The equation of the plane through $P_0 = (2,1,0)$ perpendicular to N = (3,4,5) is $\mathbf{3x} + \mathbf{4y} + \mathbf{5z} = \mathbf{10}$. A second point in the plane is $P = (0,0,\mathbf{2})$. The vector from P_0 to P is $(-\mathbf{2}, -\mathbf{1}, \mathbf{2})$, and it is **perpendicular** to N. (Check by dot product). The plane through $P_0 = (2,1,0)$ perpendicular to the z axis has $\mathbf{N} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$ and equation $\mathbf{z} = \mathbf{0}$.

The component of $\mathbf{B}$ in the direction of $\mathbf{A}$ is $|\mathbf{B}|\cos\theta$, where θ is the angle between the vectors. This is $\mathbf{A}\cdot\mathbf{B}$ divided by $|\mathbf{A}|$. The projection vector $\mathbf{P}$ is $|\mathbf{B}|\cos\theta$ times a **unit** vector in the direction of $\mathbf{A}$. Then $\mathbf{P} = (|\mathbf{B}|\cos\theta)(\mathbf{A}/|\mathbf{A}|)$ simplifies to $(\mathbf{A}\cdot\mathbf{B})\mathbf{A}/|\mathbf{A}|^2$. When $\mathbf{B}$ is doubled, $\mathbf{P}$ is **doubled**. When $\mathbf{A}$ is doubled, $\mathbf{P}$ is **not changed**. If $\mathbf{B}$ reverses direction, so does $\mathbf{P}$. If $\mathbf{A}$ reverses direction, then $\mathbf{P}$ **stays the same**.

When $\mathbf{B}$ is a velocity vector, $\mathbf{P}$ represents the **velocity in the A direction**. When $\mathbf{B}$ is a force vector, $\mathbf{P}$ **is the force component along A**. The component of $\mathbf{B}$ perpendicular to $\mathbf{A}$ equals $\mathbf{B}-\mathbf{P}$. The shortest distance from (0,0,0) to the plane $ax+by+cz=d$ is along the **normal** vector. The distance from the origin is $|\mathbf{d}|/\sqrt{\mathbf{a^2}+\mathbf{b^2}+\mathbf{c^2}}$ and the point on the plane closest to the origin is $P = (\mathbf{da},\mathbf{db},\mathbf{dc})/(\mathbf{a^2}+\mathbf{b^2}+\mathbf{c^2})$. The distance from $\mathbf{Q}=(x_1,y_1,z_1)$ to the plane is $|\mathbf{d}-\mathbf{ax_1}-\mathbf{by_1}-\mathbf{cz_1}|/\sqrt{\mathbf{a^2}+\mathbf{b^2}+\mathbf{c^2}}$.

6 The plane $y-z=0$ contains the given points (0,0,0) and (1,0,0) and (0,1,1). The normal vector is $\mathbf{N}=\mathbf{j}-\mathbf{k}$. (Certainly $P=(0,1,1)$ and $P_0=(0,0,0)$ give $\mathbf{N}\cdot(P-P_0)=0$.)

12 (a) **No**: the line where the planes (or walls) meet is not perpendicular to itself. (b) A third plane perpendicular to the first plane could make **any angle** with the second plane.

22 If $\mathbf{B}$ makes a 60° angle with $\mathbf{A}$ then the length of $\mathbf{P}$ is $|\mathbf{B}|\cos 60° = 2\cdot\frac{1}{2}=1$. Since $\mathbf{P}$ is in the direction of $\mathbf{A}$ it must be $\frac{\mathbf{A}}{|\mathbf{A}|}$.

32 The points at distance 1 from the plane $x+2y+2z=3$ fill two parallel planes $\mathbf{x}+\mathbf{2y}+\mathbf{2z}=\mathbf{6}$ and $\mathbf{x}+\mathbf{2y}+\mathbf{2z}=\mathbf{0}$. Check: The point (0,0,0) on the last plane is a distance $\frac{|d|}{|\mathbf{N}|}=\frac{3}{3}=1$ from the plane $x+2y+2z=3$.

38 The point $P=Q+t\mathbf{N}=(3+t,3+2t)$ lies on the line $x+2y=4$ if $(3+t)+2(3+2t)=4$ or $9+5t=4$ or $t=\mathbf{-1}$. Then $P=(\mathbf{2},\mathbf{1})$.

40 The drug runner takes $\frac{1}{2}$ second to go the 4 meters. You have 5 meters to travel in the same $\frac{1}{2}$ second. Your speed must be **10 meters per second**. The projection of your velocity (a vector) onto the drug runner's velocity equals **the drug runner's velocity.**

11.3 Cross Products and Determinants (page 423)

This section is mostly in three dimensions. We take the cross product of vectors $\mathbf{A}$ and $\mathbf{B}$. That produces a third vector $\mathbf{A}\times\mathbf{B}$ perpendicular to the first two. The length of this cross product is adjusted to equal $|\mathbf{A}||\mathbf{B}|\sin\theta$. This leads to a (fairly) neat formula

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3-a_3b_2)\mathbf{i} + (a_3b_1-a_1b_3)\mathbf{j} + (a_1b_2-a_2b_1)\mathbf{k}.$$

1. Find the cross product of $\mathbf{A}=\mathbf{i}$ and $\mathbf{B}=\mathbf{k}$.

   - The perpendicular direction is $\mathbf{j}$. But the cross product $\mathbf{i}\times\mathbf{k}$ is $-\mathbf{j}$. This comes from the formula above, when you substitute 1,0,0 in the second row and 0,0,1 in the third row. The minus sign also comes from the right hand rule. The length of $\mathbf{A}\times\mathbf{B}$ is $|\mathbf{A}||\mathbf{B}|\sin\theta$. The vectors $\mathbf{A}=\mathbf{i}$ and $\mathbf{B}=\mathbf{k}$ are perpendicular so $\theta=90°$ and $\sin\theta=1$.

2. Find the cross product of **k** and **i**.

   - In this opposite order the cross product changes sign: $\mathbf{k} \times \mathbf{i} = +\mathbf{j}$.

3. Find $(\mathbf{i} + \mathbf{k}) \times (\mathbf{i} + \mathbf{k})$. Also split this into $\mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{k} + \mathbf{k} \times \mathbf{i} + \mathbf{k} \times \mathbf{k}$.

   - $\mathbf{i} \times \mathbf{i}$ is the zero vector. So is $\mathbf{k} \times \mathbf{k}$. So is any $\mathbf{A} \times \mathbf{A}$ because the angle is $\theta = 0$ and its sine is zero. The splitting also gives $\mathbf{0} - \mathbf{j} + \mathbf{j} + \mathbf{0} = \mathbf{0}$.

Most determinants do not involve **i, j, k**. They are determinants of ordinary matrices. For a 2 by 2 matrix we get the ***area*** of a parallelogram. For a 3 by 3 matrix we get the ***volume*** of a parallelepiped (a box):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \text{ area with corners } (0,0), (a,b), (c,d), (a+c, b+d)$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = a_1 b_2 c_3 + \text{ five other terms } = \text{ volume of box.}$$

A 3 by 3 determinant has six terms. An n by n determinant has $n!$ terms. Half the signs are minus.

4. Compute these determinants. Interpret as areas or volumes:

$$\text{(a)} \begin{vmatrix} 2 & 5 \\ 4 & 10 \end{vmatrix} \quad \text{(b)} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \quad \text{(c)} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 6 & 8 \end{vmatrix}$$

   - (a) The determinant is $(2)(10) - (-5)(4) = 0$. Because (2,5) is parallel to (4,10), the parallelogram is crushed. It has no area. The matrix has no inverse.
   - (b) The determinant is $(1)(1)(1) = 1$ with a *plus sign*. This is also $\mathbf{j}\cdot(\mathbf{k}\times\mathbf{i})$ which is $\mathbf{j}\cdot\mathbf{j} = 1$. It is the volume of a *unit cube*. Exchange two rows and the determinant is -1 (left-handed cube).
   - (c) The six terms in the determinant are $+32 + 18 + 20 - 24 - 30 - 16$. This gives zero! The volume is zero because the 3-dimension box is completely squashed into a plane.
     The third row of this matrix is the sum of the first two rows. In such a case the determinant is zero. We never leave the plane of the first two rows. $\mathbf{A} \times \mathbf{B}$ is perpendicular to that plane, **C** is in that plane, so the determinant $(\mathbf{A} \times \mathbf{B})\cdot\mathbf{C}$ is zero.

Read-throughs and selected even-numbered solutions :

The cross product $\mathbf{A} \times \mathbf{B}$ is a **vector** whose length is $|\mathbf{A}||\mathbf{B}| \sin\theta$. Its direction is **perpendicular** to **A** and **B**. That length is the area of a **parallelogram**, whose base is $|\mathbf{A}|$ and whose height is $|\mathbf{B}| \sin\theta$. When $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, the area is $|\mathbf{a_1 b_2} - \mathbf{a_2 b_1}|$. This equals a 2 by 2 **determinant**. In general $|\mathbf{A} \cdot \mathbf{B}|^2 + |\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^{\mathbf{2}}|\mathbf{B}|^{\mathbf{2}}$.

The rules for cross products are $\mathbf{A} \times \mathbf{A} = \mathbf{0}$ and $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$ and $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$. In particular $\mathbf{A} \times \mathbf{B}$ needs the **right-hand** rule to decide its direction. If the fingers curl from **A** towards **B** (not more than 180°), then $\mathbf{A} \times \mathbf{B}$ points **along the right thumb**. By this rule $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$.

The vectors $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ have cross product $(\mathbf{a_2 b_3} - \mathbf{a_3 b_2})\mathbf{i} + (\mathbf{a_3 b_1} - \mathbf{a_1 b_3})\mathbf{j} + (\mathbf{a_1 b_2} - \mathbf{a_2 b_1})\mathbf{k}$. The vectors $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j}$ have $\mathbf{A} \times \mathbf{B} = -\mathbf{i} + \mathbf{j}$. (*This is also the 3 by 3* *determinant* $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$.) Perpendicular to the plane containing (0,0,0), (1,1,1), (1,1,0) is the normal vector **N** $= -\mathbf{i} + \mathbf{j}$. The area of the triangle with those three vertices is $\frac{1}{2}\sqrt{2}$, which is half the area of the parallelogram with fourth vertex at $(2, 2, 1)$.

Vectors **A, B, C** from the origin determine a **box**. Its volume $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ comes from a 3 by 3 **determinant.** There are six terms, **three** with a plus sign and **three** with minus. In every term each row and **column** is represented once. The rows (1,0,0), (0,0,1), and (0,1,0) have determinant $= -1$. That box is a **cube**, but its sides form a left-handed triple in the order given.

If **A, B, C** lie in the same plane then $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is **zero**. For $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the first row contains the letters **x,y,z**. So the plane containing **B** and **C** has the equation $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = 0$. When $\mathbf{B} = \mathbf{i} + \mathbf{j}$ and $\mathbf{C} = \mathbf{k}$ that equation is $\mathbf{x} - \mathbf{y} = 0$. $\mathbf{B}\times\mathbf{C}$ is $\mathbf{i} - \mathbf{j}$.

A 3 by 3 determinant splits into **three** 2 by 2 determinants. They come from rows 2 and 3, and are multiplied by the entries in row 1. With **i, j, k** in row 1, this determinant equals the **cross** product. Its **j** component is $-(\mathbf{a_1 b_3} - \mathbf{a_3 b_1})$, including the **minus** sign which is easy to forget.

10 (a) **True** ($\mathbf{A}\times\mathbf{B}$ is a vector, $\mathbf{A}\cdot\mathbf{B}$ is a number) (b) **True** (Equation (1) becomes $0 = |\mathbf{A}|^2|\mathbf{B}|^2$ so $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$) (c) **False:** $\mathbf{i}\times(\mathbf{j}) = \mathbf{i}\times(\mathbf{i}+\mathbf{j})$

16 $|\mathbf{A}\times\mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A}\cdot\mathbf{B})^2$ which is $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$. Multiplying and simplifying leads to $(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2$ which confirms $|\mathbf{A}\times\mathbf{B}|$ in eq. (6).

26 The plane has normal $\mathbf{N} = (\mathbf{i}+\mathbf{j})\times\mathbf{k} = \mathbf{i}\times\mathbf{k} + \mathbf{j}\times\mathbf{k} = -\mathbf{j} + \mathbf{i}$. So the plane is $x - y = d$. If the plane goes through the origin, its equation is $x - y = 0$.

42 The two sides going out from (a_1, b_1) are $(a_2 - a_1)\mathbf{i} + (b_2 - b_1)\mathbf{j}$ and $(a_3 - a_1)\mathbf{i} + (b_3 - b_1)\mathbf{j}$. The cross product gives the area of the parallelogram as $|(a_2 - a_1)(b_3 - b_1) - (a_3 - a_1)(b_2 - b_1)|$. Divide by 2 for triangle.

48 The triple vector product in Problem 47 is $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$. Take the dot product with **D**. The right side is easy: $(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})$. The left side is $((\mathbf{A}\times\mathbf{B})\times\mathbf{C})\cdot\mathbf{D}$ and the vectors $\mathbf{A}\times\mathbf{B}, \mathbf{C}, \mathbf{D}$ can be put in any cyclic order (see "useful facts" about volume of a box, after Theorem 11G). We choose $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$.

11.4 Matrices and Linear Equations (page 433)

A system of linear equations is written $A\mathbf{x} = \mathbf{b}$ or in this book $A\mathbf{u} = \mathbf{d}$. The coefficient matrix A multiplies the unknown vector **u**. This matrix-vector multiplication is set up so that it reproduces the given linear equations:

$$\begin{aligned} 2x + 5y &= 9 \\ x + 3y &= 5 \end{aligned} \quad \text{becomes} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix} = \mathbf{d}$$

Then the solution is $\mathbf{u} = A^{-1}\mathbf{d}$. It can be found in three or more ways:

(a) Find A^{-1} explicitly. (This matrix satisfies $A\,A^{-1} = \mathbf{I}$.) Then multiply $\mathbf{A}^{-1}$ times **d**.

(b) Use Cramer's Rule. Then x and y are ratios of determinants.

(c) Eliminate x from the second equation by subtracting $\frac{1}{2}$ of the first equation. From $3 - \frac{1}{2}(5)$ and $5 - \frac{1}{2}(9)$ you get $\frac{1}{2}y = \frac{1}{2}$. Then $y = 1$. Back substitution gives $x = 2$.

Elimination (c) is preferred for larger problems. For a 2 by 2 problem it is reasonable to know A^{-1}:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

The multiplication $A^{-1}\mathbf{d} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 5 \end{bmatrix}$ gives $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ which agrees with elimination.

Multiply by dot products $3 \cdot 9 - 5 \cdot 5 = 2$ and $-1 \cdot 9 + 2 \cdot 5 = 1$ or by linear combination of columns :

$$9\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

1. Write the equations $\begin{aligned} x + 3y &= 0 \\ 2x + 4y &= 10 \end{aligned}$ as $A\mathbf{u} = \mathbf{d}$. Solve by A^{-1} and also by elimination.

   - $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ has $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$. Multiply by $\mathbf{d} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$ to get $\mathbf{u} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$.
   - Eliminate x by subtracting two times the first equation from the second equation. This leaves $-2y = 10$. Then $y = -5$. Back substitution gives $x = 15$.

Projection onto a plane We look for the combination $\mathbf{p} = x\,\mathbf{a} + y\,\mathbf{b}$ that is closest to a given vector $\mathbf{d}$. It is the projection of $\mathbf{d}$ onto the plane of $\mathbf{a}$ and $\mathbf{b}$. The numbers x and y come from the "normal equations"

$$\begin{aligned} (\mathbf{a}\cdot\mathbf{a})\,x + (\mathbf{a}\cdot\mathbf{b})\,y &= \mathbf{a}\cdot\mathbf{d} \\ (\mathbf{b}\cdot\mathbf{a})\,x + (\mathbf{b}\cdot\mathbf{b})\,y &= \mathbf{b}\cdot\mathbf{d} \end{aligned}$$

2. Project the vector $\mathbf{d} = (1,2,4)$ onto the plane of $\mathbf{a} = (1,1,1)$ and $\mathbf{b} = (1,2,3)$. Interpret this projection as **least squares fitting by a straight line.**

   - After computing dot products, the normal equations are $3x + 6y = 7$ and $6x + 14y = 17$. Subtract 2 times the first to get $2y = 3$. Then $y = \frac{3}{2}$ and $x = -\frac{2}{3}$. The projection is $\mathbf{p} = -\frac{2}{3}(1,1,1) + \frac{3}{2}(1,2,3)$.
   - To see the line-fitting problem, write down $x\,\mathbf{a} + y\,\mathbf{b} \approx \mathbf{d}$ (this has no solution!):

$$x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + y &\approx 1 \\ x + 2y &\approx 2 \\ x + 3y &\approx 4. \end{aligned}$$

The straight line $f(t) = x + yt$ has intercept x and slope y. We are trying to make it go through the three points (1,1), (2,2), and (3,4). This is doomed to failure. If those points were on a line, we could solve our three equations. We can't. The best solution (the least squares solution) has $x = -\frac{2}{3}$ and $y = \frac{3}{2}$. That line $f(t) = -\frac{2}{3} + \frac{3}{2}t$ comes closest to the three points. It minimizes the sum of squares of the three errors.

Read-throughs and selected even-numbered solutions :

The equations $3x + y = 8$ and $x + y = 6$ combine into the vector equation $\mathbf{x}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \mathbf{y}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \mathbf{d}$. The left side is $A\mathbf{u}$ with coefficient matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and unknown vector $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$. The determinant of $\mathbf{A}$ is **2**, so this problem is not **singular**. The row picture shows two intersecting **lines**. The column picture shows

$x\mathbf{a} + y\mathbf{b} = \mathbf{d}$, where $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The inverse matrix is $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$. The solution is $\mathbf{u} = A^{-1}\mathbf{d} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

A matrix-vector multiplication produces a vector of dot **products** from the rows, and also a combination of the **columns**:

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}\begin{bmatrix} \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\cdot\mathbf{u} \\ \mathbf{B}\cdot\mathbf{u} \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\mathbf{a} + y\mathbf{b} \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

If the entries are a, b, c, d, the determinant is $D = \mathbf{ad} - \mathbf{bc}$. A^{-1} is $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ divided by D. Cramer's Rule shows components of $\mathbf{u} = A^{-1}\mathbf{d}$ as ratios of determinants: $x = (\mathbf{b_2 d_1} - \mathbf{b_1 d_2})/D$ and $y = (\mathbf{a_1 d_2} - \mathbf{a_2 d_1})/D$.

A matrix-matrix multiplication MV yields a matrix of dot products, from the rows of M and columns of $\mathbf{V}$:

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}\cdot\mathbf{v}_1 & \mathbf{A}\cdot\mathbf{v}_2 \\ \mathbf{B}\cdot\mathbf{v}_1 & \mathbf{B}\cdot\mathbf{v}_2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}.$$

The last line contains the **identity** matrix, denoted by I. It has the property that $IA = AI = \mathbf{A}$ for every matrix A, and $I\mathbf{u} = \mathbf{u}$ for every vector $\mathbf{u}$. The inverse matrix satisfies $A^{-1}A = \mathbf{I}$. Then $A\mathbf{u} = \mathbf{d}$ is solved by multiplying both sides by $\mathbf{A^{-1}}$, to give $\mathbf{u} = \mathbf{A^{-1}d}$. There is no inverse matrix when $\mathbf{det\ A = 0}$.

The combination $x\mathbf{a} + y\mathbf{b}$ is the projection of $\mathbf{d}$ when the error $\mathbf{d - xa - yb}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{a}$ = (1,1,1), $\mathbf{b}$ = (1,2,3), and $\mathbf{d}$ = (0,8,4), the equations for x and y are $\mathbf{3x + 6y = 12}$ **and** $\mathbf{6x + 14y = 28}$. Solving them also gives the closest **line** to the data points (1,0), **(2,8)**, and (3,4). The solution is $x = 0, y = 2$, which means the best line is **horizontal**. The projection is $0\mathbf{a} + 2\mathbf{b} = (\mathbf{2, 4, 6})$. The three error components are $(\mathbf{-2, 4, -2})$. Check perpendicularity: $(\mathbf{1,1,1})\cdot(\mathbf{-2,4,-2}) = 0$ and $(\mathbf{1,2,3})\cdot(\mathbf{-2,4,-2}) = 0$. Applying calculus to this problem, x and y minimize the sum of squares $E = (\mathbf{-x - y})^2 + (\mathbf{8 - x - 2y})^2 + (\mathbf{4 - x - 3y})^2$.

8 The solution is $x = \frac{d-b}{ad-bc}, y = \frac{a-c}{ad-bc}$ (ok to use Cramer's Rule). The solution breaks down if $ad = bc$.

$$\frac{d-b}{ad-bc}\begin{bmatrix} a \\ c \end{bmatrix} + \frac{a-c}{ad-bc}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

12 With $A = I$ the equations are $\begin{matrix} 1x + 0y = d_1 \\ 0x + 1y = d_2 \end{matrix}$. Then $x = \dfrac{\begin{vmatrix} d_1 & 0 \\ d_2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = d_1$ and $y = \dfrac{\begin{vmatrix} 1 & d_1 \\ 0 & d_2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = d_2$.

22 Problem 21 is $\begin{matrix} .96x + .02y = d_1 \\ .04x + .98y = d_2. \end{matrix}$ The sums down the columns of $\mathbf{A}$ are .96 +.04 = 1 and .02 + .98 = 1. Reason: Everybody has to be accounted for. Nobody is lost or gained. Then $x + y$ (total population before move) equals $d_1 + d_2$ (total population after move).

34 $AB = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$ has $(AB)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix}$. Check that this is $B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$.

40 Compute $\mathbf{a}\cdot\mathbf{a} = 3$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = 2$ and $\mathbf{b}\cdot\mathbf{b} = 6$ and $\mathbf{a}\cdot\mathbf{d} = 5$ and $\mathbf{b}\cdot\mathbf{d} = 6$. The normal equation (14) is $\begin{matrix} 3x + 2y = 5 \\ 2x + 6y = 6 \end{matrix}$ with solution $x = \frac{18}{14} = \frac{9}{7}$ and $y = \frac{8}{14} = \frac{4}{7}$. The nearest combination $x\mathbf{a} + y\mathbf{b}$ is $\mathbf{p} = (\frac{5}{7}, \frac{13}{7}, \frac{17}{7})$. The vector of three errors is $\mathbf{d} - \mathbf{p} = (\frac{2}{7}, -\frac{6}{7}, \frac{4}{7})$. It is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. The best straight line is $f = x + yt = \frac{9}{7} + \frac{4}{7}t$.

11.5 Linear Algebra (page 443)

This section is about 3 by 3 matrices A, leading to three equations $A\mathbf{u} = \mathbf{d}$ in three unknowns. The ideas of A^{-1} and determinants and elimination still apply – but the formulas are beginning to get complicated. The determinant formula (six terms) we know from Section 11.3. The inverse divides by this determinant D. ***When*** $D = 0$ ***the matrix A has no inverse.*** Such a matrix is called ***singular***. Otherwise see pages 438-439 for A^{-1}.

Again we can solve $A\mathbf{u} = \mathbf{d}$ in three ways: use A^{-1} which has D in the denominator, or use Cramer's Rule which has D in *every* denominator, or use elimination which will fail if $D = 0$.

1. Solve the three equations $\begin{matrix} x & + & y & +z & = & 5 \\ x & + & 2y & +2z & = & 9 \\ & & y & -z & = & 0 \end{matrix}$

   - My first choice is elimination. Subtract equation 1 from equation 2 to get $y + z = 4$. Subtract this from equation 3 to get $-2z = -4$. Then $z = 2$. Then $y = 2$. Then $x = 1$. The solution is $\mathbf{u} = (1,2,2)$.

   The determinant of $\mathbf{A}$ is $D = -2 + 1 + 0 - 2 + 1 + 0 = -2$. Find A^{-1} from the formulas:

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} -4 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}. \text{ Check } AA^{-1} = I. \text{ Then multiply } A^{-1}\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

   The 1993 textbook "Introduction to Linear Algebra" by Gilbert Strang goes beyond these formulas. In calculus we are seeing lines and planes as special cases of curves and surfaces. In linear algebra we see lines and planes as special cases of n-dimensional vector spaces. Please take that course – it is truly useful.

Read-throughs and selected even-numbered solutions :

Three equations in three unknowns can be written as $A\mathbf{u} = \mathbf{d}$. The **vector u** has components x, y, z and **A** is a **3 by 3 matrix.** The row picture has a **plane** for each equation. The first two planes intersect in a **line,** and all three planes intersect in a **point,** which is **u**. The column picture starts with vectors **a,b,c** from the columns of **A** and combines them to produce $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$. The vector equation is $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$.

The determinant of **A** is the triple product $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$. This is the volume of a box, whose edges from the origin are $\mathbf{a}, \mathbf{b}, \mathbf{c}$. If det $A = \mathbf{0}$ then the system is **singular**. Otherwise there is an **inverse** matrix such that $A^{-1}A = \mathbf{I}$ (the **identity** matrix). In this case the solution to $A\mathbf{u} = \mathbf{d}$ is $\mathbf{u} = \mathbf{A}^{-1}\mathbf{d}$.

The rows of A^{-1} are the cross products $\mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}, \mathbf{a}\times\mathbf{b}$, divided by D. The entries of A^{-1} are 2 by 2 **determinants**, divided by D. The upper left entry equals $(\mathbf{b_2c_3} - \mathbf{b_3c_2})/\mathbf{D}$. The 2 by 2 determinants needed for a row of A^{-1} do not use the corresponding **column** of A.

The solution is $\mathbf{u} = A^{-1}\mathbf{d}$. Its first component x is a ratio of determinants, $|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|$ divided by $|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|$. Cramer's Rule breaks down when det $A = \mathbf{0}$. Then the columns **a,b,c** lie in the same **plane.** There is no solution to $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$, if $\mathbf{d}$ is not on that **plane.** In a singular row picture, the intersection of planes 1 and 2 is **parallel** to the third plane.

In practice $\mathbf{u}$ is computed by **elimination.** The algorithm starts by subtracting a multiple of row 1 to eliminate x from **the second equation.** If the first two equations are $x - y = 1$ and $3x + z = 7$, this elimination step leaves $\mathbf{3y + z = 4}$. Similarly x is eliminated from the third equation, and then $\mathbf{y}$ is eliminated. The equations are solved by back **substitution.** When the system has no solution, we reach an impossible equation like $\mathbf{1} = \mathbf{0}$. The example $x - y = 1, 3x + z = 7$ has no solution if the third equation is $\mathbf{3y + z = 5}$.

8
$$\begin{aligned} x + 2y + 2z &= 0 \\ 2x + 3y + 5z &= 0 \\ 2y + 2z &= 8 \end{aligned} \;\longrightarrow\; \begin{aligned} x + 2y + 2z &= 0 \\ -y + z &= 0 \\ 2y + 2z &= 8 \end{aligned} \;\longrightarrow\; \begin{aligned} x + 2y + 2z &= 0 \\ -y + z &= 0 \\ 4z &= 8 \end{aligned} \;\longrightarrow\; \begin{aligned} x &= -8 \\ y &= 2 \\ z &= 2 \end{aligned}$$

18 Choose $\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as right side. The same steps as in Problem 16 end with $y - 3z = 0$ and $-y + 3z = 1$.

Addition leaves $\mathbf{0} = \mathbf{1}$. *No solution. Note:* The left sides of the three equations add to zero.

There is a solution only if the right sides (components of $\mathbf{d}$) also add to zero.

20 $BC = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -6 \\ 2 & 2 & -18 \end{bmatrix}$ and $CB = \begin{bmatrix} -20 & -13 & 1 \\ 4 & 2 & -1 \\ 16 & 11 & 0 \end{bmatrix}$. The columns of CB add to zero

(they are combinations of columns of C, and those add to zero). BC and CB are singular because C is.

22 $2A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 4 & 12 \\ 0 & 0 & 6 \end{bmatrix}$ has determinant 48 which is **8** times det A. If an n by n matrix is multiplied by 2,

the determinant is multiplied by 2^n. Here $2^3 = 8$.

28 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ These are "**even**"

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ These are "**odd**"

11 Chapter Review Problems

Review Problems

R1 If the vectors $\mathbf{v}$ and $\mathbf{w}$ form two sides of a triangle, the third side has squared length $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} - 2\mathbf{v}\cdot\mathbf{w}$. How is this connected to the Law of Cosines?

R2 Find unit vectors in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

R3 Explain why the derivative of a dot product $\mathbf{v}(t)\cdot\mathbf{w}(t)$ is $\mathbf{v}(t)\cdot\mathbf{w}'(t) + \mathbf{v}'(t)\cdot\mathbf{w}(t)$.

R4 Find a plane through (1,1,1) that is perpendicular to the plane $2x + 2y - z = 3$.

R5 How far is the origin from the plane through (1,0,0), (0,2,0), (0,0,4)? What is the nearest point?

R6 Project $\mathbf{B} = (1,4,4)$ onto the vector $\mathbf{A} = (2,2,-1)$. Then project the projection back onto $\mathbf{B}$.

R7 Find the dot product and cross product of $\mathbf{A} = (3,1,2)$ and $\mathbf{B} = (1,-1,-1)$.

R8 Starting from a 2 by 2 matrix with entries a, b, c, d, fill out the other 5 entries of a 3 by 3 matrix that has the same determinant.

R9 Multiply $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$. Compute the determinant of AB and check that it equals the determinant of A times the determinant of B.

R10 Fit the three points (0,1), (1,3), (2,3) by the closest line $f = x + yt$ (least squares).

Drill Problems

D1 Find the lengths and dot products of $\mathbf{A} = (1,2,1)$ and $\mathbf{B} = (1,1,2)$. Check the inequality $|\mathbf{A}\cdot\mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|$.

D2 Find the normal vector to the plane $3x - y - z = 0$ and find a vector *in* the plane.

D3 Multiply $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ times A^{-1} after computing A^{-1}.

D4 For the same matrix compute A^2 and $2A$ and their determinants.

D5 Find the area of the parallelogram with $\mathbf{u} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j}$ as two sides.

D6 Find the volume of the box with those sides $\mathbf{u}$ and $\mathbf{v}$ and third side $\mathbf{w} = \mathbf{j} + 2\mathbf{k}$.

D7 Write down two equations in x and y that have no solution. Change the right sides of those equations to produce infinitely many solutions. Change the left side to produce one unique solution and find it.