

CHAPTER 10 INFINITE SERIES

10.1 The Geometric Series (page 373)

The advice in the text is: *Learn the geometric series.* This is the most important series and also the simplest. The pure geometric series starts with the constant term 1: $1 + x + x^2 + \dots = \frac{1}{1-x}$. With other symbols $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. The ratio between terms is x (or r). Convergence requires $|x| < 1$ (or $|r| < 1$).

A closely related geometric series starts with any number a , instead of 1. Just multiply everything by a : $a + ax + ax^2 + \dots = \frac{a}{1-x}$. With other symbols $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. As a particular case, choose $a = x^{N+1}$. The geometric series has its first terms chopped off:

$$\text{Tail end: } x^{N+1} + x^{N+2} + \dots = \frac{x^{N+1}}{1-x} \quad \text{or} \quad \sum_{n=N+1}^{\infty} r^n = \frac{r^{N+1}}{1-r}.$$

Now do the opposite. *Keep the first terms and chop off the last terms.* Instead of an infinite series you have a finite series. The sum looks harder at first, but not after you see where it comes from:

$$\text{Front end: } 1 + x + \dots + x^N = \frac{1-x^{N+1}}{1-x} \quad \text{or} \quad \sum_{n=0}^N r^n = \frac{1-r^{N+1}}{1-r}.$$

The proof is in one line. *Start with the whole series and subtract the tail end.* That leaves the front end. It is the difference between $\frac{1}{1-x}$ and $\frac{x^{N+1}}{1-x}$ – complete sum minus tail end.

Again you can multiply everything by a . The front end is a finite sum, and the restriction $|x| < 1$ or $|r| < 1$ is now omitted. We only avoid $x = 1$ or $r = 1$, because the formula gives $\frac{0}{0}$. (But with $x = 1$ the front end is $1 + 1 + \dots + 1$. We don't need a formula to add up 1's.)

Personal note: I just heard a lecture on the mathematics of finance. After two minutes the lecturer wrote down a series – you can guess what it was. The geometric series is the most important there too. It gives the cost of a contract to pay you \$1 every year, *forever*. You don't have to pay an infinite amount $1 + 1 + 1 + \dots$ up front, because of interest. Suppose the interest rate per year is i . What you pay now is multiplied by $(1+i)$ every year. By paying $\frac{1}{1+i}$ at present, you can collect \$1 after a year. Also pay $\frac{1}{(1+i)^2}$ to have another \$1 waiting after two years. The “present value” of \$1 to be collected after every year comes out to be $\frac{1}{i}$:

$$\frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \dots = \frac{\frac{1}{1+i}}{1 - \frac{1}{1+i}}. \quad \text{Multiply by } \frac{1+i}{1+i} \text{ to find } \frac{1}{i}.$$

Here is a quick way to explain that. If you pay $\frac{1}{i}$ now, it earns $i(\frac{1}{i}) = \$1$ interest after a year. So you collect that dollar, and leave the $\frac{1}{i}$. The second year interest is again \$1, which you collect. This goes on forever. We are assuming constant interest rate and no bank charge, both unfortunately false.

What if you only want \$1 for N years? Now the annuity is not “perpetual” – it ends. What you pay now is smaller. Use the formula for a finite sum $a + a^2 + \dots + a^N$ (the front end, starting with $a = \frac{1}{1+i}$). The N -year annuity costs

$$\frac{1}{1+i} + \frac{1}{1+i^2} + \dots + \frac{1}{(1+i)^N} = \frac{\frac{1}{1+i} - \frac{1}{(1+i)^{N+1}}}{1 - \frac{1}{1+i}} = \frac{1}{i} \left[1 - \frac{1}{(1+i)^N} \right].$$

Again the management professor had a good way to understand this. The contract that pays forever costs $\frac{1}{i}$. But you only hold it for N years. Then you give the tail end back and stop collecting the annual dollar. At that future time the tail end is worth $\frac{1}{i}$, because it still pays \$1 forever. But something worth $\frac{1}{i}$ then is only worth $\frac{1}{i} \frac{1}{(1+i)^N}$ now. That is the present value of what you are giving back, so subtract it from the present value $\frac{1}{i}$ of the infinite contract to get your cost $\frac{1}{i} \left(1 - \frac{1}{(1+i)^N} \right)$.

EXAMPLE. Suppose the interest rate is 100%, so $i = 1$. An investment of $\frac{1}{i} = \$1$ will bring you a dollar each year forever. If you only want that dollar for $N = 10$ years, your present investment can be a little smaller. It is smaller by $\frac{1}{i} \frac{1}{(1+i)^N} = \frac{1}{2^{10}}$, which is not much! This is $\frac{1}{1024}$ dollars or about $\frac{1}{10}$ cent. For a ten-year annuity you pay 99.9 cents, for a perpetual annuity you pay the full 100 cents.

The reason for those strange numbers is the 100% interest rate. A dollar in ten years is only worth $\frac{1}{10}$ cent now. With galloping inflation like that, all calculations look unreasonable. You see why 100% interest or more (which does happen!) can destroy an economy.

NOTE: Annuities are also discussed in Section 6.6. You could redo the calculation based on 10% interest, so $i = 0.1$. How much is a contract for \$1 a year in perpetuity? (That means forever.) How much do you pay now to receive \$1 a year for $N = 10$ years?

In Problems 1 – 5, find the sum of the geometric series from the formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. The ratio r of successive terms must satisfy $|r| < 1$ for the series to converge.

1. $24 - 12 + 6 - 3 + \dots$ This has $r = -\frac{1}{2}$.

2. $\sqrt{125} + \sqrt{25} + \sqrt{5} + \dots$ This has $r = \frac{1}{\sqrt{5}}$.

3. $1 + (x - \pi) + (x - \pi)^2 + (x - \pi)^3 + \dots$ This has $r = x - \pi$.

4. $6 - \frac{12}{x} + \frac{24}{x^2} - \frac{48}{x^3} + \dots$ This has $r = -\frac{2}{x}$.

5. $5.282828 \dots$ (subtract 5, then add it back) This has $r = \frac{1}{100}$.

- 1. Starting from $a = 24$ with $r = -\frac{1}{2}$ the sum is $\frac{24}{1-(-\frac{1}{2})} = \frac{24}{\frac{3}{2}} = (\frac{2}{3})24 = 16$.

- 2. The common ratio is $\frac{1}{\sqrt{5}}$. The sum is $\frac{\sqrt{125}}{1-\frac{1}{\sqrt{5}}} = \frac{25}{\sqrt{5}-1}$.

- 3. Each term equals the previous term multiplied by $(x - \pi)$, so $r = x - \pi$. The sum is $\frac{1}{1-(x-\pi)} = \frac{1}{\pi+1-x}$. This does not work for all values of x ! The formula $\frac{a}{1-r}$ is only correct when $|r| = |x - \pi| < 1$, or $\pi - 1 < x < \pi + 1$. For other values of x the series does not have a sum.

- 4. The common ratio is $r = -\frac{2}{x}$, so the sum is $\frac{6}{1+\frac{2}{x}} = \frac{6x}{x+2}$. This is valid only when $|r| = |\frac{2}{x}| < 1$, or $|x| > 2$. For example, the series diverges for $x = 1.5$ but for $x = -4$ the sum is 12.

- 5. Every repeating decimal involves a geometric series. The number $5.282828 \dots$ can be written $5 + \frac{28}{100} + \frac{28}{10000} + \dots$. Subtract the 5 because it is not part of the pattern. The series $\frac{28}{100} + \frac{28}{10000} + \dots$ converges to $\frac{\frac{28}{100}}{1-\frac{1}{100}} = \frac{28}{99}$. Now add back the 5 to get the answer $5\frac{28}{99}$.

6. A ball is dropped from a height of 8 feet and rebounds 6 feet. After each bounce it rebounds $\frac{3}{4}$ of the distance it fell. How far will the ball travel before it comes to rest?

- The ball goes down 8, up 6 , down 6 , up $\frac{3}{4}(6) = \frac{9}{2}$, down $\frac{9}{2}$, up $\frac{27}{8}, \dots$. Every distance after the original 8 is repeated. The total is $8 + 2(6) + 2(\frac{9}{2}) + 2(\frac{27}{8}) + \dots$. Subtract the 8, because it doesn't fit the pattern. The rest of the series is $2[6 + 6(\frac{3}{4}) + 6(\frac{3}{4})^2 + \dots] = 2[\frac{6}{1-\frac{3}{4}}] = 48$. The total distance is $8 + 48 = 56$ feet.

7. (Draw your own figure above) A robot is programmed to move in this way: First go 1 km east. Turn 90° to go $\frac{1}{2}$ km north. Turn left again to go $\frac{1}{4}$ km west. After each left turn, the distance is half the previous step. If the robot starts at $(0, 0)$ where does it end up?

- First look at the east-west coordinate x . This changes every other step following the pattern $1, 1 - \frac{1}{4}, 1 - \frac{1}{4} + \frac{1}{16}, 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64}, \dots$. The limiting x coordinate is $1 - \frac{1}{4} + \frac{1}{16} - \dots = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5}$.

Similarly, the y coordinate ends up at $\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \dots = \frac{\frac{1}{2}}{1 + \frac{1}{4}} = \frac{2}{5}$. The robot “spirals” inward to the point $(\frac{4}{5}, \frac{2}{5})$.

8. (This is 10.1.6) Multiply $(1 + x + x^2 + \dots)$ times $(1 - x + x^2 - \dots)$. Compare with equation (14).

- The first terms are $1(1 - x + x^2 - \dots) + x(1 - x + x^2 - \dots)$. Those combine to give 1. The next terms are $x^2(1 - x + x^2 - \dots) + x^3(1 - x + x^2 - \dots)$. Those combine to give x^2 . By grouping the multiplication two steps at a time, the final answer becomes $1 + x^2 + x^4 + \dots$. This answer equals $\frac{1}{1-x^2}$. It is the geometric series in equation (14) with $r = x^2$. Check: The original series add to $\frac{1}{1+x}$ and $\frac{1}{1-x}$. Multiplication gives $\frac{1}{1-x^2}$.

9. Find two infinite series that add up to $\ln 7$.

- There are three candidates: (10 a), (10 b), and (13) on pages 370 – 371. Formula (10 b) cannot be used directly for $\ln(1 + x) = \ln 7$, because we would need $x = 6$. *These formulas do not apply for $|x| > 1$.* One way out is to use (10a) with $x = \frac{6}{7}$. This would give $-\ln(1 - \frac{6}{7}) = -\ln(\frac{1}{7}) = \ln 7$. So one answer is $(\frac{6}{7}) + \frac{1}{2}(\frac{6}{7})^2 + \frac{1}{3}(\frac{6}{7})^3 + \dots = 1.946$.

A faster method is to use the series (13) for $\ln(\frac{1+x}{1-x})$. Set $\frac{1+x}{1-x} = 7$ to get $x = \frac{3}{4}$. Substitute this x in the series: $\ln 7 = 2[(\frac{3}{4}) + \frac{1}{3}(\frac{3}{4})^3 + \frac{1}{5}(\frac{3}{4})^5 + \dots] = 1.946$.

For any positive number a , we solve $\frac{1+x}{1-x} = a$ to find $x = \frac{a-1}{a+1}$. Substitute this x into the series (13) to get $\ln(\frac{1+x}{1-x}) = \ln a$. The number x will always be between -1 and 1 .

10. Find a series for $\frac{1}{(1-x)^3}$ that converges for $|x| < 1$.

- Here are two ways. Equation (5) on page 369 gives a series for $\frac{1}{(1-x)^2}$. Multiply by the series $1 + x + x^2 + \dots = \frac{1}{1-x}$ to obtain the series for $\frac{1}{(1-x)^3}$. A better way is to take the *derivative* of the series for $\frac{1}{(1-x)^2}$ to get $\frac{2}{(1-x)^3}$:
The derivative of $1 + 2x + 3x^2 + 4x^3 + \dots + (n+1)x^n + \dots$ is $2 + 6x + 12x^2 + \dots + n(n+1)x^{n-1} + \dots$. Dividing by 2 gives $\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + \dots + \frac{n(n+1)}{2}x^{n-1} + \dots$.

10. What is the average number of times you would have to roll a pair of dice to get a seven?

- Six out of the 36 combinations on the two dice add to seven: $1 + 6, 2 + 5, 3 + 4, \dots, 6 + 1$. On the first roll, your chances of getting a seven are $p_1 = \frac{1}{6}$. The probability of *not* getting a seven on the first roll but getting one on the second roll is $p_2 = (\frac{5}{6})(\frac{1}{6})$. The chance of two failures followed by success is $p_3 = (\frac{5}{6})^2(\frac{1}{6})$.
- The mean number of rolls is $\sum np_n = 1(\frac{1}{6}) + 2(\frac{5}{6})(\frac{1}{6}) + 3(\frac{5}{6})^2(\frac{1}{6}) + \dots$. This infinite series fits the pattern of equation (5) on page 369 with $x = \frac{5}{6}$:
 $\frac{1}{6}(1 + 2(\frac{5}{6}) + 3(\frac{5}{6})^2 + 4(\frac{5}{6})^3 + \dots) = \frac{1}{6}(\frac{1}{(1-\frac{5}{6})^2}) = \frac{1}{6}(6^2) = \text{average of 6 rolls.}$

11. Use a series to compute $\int_0^{1/2} \tan^{-1} x \, dx$, correct to 4 places.

- Equation (18) on page 371 tells us that $\tan^{-1} x = x - \frac{1}{3}x^3 + 15x^5 - \dots$ provided $|x| < 1$. Integrate from 0 to $\frac{1}{2}$:

$$\begin{aligned} \int_0^{1/2} \tan^{-1} x \, dx &= [\frac{1}{2}x^2 - \frac{1}{3 \cdot 4}x^3 + \frac{1}{5 \cdot 6}x^5 - \dots]_0^{1/2} \\ &\approx \frac{1}{2}(\frac{1}{2})^2 - \frac{1}{3 \cdot 4}(\frac{1}{2})^4 + \frac{1}{5 \cdot 6}(\frac{1}{2})^6 - \frac{1}{7 \cdot 8}(\frac{1}{2})^8 + \frac{1}{9 \cdot 10}(\frac{1}{2})^{10} \approx .1203 \end{aligned}$$

Read-throughs and selected even-numbered solutions :

The geometric series $1 + x + x^2 + \dots$ adds up to $1/(1 - x)$. It converges provided $|x| < 1$. The sum of n terms is $(1 - x^{n+1})/(1 - x)$. The derivatives of the series match the derivatives of $1/(1 - x)$ at the point $x = 0$, where the n th derivative is $n!$ The decimal 1.111... is the geometric series at $x = .1$ and equals the fraction $10/9$. The decimal .666... multiplies this by .6. The decimal .999... is the same as 1.

The derivative of the geometric series is $1/(1 - x)^2 = 1 + 2x + 3x^2 + \dots$. This also comes from squaring the geometric series. By choosing $x = .01$, the decimal 1.02030405 is close to $(100/99)^2$. The differential equation $dy/dx = y^2$ is solved by the geometric series, going term by term starting from $y(0) = 1$.

The integral of the geometric series is $-\ln(1 - x) = x + x^2/2 + \dots$. At $x = 1$ this becomes the harmonic series, which diverges. At $x = \frac{1}{2}$ we find $\ln 2 = \frac{1}{2} + (\frac{1}{2})^2/2 + (\frac{1}{2})^3/3 + \dots$. The change from x to $-x$ produces the series $1/(1 + x) = 1 - x + x^2 - x^3 + \dots$ and $\ln(1 + x) = x - x^2/2 + x^3/3 - \dots$.

In the geometric series, changing to x^2 or $-x^2$ gives $1/(1 - x^2) = 1 + x^2 + x^4 + \dots$ and $1/(1 + x^2) = 1 - x^2 + x^4 - \dots$. Integrating the last one yields $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots = \tan^{-1}x$. The angle whose tangent is $x = 1$ is $\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \dots$. Then substituting $x = 1$ gives the series $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \dots)$.

- 4 $1 + (1 - x) + (1 - x)^2 + \dots = \frac{1}{1 - (1 - x)} = \frac{1}{x}$; integration gives $\ln x = x - \frac{(1 - x)^2}{2} - \frac{(1 - x)^3}{3} - \dots + C$ and at $x = 1$ we find $C = -1$. Therefore $\ln x = -[(1 - x) + \frac{(1 - x)^2}{2} + \frac{(1 - x)^3}{3} + \dots]$. At $x = 0$ this is $-\infty = -\infty$.
- 18 $\frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{8}x^3 - \dots = \frac{x}{2} - (\frac{x}{2})^2 + (\frac{x}{2})^3 - \dots = \frac{\frac{x}{2}}{1 + \frac{x}{2}} = \frac{x}{2 + x}$.
- 20 $x - 2x^2 + 3x^3 - \dots = x(1 - 2x + 3x^2 - \dots) = [\text{change } x \text{ to } -x \text{ in equation (5)}] = \frac{x}{(1 + x)^2}$.
- 26 $\int(1 + x^2 + x^4 + \dots)dx = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \frac{1}{2} \ln \frac{1 + x}{1 - x}$ by equation (13). This is $\int \frac{1}{1 - x^2} dx$ which is also $\tanh^{-1}x (+ C)$.
- 36 $y'(0) = [y(0)]^2 = b^2$; $y''(0) = 2y(0)y'(0) = 2b^3$; $y'''(0) = (\text{from second derivative of } y' = y^2) = 2y(0)y''(0) + 2y'(0)y'(0) = 6b^4$. Then $y(x) = b + b^2x + 2b^3(\frac{x^2}{2}) + 6b^4(\frac{x^3}{6}) + \dots = b(1 + bx + b^2x^2 + b^3x^3 + \dots)$ which is $y(x) = \frac{b}{1 - bx}$.
- 40 Note: The equations referred to should be (10) and (13). Choose $x = \frac{2}{3}$ in (10a): $\frac{2}{3} + \frac{1}{2}(\frac{2}{3})^2 + \dots = -\ln \frac{1}{3}$.

- (b) Each term of $\sum_1^\infty \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{16} + \dots$ is smaller than the corresponding term of $\sum_1^\infty \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$. The latter is a convergent geometric series. Therefore the series $\sum_1^\infty \frac{1}{n2^n}$ also converges.
- (c) Each term $\frac{1}{(n^4+n)^{1/3}}$ is smaller than $\frac{1}{n^{4/3}}$. Comparison with p -series $\sum \frac{1}{n^{4/3}}$ with $p = \frac{4}{3}$ gives convergence.
Other series like $\sum \frac{1}{(n^4-2n)^{1/3}}$ or $\sum \frac{1}{(n^4-n^3-1)^{1/3}}$ converge by the same comparison. (These are slightly larger than $\sum \frac{1}{n^{4/3}}$ but as $n \rightarrow \infty$ the behavior is the same. The "limit comparison test" applies.)
- (d) $\sum \frac{1}{e^n}$ is a convergent geometric series, because $r = \frac{1}{e}$ is less than 1. Our series is $\sum \frac{n}{e^n}$. When n gets large, e^n completely dominates n and the series remains convergent. To be more precise, notice that $n < 2^n$. Then $\frac{n}{e^n} < \frac{2^n}{e^n} = (\frac{2}{e})^n$. Since $\frac{2}{e} < 1$, we know that $\sum (\frac{2}{e})^n$ is a convergent geometric series. By comparison $\sum \frac{n}{e^n}$ also converges.
- (e) The terms $\frac{\sqrt{n+6}}{n(n+1)(n+2)}$ behave like $\frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$ when n gets large. Since $\sum \frac{1}{n^{5/2}}$ converges, so does the original series. Use the limit comparison test or show directly that $\sum \frac{3}{n^{5/2}}$ is larger. Increase $\sqrt{n+6}$ to $3\sqrt{n}$. Decrease the denominator $n(n+1)(n+2)$ to n^3 :
 $\frac{\sqrt{n+6}}{n(n+1)(n+2)} < \frac{3\sqrt{n}}{n^3} = \frac{3}{n^{5/2}}$. The larger series $\sum \frac{3}{n^{5/2}}$ converges. Therefore (e) converges.
- (f) The angle $\tan^{-1} n$ approaches $\frac{\pi}{2}$ from below. Therefore $\sum \frac{\tan^{-1} n}{n^2}$ is less than $\frac{\pi}{2} \sum \frac{1}{n^2}$. This is a convergent p -series (with $p = 2$.)

10C. Replace n by x and use the **Integral Test** 10C – provided the resulting function of x is decreasing to zero, and you can find its integral. The series $\sum_1^\infty f(n)$ converges or diverges together with the integral $\int_1^\infty f(x)dx$.

3. Determine the convergence of (a) $\sum_2^\infty \frac{1}{n(\ln n)^2}$ (b) $\sum_1^\infty \frac{n}{e^n}$ (c) $\sum \frac{6n}{(n^2+8)^{2/3}}$.

- (a) $\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty \frac{du}{u^2} = [-\frac{1}{u}]_{\ln 2}^\infty = \frac{1}{\ln 2}$. The integral converges so the series converges.
- (b) $\int_1^\infty \frac{x dx}{e^x} = \int_1^\infty x e^{-x} dx = [-x e^{-x} - e^{-x}]_1^\infty = \frac{2}{e}$. The series converges because the integral converges. Compare with problem 2(d) above. *There we had to find a comparison series. Here the integral test gives a comparison automatically.*
- (c) $\int_1^\infty \frac{6x dx}{(x^2+8)^{2/3}} = 3 \int_9^\infty u^{-2/3} du = [9u^{1/3}]_9^\infty = \infty$. The series diverges because the integral diverges. It could have been compared to the divergent p -series $\sum \frac{1}{n^{1/3}}$.

10D. **Ratio test** Compute the limit L of $\frac{a_{n+1}}{a_n}$, the ratio of the $(n+1)$ st term to the n th term. If $L < 1$ then the series converges. If $L > 1$ then the series diverges. If $L = 1$, the ratio test doesn't decide; look for a more careful comparison test. The ratio test is generally successful when the terms include a factorial. (The ratio of $(n+1)!$ to $n!$ is $n+1$).

4. Use the ratio test to determine the convergence of these five series:

(a) $\sum \frac{2^n}{(n+1)!}$ (b) $\sum n(\frac{2}{3})^n$ (c) $\sum \frac{n^3}{(\ln 2)^n}$ (d) $\sum \frac{(n+2)!5!}{n!5^n}$ (e) $\sum_1^\infty \frac{n^n}{n!}$.

- (a) The ratio $\frac{a_{n+1}}{a_n}$ is $\frac{2^{n+1}}{(n+2)!} \div \frac{2^n}{(n+1)!} = \frac{2^{n+1}(n+1)!}{(n+2)!2^n} = \frac{2}{n+2}$. This ratio $\frac{2}{n+2}$ goes to zero. Since the limit is $L = 0$, the series converges.
- (b) The ratio is $(n+1)(\frac{2}{3})^{n+1}$ divided by $(n)(\frac{2}{3})^n$. This is $(\frac{n+1}{n}) \cdot \frac{2}{3} \rightarrow \frac{2}{3}$. Since $L = \frac{2}{3}$, the series converges.

- (c) $\frac{(n+1)^3}{(\ln 2)^{n+1}} \div \frac{n^3}{(\ln 2)^n} = \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{\ln 2}$. The limit is $L = \frac{1}{\ln 2} > 1$, so this series diverges. (In fact the terms $\frac{n^3}{(\ln 2)^n}$ approach infinity. They fail the very first test, that $a_n \rightarrow 0$.)
- (d) Look at the terms $\frac{(n+2)!5!}{n!5^n}$. The first part $\frac{(n+2)!}{n!}$ is $(n+2)(n+1)$. This is about n^2 . But 5^n grows much faster. We expect convergence – now prove it by the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+3)!5!}{(n+1)!5^{n+1}} \cdot \frac{n!5^n}{(n+2)!5!} = \frac{(n+3)}{5(n+1)} \rightarrow L = \frac{1}{5}$$
. Convergence.
- (e) The ratio test for $\frac{n^n}{n!}$ leads to $L = e$ (divergence because n^n grows faster than $n!$).

$$\frac{(n+1)^{n+1}}{(n+1)!} \div \frac{n^n}{n!} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} = \left(\frac{n+1}{n}\right)^n \cdot \frac{n+1}{n+1} = \left(1 + \frac{1}{n}\right)^n \rightarrow e$$
. And $Le > 1$.

10E. The **Root Test** is useful when the terms of the series include an expression $(\quad)^n$. The test looks at the limit of $(a_n)^{1/n}$. If the limit L is less than 1, the series converges. If $L > 1$ the series diverges. As with the ratio test, the root test gives no answer if $L = 1$ – and this often happens.

5. Test these series for convergence: (a) $\sum \frac{1}{(\ln n)^n}$ (b) $\sum \frac{2^{n+1}}{n^n}$ (c) $\sum_1^\infty \frac{2^n}{n^{10}}$

- (a) The n th root of $a_n = \frac{1}{(\ln n)^n}$ is $\frac{1}{\ln n}$. Since $\frac{1}{\ln n} \rightarrow L = 0$, the series converges.
- (b) The n th root of $\frac{2^{n+1}}{n^n}$ is $\frac{2^{n+1}}{n^n} \rightarrow L = 0$. (Certainly $\sqrt[n]{2}$ approaches 1.) The series converges.
- (c) The n th root of $\frac{2^n}{n^{10}}$ is $\frac{2}{n^{10/n}}$. Its limit is $L = 2$ and the series diverges. (We used the fact that $n^{1/n} \rightarrow 1$. Its tenth power $n^{10/n}$ goes to $1^{10} = 1$.) This series also fails test 10A – its terms don't go to zero. That is always true if $L > 1$!

10F. The **Limit Comparison Test** shows that $\sum \frac{\ln n}{n^{1.7}}$ converges. Compare $a_n = \frac{\ln n}{n^{1.7}}$ with $c_n = \frac{1}{n^{1.1}}$. Note that $\sum c_n$ is convergent (p -series with $p = 1.1$). Since $\frac{a_n}{c_n} = \frac{\ln n}{n^{1.1}} \rightarrow 0$, the original series $\frac{\ln n}{n^{1.7}}$ converges.

In general we look at $L = \lim \frac{a_n}{c_n}$. Look for a comparison series $\sum c_n$ so that L is a finite number. *This test does not require $L < 1$.* If $\sum c_n$ converges then $\sum a_n$ converges. If $\sum c_n$ diverges and $L > 0$, the test gives the opposite answer: $\sum a_n$ also diverges.

6. Apply the limit comparison test to (a) $\sum \frac{n}{n^3+2}$ (divergent) and (b) $\sum \frac{n+7}{n^3-2}$ (convergent).

The ratio test and root test gives no decision. They find $L = 1$. **Comparison tests and integral tests are more precise than ratio and root tests.**

- (a) The terms $a_n = \frac{n}{n^3+2}$ behave like $c_n = \frac{1}{n^2}$ when n is large. The ratio is $\frac{a_n}{c_n} = \frac{n}{n^3+2} \div \frac{1}{n} = \frac{n^2}{n^3+2} \rightarrow L = 1$. The comparison series $\sum \frac{1}{n^2}$ diverges (p -series with $p = 1$). Then $\sum a_n$ diverges.
- (b) The terms $a_n = \frac{n+7}{n^3-2}$ behave like $c_n = \frac{1}{n^2}$ when n is large. The ratio is $\frac{a_n}{c_n} = \frac{n+7}{n^3-2} \div \frac{1}{n^2} = \frac{n^3+7n^2}{n^3-2} \rightarrow L = 1$. The comparison series $\sum \frac{1}{n^2}$ converges (p -series with $p = 2$). Then $\sum a_n$ converges.

Read-throughs and selected even-numbered solutions :

The convergence of $a_1 + a_2 + \dots$ is decided by the partial sums $s_n = a_1 + \dots + a_n$. If the s_n approach s , then $\sum a_n = s$. For the geometric series $1 + x + \dots$ the partial sums are $s_n = (1 - x^{n+1})/(1 - x)$. In that case $s_n \rightarrow 1/(1 - x)$ if and only if $|x| < 1$. In all cases the limit $s_n \rightarrow s$ requires that $a_n \rightarrow 0$. But the harmonic series $a_n = 1/n$ shows that we can have $a_n \rightarrow 0$ and still the series diverges.

The comparison test says that if $0 \leq a_n \leq b_n$ then $\sum a_n$ converges if $\sum b_n$ converges. In case a decreasing $y(x)$ agrees with a_n at $x = n$, we can apply the integral test. The sum $\sum a_n$ converges if and only if

$\int_1^\infty \mathbf{y}(\mathbf{x})d\mathbf{x}$ converges. By this test the p -series $\sum 1/n^p$ converges if and only if p is greater than 1. For the harmonic series ($p = 1$), $s_n = 1 + \cdots + 1/n$ is near the integral $f(n) = \ln n$.

The ratio test applies when $a_{n+1}/a_n \rightarrow L$. There is convergence if $|L| < 1$, divergence if $|L| > 1$, and no decision if $L = 1$ or -1 . The same is true for the root test, when $(a_n)^{1/n} \rightarrow L$. For a geometric- p -series combination $a_n = x^n/n^p$, the ratio a_{n+1}/a_n equals $x(n+1)^p/n^p$. Its limit is $L = x$ so there is convergence if $|x| < 1$. For the exponential $e^x = \sum x^n/n!$ the limiting ratio a_{n+1}/a_n or $x/(n+1)$ is $L = 0$. This series always converges because $n!$ grows faster than any x^n or n^p .

There is no sharp line between convergence and divergence. But if $\sum b_n$ converges and a_n/b_n approaches L , it follows from the limit comparison test that $\sum a_n$ also converges.

- 2 The series $\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots$ converges because it is below the comparison series $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} \cdots = .999 \cdots = 1$.
- 12 $\sum \frac{1}{\sqrt{n^2+10}}$ diverges by comparison with the smaller series $\sum \frac{1}{n+10}$ which diverges. Check that $\sqrt{n^2+10}$ is less than $n+10$.
- 24 The terms are $(\frac{n-1}{n})^n = (1 - \frac{1}{n})^n \rightarrow e^{-1}$ so the ratio approaches $L = \frac{e^{-1}}{e^{-1}} = 1$. (Divergent series because its terms don't approach zero.)
- 38 $\sum \frac{\ln n}{n}$ diverges by comparison with $\int_1^\infty \frac{\ln x}{x} dx = [\frac{1}{2}(\ln x)^2]_1^\infty = \infty$. ($\frac{\ln x}{x}$ decreases for $x > e$ and these later terms decide divergence. Another comparison is with $\sum \frac{1}{n}$.)
- 46 The first term is $\frac{1}{2 \ln 2}$. After that $\frac{1}{n \ln n} < \int_{n-1}^n \frac{dx}{x \ln x}$. The sum from 3 to n is below $\int_2^n \frac{dx}{x \ln x} = \ln(\ln n) - \ln(\ln 2)$. By page 377 the computer has not reached, $n = 3.2 \cdot 10^{19}$ in a million years. So the sum has not reached $\frac{1}{2 \ln 2} + \ln(\ln 3.2 \cdot 10^{19}) - \ln(\ln 2) < 5$.
- 58 $\sum \frac{1}{n^{\ln n}}$ converges by comparison with $\sum \frac{2}{n^2}$ (note $\ln n > 2$ beyond the 8th term). (Also $\sum \frac{1}{(\ln n)^{\ln n}}$ converges!)
- 60 $\ln 10^n = n \ln 10$ so the sum is $\frac{1}{\ln 10} \sum \frac{1}{n} = \infty$ (harmonic series diverges)
- 62 $n^{-1/n}$ approaches 1 so the series cannot converge

10.3 Convergence Tests: All Series (page 384)

We turn to series whose terms may be positive or negative. If you take absolute values to make all terms positive, you have the "absolute series." When the absolute series converges, the original series also converges. In this case the original series is said to **converge absolutely**. Some series converge but do not converge absolutely. These are **conditionally convergent**. They depend on some plus-minus cancellation for their partial sums to approach a limit. It is easiest to prove convergence when their signs are alternating and they pass the *alternating series test*: *The terms decrease in absolute value to zero.*

Such a series is at least conditionally convergent. It may or may not be absolutely convergent.

Remember to think about the limit of the terms: Unless $a_n \rightarrow 0$, the series cannot possibly converge. The

shrinking test comes first! In problems 1-6, decide if the alternating series converges: absolutely or conditionally?

1. $\sum (-1)^n \left(\frac{n^2-1}{n^2+1}\right)$ 2. $\sum (-1)^n \frac{n^2-1}{n^3+1}$ 3. $\sum (-1)^n \left(\frac{n^2-1}{\sqrt{n^7+1}}\right)$.

- Series 1 fails the shrinking test because $\frac{n^2-1}{n^2+1}$ does not approach zero.
- For series 2, the absolute series $\sum \frac{n^2-1}{n^3+1}$ diverges by comparison with $\sum \frac{1}{n}$. *The alternating series 2 is conditionally convergent.* Its terms decrease in absolute value, approaching zero. The decrease is not fast enough for the absolute series to converge. But the alternating series test is passed.
- Series 3 is absolutely convergent. Compare the terms with $\frac{n^2}{n^{7/2}} = \frac{1}{n^{3/2}}$. And $\sum \frac{1}{n^{3/2}}$ converges.

4. $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ 5. $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ 6. $\sum_2^{\infty} \frac{\sin n}{n \ln n}$.

- The absolute values $\frac{1}{\ln n}$ are decreasing to zero. By the alternating series test, series 4 converges. However, since $\sum \frac{1}{\ln n} > \sum \frac{1}{n} = \infty$, series 4 is not absolutely convergent.
- The integral test shows that $\sum \frac{1}{n \ln n}$ is divergent, because $\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u = \ln(\ln x) \rightarrow \infty$. But series 5 is conditionally convergent. The alternating terms approach zero.
- Series 6 has both positive and negative terms. *But it is not strictly alternating.* Is this one that you and your instructor cannot decide?

Important fact when the alternating series test is passed: If you use the first n terms to estimate the sum, the error can be no larger than the $(n+1)$ st term. By “error” we mean $|s_n - s| = |\text{partial sum minus infinite sum}|$. Figure 10.3 on page 383 explains why this is true. Suppose you use $a_1 - a_2 + a_3 - a_4$ to estimate the sum s . *The error is smaller than a_5* (because adding a_5 throws us beyond the final sum s).

Another way to express it: *s is between the partial sums s_n and s_{n+1} .* Use this in problems 7-9.

7. Section 10.1 derived $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$. The quadratic approximation is $\ln(1+x) \approx x - \frac{1}{2}x^2$. How close is this for $\ln 1.2$, when x is 0.2?

- The number $\ln 1.2$ is $\ln(1+0.2) = (0.2) - \frac{1}{2}(0.2)^2 + \frac{1}{3}(0.2)^3 - \dots$. This is a convergent alternating series. Its terms decrease to zero. The quadratic approximation $(0.2) - \frac{1}{2}(0.2)^2 = 0.18$ uses the first two terms. The error is no larger than the next term and has the same positive sign: $0 < \text{error} < \frac{1}{3}(0.2)^3 < 0.0027$.

In other words $0.18 < \ln 1.2 < 0.1827$. The third term 0.0027 throws us beyond $\ln 1.2 = 0.1823\dots$.

8. Section 10.4 will show that $\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots$. Compute $\frac{1}{e}$ to 4 places.

- The question is, how many terms do we need? The error is to be less than $0.00005 = 5 \times 10^{-5}$. Since $\frac{1}{7!} \approx 2.0 \times 10^{-4}$ and $\frac{1}{8!} \approx 2.5 \times 10^{-5}$, we do *not* need $\frac{1}{8!}$. Then $\frac{1}{e} \approx \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \approx 0.36786$.

9. How many terms of $\sum (-1)^{n+1} \frac{1}{n^3}$ give the sum to five decimal place accuracy?

- Five-place accuracy means error $< 0.00005 = 5 \times 10^{-6}$. If the series stops at $\pm \frac{1}{n^3}$, the error is less than the next term $\frac{1}{(n+1)^3}$. Set $\frac{1}{(n+1)^3} < 5 \times 10^{-6}$ to find $(n+1)^3 > 2 \times 10^5$ and $n+1 > 58.8$. Using $n = 58$ terms gives five-place accuracy.

Read-throughs and selected even-numbered solutions :

The series $\sum a_n$ is absolutely convergent if the series $\sum |a_n|$ is convergent. Then the original series $\sum a_n$ is also convergent. But the series $\sum a_n$ can converge without converging absolutely. That is called **conditional convergence**, and the series $1 - \frac{1}{2} + \frac{1}{3} - \dots$ is an example.

For alternating series, the sign of each a_{n+1} is **opposite** to the sign of a_n . With the extra conditions that $|a_{n+1}| \leq |a_n|$ and $a_n \rightarrow 0$, the series converges (at least conditionally). The partial sums s_1, s_3, \dots are **decreasing** and the partial sums s_2, s_4, \dots are **increasing**. The difference between s_n and s_{n-1} is a_n . Therefore the two series converge to the same number s . An alternating series that converges absolutely [conditionally] (not at all) is $\sum (-1)^{n+1}/n^2$ [$\sum (-1)^{n+1}/n$] [$\sum (-1)^{n+1}$]. With absolute [conditional] convergence a reordering cannot [can] change the sum.

10 $\sum (-1)^{n+1} 2^{1/n}$ **diverges** (terms don't approach zero)

16 The terms alternate in sign but do not decrease to zero. The positive terms $\frac{2}{3}, \frac{4}{7}, \frac{6}{11}, \dots$ approach $\frac{1}{2}$ and so does the sequence $\frac{3}{5}, \frac{5}{9}, \frac{7}{13}, \dots$

20 The difference between s and s_{100} is less than $\frac{1}{101^2}$, the next term in the series (because after that term comes $-\frac{1}{102^2}$ and the sums stay between s_{100} and s_{101}).

24 The series $a_1 + a_2 - a_3 + a_4 + a_5 - a_6 + \dots$ is sure to converge (*conditionally*) if $0 \leq a_{3n+3} < a_{3n+1} + a_{3n+2} < a_{3n}$ for every n . Then it passes the alternating series test when each pair of positive terms is combined. (The series could converge without passing this particular test.)

28 Shorter answer than expected: $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6}$ comes from rearranging $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$. Continue this way, **six terms at a time**. The partial sums s_6, s_{12}, \dots are not changed and still approach $\ln 2$. The partial sums in between also approach $\ln 2$ because the six terms in each group approach zero.

36 If $\sum a_n$ is conditionally but not absolutely convergent, take positive terms until the sum exceeds 10. Then take one negative term. Then positive terms until the sum exceeds 20. Then one negative term, and so on. The partial sums approach $+\infty$ (because the single negative terms go to zero, otherwise no conditional convergence in the first place).

40 For $s = -1$ choose all minus signs: $-\frac{1}{2} - \frac{1}{4} - \dots = -1$. For $s = 0$ choose one plus sign and then all minus: $\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \dots = 0$. For $s = \frac{1}{3}$ choose alternating signs: $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \dots = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}$.

10.4 The Taylor Series for e^x , $\sin x$ and $\cos x$ (page 390)

In Problems 1 – 4, begin with these three known series (valid for all x):

$$\begin{aligned}
 e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \\
 \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots \\
 \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots
 \end{aligned}$$

Write down the corresponding power series for these four functions:

1. e^{-x}
2. $\sin 2x$
3. $\cos^2 x$
4. $x \sin x^2$

- Substitute $-x$ for x in the series for e^x to get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + \frac{(-1)^n x^n}{n!} + \dots$$

- Substitute $2x$ for x in the series for $\sin x$ to get

$$\sin 2x = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \dots + \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} + \dots$$

- You don't want to square the series for $\cos x$ if you can avoid it. Here is how to avoid it. Use $\cos 2x = 2 \cos^2 x - 1$ or $\cos^2 x = \frac{1 + \cos 2x}{2}$:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x = \frac{1}{2} + \frac{1}{2} \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots + \frac{(-1)^n (2x)^{2n}}{(2n)!} \right) + \dots$$

Take $x = \frac{\pi}{4}$ as a check. We expect $\cos^2(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$. The first four terms give 0.5000.

- To obtain $x \sin x^2$, substitute x^2 for x in the series for $\sin x$, then multiply each term by x :

$$x \sin x^2 = x \left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \dots \right) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \dots$$

5. Find the Taylor series for $f(x) = \frac{1}{1-x}$ about the basepoint $x = 0$. Compare with Section 10.1.

Formula 10K on page 387 gives the pattern for Taylor series. We need the value of $f(x)$ and all its derivatives at $x = 0$, so the series can be made to agree with the function. A table helps to organize this:

n	n th derivative	set $x = 0$ for $f^{(n)}(0)$	n th term	$= \frac{f^{(n)}(0)}{n!} x^n$
0	$\frac{1}{1-x}$	$1 = 0!$	$\frac{0!}{0!} x^0$	$= 1$
1	$\frac{1}{(1-x)^2}$	$1 = 1!$	$\frac{1!}{1!} x$	$= x$
2	$\frac{2}{(1-x)^3}$	$2 = 2!$	$\frac{2!x^2}{2!}$	$= x^2$
3	$\frac{2 \cdot 3}{(1-x)^4}$	$2 \cdot 3 = 3!$	$\frac{3!x^3}{3!}$	$= x^3$
4	$\frac{2 \cdot 3 \cdot 4}{(1-x)^5}$	$2 \cdot 3 \cdot 4 = 4!$	$\frac{4!x^4}{4!}$	$= x^4$

The n th derivative is $f^{(n)}(0) = n!$ so the n th term of the Taylor series is x^n . This means that $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$. That is the geometric series from Section 10.1.

6. Find the Taylor series for $y = \ln x$ about the point $x = 5$. You need to know all the derivatives of $f(x) = \ln x$ evaluated at $x = 5$:

n	$f^{(n)}(x)$	$f^{(n)}(5)$	n th term	$= \frac{f^{(n)}(5)}{n!} (x-5)^n$
0	$\ln x$	$\ln 5$	$\ln 5$	
1	x^{-1}	$\frac{1}{5}$	$\frac{1}{5}(x-5)$	
2	$-x^{-2}$	$\frac{-1}{5^2}$	$\frac{-1}{2 \cdot 5^2}(x-5)^2$	
3	$2x^{-3}$	$\frac{2}{5^3}$	$\frac{2!}{3!5^3}(x-5)^3$	

The pattern is $\ln x = \ln 5 + \frac{1}{5}(x-5) - \frac{1}{2 \cdot 5^2}(x-5)^2 + \frac{1}{3 \cdot 5^3}(x-5)^3 + \dots$

7. (This is 10.4.40) Find the first two nonzero terms of the Taylor series for $f(x) = \ln(\cos x)$ around $x = 0$.

- The derivatives of $f(x) = \ln \cos x$ are $f'(x) = -\tan x$, $f''(x) = -\sec^2 x$, $f'''(x) = -2 \sec^2 x \tan x$, $f^4(x) = -2 \sec^4 x - 4 \sec^2 x \tan^2 x$. Their values at $x = 0$ are $0, 0, -1, 0, -2$. (We go out to the fourth power of x in order to get two nonzero terms.) So the series for $\ln \cos x$ begins $0 + 0 + \frac{(-1)}{2!}x^2 + 0 + \frac{(-2)x^4}{4!} = -\frac{x^2}{2} - \frac{x^4}{12}$.

If you have a graphing calculator, graph $y = \ln \cos x$ together with $y = -\frac{x^2}{2} - \frac{x^4}{12}$ on the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$. You will see how closely the polynomial approximates the function.

8. (This is 10.4.13) Find the infinite series (powers of x) that solves $y'' = 2y' - y$ starting from $y = 0$ and $y' = 1$ at $x = 0$.

- The equation $y'' = 2y' - y$ will let us figure out the n th derivative of $y = f(x)$ at $x = 0$:

n	$f^{(n)}$	$f^{(n)}$ at $x = 0$	n th term = $\frac{f^{(n)}(0)}{n!}x^n$
0	y	0(given)	0
1	y'	1(given)	x
2	$y'' = 2y' - y$	$2(1) - 0 = 2$	$\frac{2}{2!}x^2 = x^2$
3	$y''' = 2y'' - y'$	$2(2) - 1 = 3$	$\frac{3}{3!}x^3 = \frac{1}{2!}x^3$
4	$y^{(4)} = 2y^{(3)} - y''$	$2(3) - 2 = 4$	$\frac{4}{4!}x^4 = \frac{1}{3!}x^4$

The n th derivative of $y(x)$ at $x = 0$ is n . The n th term is $\frac{n}{n!}x^n = \frac{1}{(n-1)!}x^n$. Then

$$y(x) = 0 + x + x^2 + \frac{1}{2!}x^3 + \frac{1}{3!}x^4 + \dots + \frac{1}{(n-1)!}x^n + \frac{1}{n!}x^{n+1} + \dots$$

This is the series for e^x multiplied by x . Therefore $y = xe^x$ solves the differential equation. You can check by substituting $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ into the equation.

9. (This is part of 10.4.47) Find the real and imaginary parts and the 99th power of $e^{-i\pi/6}$.

- Use Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ to find the real and imaginary parts:

$$e^{i\pi/6} = \cos\left(\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{i}{2}. \text{ The 99th power is } e^{-99i\pi/6} = \cos\left(-\frac{99\pi}{6}\right) + i \sin\left(-\frac{99\pi}{6}\right).$$

But $\frac{99\pi}{6} = \frac{33\pi}{2} = 16\pi + \frac{\pi}{2}$. We can ignore 16π (why?). The answer is $\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = -i$.

Read-throughs and selected even-numbered solutions :

The **Taylor** series is chosen to match $f(x)$ and all its **derivatives** at the basepoint. Around $x = 0$ the series begins with $f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$. The coefficient of x^n is $f^{(n)}(0)/n!$. For $f(x) = e^x$ this series is $\sum \mathbf{x}^n/n!$. For $f(x) = \cos x$ the series is $\mathbf{1} - \mathbf{x}^2/2! + \mathbf{x}^4/4! - \dots$. For $f(x) = \sin x$ the series is $\mathbf{x} - \mathbf{x}^3/3! + \dots$. If the signs were positive in those series, the functions would be **cosh** x and **sinh** x . Addition gives **cosh** $x + \mathbf{sinh} x = \mathbf{e}^x$.

In the Taylor series for $f(x)$ around $x = a$, the coefficient of $(x-a)^n$ is $b_n = f^{(n)}(a)/n!$. Then $b_n(x-a)^n$ has the same **derivatives** as f at the basepoint. In the example $f(x) = x^2$, the Taylor coefficients are $b_0 = \mathbf{a}^2, b_1 = \mathbf{2a}, b_2 = \mathbf{1}$. The series $b_0 + b_1(x-a) + b_2(x-a)^2$ agrees with the original \mathbf{x}^2 . The series for e^x around $x = a$ has $b_n = \mathbf{e}^a/n!$. Then the Taylor series reproduces the identity $e^x = (\mathbf{e}^a)(\mathbf{e}^{x-a})$.

We define $e^x, \sin x, \cos x$, and also $e^{i\theta}$ by their series. The derivative $d/dx(1 + x + \frac{1}{2}x^2 + \dots) = 1 + x + \dots$ translates to $d/dx(\mathbf{e}^x) = \mathbf{e}^x$. The derivative of $1 - \frac{1}{2}x^2 + \dots$ is $-\mathbf{x} + \mathbf{x}^3/3! - \dots$. Using $i^2 = -1$ the series

$1 + i\theta + \frac{1}{2}(i\theta)^2 + \dots$ splits into $e^{i\theta} = \cos \theta + i \sin \theta$. Its square gives $e^{2i\theta} = \cos 2\theta + i \sin 2\theta$. Its reciprocal is $e^{-i\theta} = \cos \theta - i \sin \theta$. Multiplying by r gives $re^{i\theta} = r \cos \theta + ir \sin \theta$, which connects the polar and rectangular forms of a complex number. The logarithm of $e^{i\theta}$ is $i\theta$.

14 At $x = 0$ the equation gives $y'' = y = 1$, $y''' = y' = 0$ and $y'''' = y'' = y = 1$ (even derivatives equal 1, odd derivatives equal 0). The Taylor series is $y(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \frac{1}{2}(e^x + e^{-x}) = \cosh x$.

18 At $x = 2\pi$ the cosine and its derivatives are $1, 0, -1, 0, 1, \dots$. The Taylor series is $\cos x = 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \dots$. At $x = 0$ the function $\cos(x - 2\pi)$ and its derivatives again equal $1, 0, -1, 0, 1, \dots$.

Now the Taylor series is $\cos(x - 2\pi) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$.

22 At $x = 1$ the function x^4 and its derivatives equal $1, 4, 12, 24, 0, 0, \dots$. The Taylor series has five nonzero terms: $x^4 = 1 + 4(x-1) + \frac{12}{2}(x-1)^2 + \frac{24}{6}(x-1)^3 + \frac{24}{24}(x-1)^4$.

26 $\cos \sqrt{x} = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \dots = 1 - \frac{x}{2} + \frac{x^2}{24} - \dots$. (Note that $\sin \sqrt{x}$ would not succeed; the terms $\sqrt{x}, (\sqrt{x})^3, \dots$ are not acceptable in a Taylor series. The function has no derivative at $x = 0$.)

28 $\frac{\sin x}{x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \dots}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots$.

46 $(e^{i\theta})^2 = e^{2i\theta}$ equals $\cos 2\theta + i \sin 2\theta$, so neither of the proposed answers is correct.

50 $(2e^{i\pi/3})^2 = 4e^{2\pi i/3}$ and also $(1 + \sqrt{3}i)(1 + \sqrt{3}i) = 1 + 2\sqrt{3}i - 3 = -2 + 2\sqrt{3}i$; $(4e^{i\pi/4})^2 = 16e^{i\pi/2}$ and also $(2\sqrt{2} + i2\sqrt{2})(2\sqrt{2} + i2\sqrt{2}) = 8 + 16i - 8 = 16i$.

10.5 Power Series (page 395)

1. This problem compares the convergence of a power series when x is close to and far from the basepoint. How many terms are needed to estimate $\sin(\frac{1}{2})$ and $\sin 10$ to 6-place accuracy?

- With $x = 0$ as basepoint, the series is $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$. This is a convergent alternating series when x is positive. *The first unused term is larger than the error* (provided the terms decrease as in the alternating series test). We want the error to be less than 5×10^{-7} .

For $\sin \frac{1}{2} = \frac{1}{2} - \frac{1}{3!}(\frac{1}{2})^3 + \frac{1}{5!}(\frac{1}{2})^5 - \frac{1}{7!}(\frac{1}{2})^7$, the error is less than $\frac{1}{9!}(\frac{1}{2})^9 \approx 5 \times 10^{-9}$. These four terms give the desired accuracy. However, for $\sin(10) = 10 - \frac{10^3}{3!} + \frac{10^5}{5!} - \dots$ we are forced to go out to $\frac{-10^{37}}{37!}$. It requires nineteen nonzero terms to guarantee an error below $\frac{10^{39}}{39!} \approx 4 \times 10^{-8}$.

To estimate $\sin 10$, it would be much more efficient to move the basepoint to 3π (because 3π is close to 10). The series has powers of $(x - 3\pi)$. The constant term is $\sin 3\pi = 0$. Then $\sin 10 = ???$

2. Find the convergence interval of x 's for the series $(x+2) + \frac{1}{\sqrt{2}}(x+2)^2 + \frac{1}{\sqrt{3}}(x+2)^3 + \dots$.

- This is a power series with basepoint $a = -2$ (it has powers of $x+2$ which is $x-a$). **The interval of convergence is always centered at the basepoint.** Perform the ratio test on the *absolute series*, treating x as a constant:

$$\left| \frac{(x+2)^{n+1}}{\sqrt{n+1}} / \frac{(x+2)^n}{\sqrt{n}} \right| = \frac{\sqrt{n}}{\sqrt{n+1}} |x+2| \text{ approaches } L = |x+2| \text{ as } n \rightarrow \infty.$$

When the limit $L = |x+2|$ is below 1, we have absolute convergence. This interval is from $x = -3$ to $x = -1$. The radius of convergence is $r = 1$, the half-length of the interval. The endpoints $x = -3$ (conditional convergence) and $x = -1$ (no convergence) need separate discussion.

3. Find the interval of x 's for which $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 3^n}$ converges.

- The absolute value ratios

$$\left| \frac{x^{n+1}}{(n+1)3^{n+1}} / \frac{x^n}{n 3^n} \right| = \frac{n 3^n}{(n+1)3^{n+1}} \left| \frac{x^{n+1}}{x^n} \right| \text{ approach } L = \frac{|x|}{3}.$$

Absolute convergence occurs when $\frac{|x|}{3} < 1$, or $|x| < 3$. The series diverges when $|x| > 3$. At the right hand endpoint $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is the alternating harmonic series, which converges conditionally to $\ln 2$. (See pages 370-371.) At the left endpoint $x = -3$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n (-3)^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series that diverges.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 3^n}$ converges if $-3 < x \leq 3$. The convergence radius is $r = 3$.

4. The series for $y = \cos x$ around the basepoint $x = \pi$ is $-1 + 0 + \frac{1}{2}(x - \pi)^2 + 0 - \frac{1}{4!}(x - \pi)^4 + R_4$. Estimate the remainder R_4 when $x = 3$.

- Equation (2) on page 392 gives $R_4(x) = \frac{f^{(5)}(c)(x-\pi)^5}{5!}$. The unknown point c is between x and π . For the function $y = \cos x$, the fifth derivative is $f^{(5)}(x) = -\sin x$. We don't know $\sin c$, but we know it lies between -1 and 1 . Therefore $|R_4(x)| \leq \frac{1}{5!}|x - \pi|^5$. For $x = 3$ this remainder $|R_4(3)|$ is less than $\frac{1}{5!}|3 - \pi|^5 < 5 \times 10^{-7}$.

5. Use the binomial series on page 394 to estimate $(242)^{1/5}$ to six-place accuracy.

- With $p = \frac{1}{5}$, the binomial series gives $(1+x)^{1/5}$. But we can't use formula (6)! The series for $(1+x)^p$ has the requirement $|x| < 1$, and for $242^{1/5} = (1+241)^{1/5}$ we would have $x = 241$. The series would quickly diverge – terms would get bigger. We need to start from a known answer near $242^{1/5}$ (much nearer than $1^{1/5}$), and then use the series to make a small correction.

The nearest convenient value is $(243)^{1/5}$ which is exactly 3. The fifth power is $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$. So move the problem to $(242)^{1/5} = (243)^{1/5} \left(\frac{242}{243}\right)^{1/5} = 3\left(1 - \frac{1}{243}\right)^{1/5}$. Now we have $x = \frac{-1}{243}$ and the binomial series is good:

$$3(1+x)^{1/5} = 3\left(1 + \frac{1}{5}x + \frac{1}{5}\left(\frac{-4}{5}\right)\frac{x^2}{2} + \dots\right) = 3 + \frac{3}{5}\left(\frac{-1}{243}\right) - \frac{12}{50}\left(\frac{-1}{243}\right)^2 + \dots = 2.99753.$$

6. Integrate the binomial series for $\frac{1}{\sqrt{1-x^2}}$ to find the power series for $\sin^{-1} x$.

Since $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$, use the binomial series (6) with $p = -\frac{1}{2}$. And also replace x in that series by $-x^2$. The result is

$$\begin{aligned} (1-x^2)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \dots \\ &= 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots \end{aligned}$$

Integrating each term gives $\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{7 \cdot 16}x^7 + \dots$

7. (This is 10.5.46.) Compute $\int_0^1 e^{-x^2} dx$ to three decimal places by integrating the power series.

- Start with $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$. Replace x by $-x^2$ to get $e^{-x^2} = 1 - \frac{x^2}{1} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$. Integrate each term of this series from $x = 0$ to $x = 1$:

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \times 2!} - \frac{x^7}{7 \times 3!} + \dots \right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{5 \times 2!} - \frac{1}{7 \times 3!} + \frac{1}{9 \times 4!} - \frac{1}{11 \times 5!} \approx 0.747. \end{aligned}$$

For three-place accuracy, we need an error less than 5×10^{-4} . Since the series is alternating with decreasing terms, the actual error is less than the first unused term $\frac{1}{13 \times 6!} \approx 10^{-4}$.

If $|x| < |X|$ and $\sum a_n X^n$ converges, then the series $\sum a_n x^n$ also converges. There is convergence in a symmetric interval around the origin. For $\sum (2x)^n$ the convergence radius is $r = \frac{1}{2}$. For $\sum x^n/n!$ the radius is $r = \infty$. For $\sum (x-3)^n$ there is convergence for $|x-3| < 1$. Then x is between 2 and 4.

Starting with $f(x)$, its Taylor series $\sum a_n x^n$ has $a_n = f^{(n)}(0)/n!$. With basepoint a , the coefficient of $(x-a)^n$ is $f^{(n)}(a)/n!$. The error after the x^n term is called the remainder $R_n(x)$. It is equal to $f^{(n+1)}(c)(x-a)^{n+1}/(n+1)!$ where the unknown point c is between a and x . Thus the error is controlled by the $(n+1)$ st derivative.

The circle of convergence reaches out to the first point where $f(x)$ fails. For $f = 4/(2-x)$, that point is $x = 2$. Around the basepoint $a = 5$, the convergence radius would be $r = 3$. For $\sin x$ and $\cos x$ the radius is $r = \infty$.

The series for $\sqrt{1+x}$ is the binomial series with $p = \frac{1}{2}$. Its coefficients are $a_n = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2}) \dots /n!$. Its convergence radius is 1. Its square is the very short series $1+x$.

- 2 In the geometric series $\frac{1}{1-x} = 1 + x + x^2 + \dots$ change x to $4x^2$: $\frac{1}{1-4x^2} = 1 + 4x^2 + 16x^4 + \dots$. Convergence fails when $4x^2$ reaches 1 (thus $x = \frac{1}{2}$ or $x = -\frac{1}{2}$). The radius of convergence is $r = \frac{1}{2}$.
- 4 $\tan x$ has derivatives $\sec^2 x$, $2 \sec^2 x \tan x$, $2 \sec^4 x + 4 \sec^2 x \tan^2 x$. At $x = 0$ the series is $1 + 0x + \frac{2}{3}x^2 + 0x^3 = 1 + x^2$. The function $\tan x = \frac{\sin x}{\cos x}$ is infinite when $\cos x = 0$, at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$. Then $r = \frac{\pi}{2}$.
- 6 In the geometric series replace x by $-4x^2$. Then $\frac{1}{1+4x^2} = 1 - 4x^2 + 16x^4 - \dots$. Convergence fails when $|4x^2|$ reaches 1. The function blows up when $4x^2 = -1$, at $x = \frac{i}{2}$ and $x = -\frac{i}{2}$. The radius of convergence is $r = \frac{1}{2}$.
- 14 (a) Combine $x + x^4 + x^7 + \dots = \frac{x}{1-x^3}$ and $x^2 + x^5 + x^8 + \dots = \frac{x^2}{1-x^3}$ and $-(x^3 + x^6 + \dots) = -\frac{x^3}{1-x^3}$ to get $\frac{x+x^2-x^3}{1-x^3}$. (b) Adding the series for $\cos x$ and $\cosh x$ leads to $1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \dots = \frac{1}{2}(\cos x + \cosh x)$. (c) $\ln(x-1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \dots$ so changing x to $x-1$ gives the series for $\ln(x-2)$ around $a = 1$.
- 22 The remainder after n terms of the series for e^x around $a = 1$ is $R_n(x) = e^c \frac{(x-1)^{n+1}}{(n+1)!}$. The factor e^c is between 1 and e^x . As $n \rightarrow \infty$ the factorial assures that $R_n(x) \rightarrow 0$ and the series converges to e^x .
- 26 The derivatives of $(1-x)^{-1/2}$ are $\frac{1}{2}(1-x)^{-3/2}$, $\frac{1 \cdot 3}{2 \cdot 2}(1-x)^{-5/2}$, \dots , $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n}(1-x)^{-(2n+1)/2}$. At $x = 0$ this n th derivative divided by $n!$ is the coefficient a_n .
- 50 If $a_n^{1/n}$ approaches L then $(a_n x^n)^{1/n}$ approaches $\frac{x}{L}$. By the root test the series $\sum a_n x^n$ converges when $|\frac{x}{L}| < 1$ and diverges when $|\frac{x}{L}| > 1$. So the radius of convergence is $r = L$.

10 Chapter Review Problems

Review Problems

- R1** Define the convergence of a sequence s_1, s_2, s_3, \dots and the convergence of a series $\sum a_n$.
- R2** Find the sums of (a) $\frac{2}{5} - \frac{4}{25} + \frac{8}{125} - \dots$ (b) $3.646464 \dots$ (c) $\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \dots$
- R3** When is $a_0 + a_1 + a_2 + \dots$ a geometric series? When does it converge?
- R4** Explain these tests for the convergence (or divergence!) of the infinite series $\sum a_n$ with all $a_n \geq 0$: Comparison test, Integral test, Ratio test, Root test.
- R5** When can you be sure that an *alternating series* converges? Give examples of absolute convergence, conditional convergence, and divergence.
- R6** Find the Taylor Series for $y = \cos x$ around the basepoint $a = \frac{\pi}{3}$.
- R7** If $a_n \rightarrow 0$ show by comparison that $\sum a_n x^n$ converges at least for $-1 < x < 1$.
- R8** What is a p -series? Which p -series converge and which diverge?
- R9** In using the Taylor series to evaluate $f(x)$, why should the basepoint a be close to the evaluation point?
- R10** If the root test for $\sum a_n(x-4)^n$ yields $L = 3|x-4|$, what is the convergence interval $|x-4| < r$?

Drill Problems

Determine whether the positive series **D1** – **D8** converge or diverge. Name your test.

- D1** $\sum_1^{\infty} \frac{1}{\sqrt{2n+1}}$ **D2** $\sum_1^{\infty} \frac{1}{2n+1}$ **D3** $\sum_1^{\infty} \frac{1}{3^{n+1}}$ **D4** $\sum \frac{n - \ln n}{n^2}$
- D5** $\sum \frac{(\ln n)^3}{n^2}$ **D6** $\sum \frac{2^n}{n^2}$ **D7** $\sum \frac{n^2}{3n^3+7n}$ **D8** $\frac{e}{3} + \frac{e^2}{9} + \frac{e^3}{27} + \dots$

Decide whether the alternating series **D9** – **D12** diverge, converge absolutely, or converge conditionally.

- D9** $\sum \frac{(-1)^n}{\sqrt{n}}$ **D10** $1.1 - 1.01 + 1.001 - \dots$ **D11** $\sum \frac{(-1)^n}{n^2}$ **D12** $\sum \frac{1}{n} \sin \frac{2\pi}{n}$

- D13** Find the first four terms of the Taylor series for $f(x) = \ln(\sin x)$ about $a = \frac{\pi}{4}$.

- D14** Write out three non-zero terms of the Taylor series for $f(x) = \tan x$ about $a = 0$.
- D15** Use your answer from **D14** to find the first three terms for $f(x) = \sec^2 x$.
- D16** Use your answer from **D14** to find the first three terms for $\ln \cos x$.
- D17** Find the geometric series for $f(x) = \frac{1}{1-2x}$. What is the interval of convergence?
- D18** Find three terms of the binomial series for $(1+x)^{2/3}$.
Find the interval of convergence for the power series **D19** to **D23**.
- D19** $\sum_1^\infty \frac{(2x-3)^n}{n^2}$ **D20** $\sum_2^\infty \frac{x^n}{\ln n}$ **D21** $\sum \frac{n^2}{n!} x^n$ **D22** $\sum \frac{(-1)^n (x+4)^n}{3^n}$ **D23** $\sum_1^\infty n! x^n$
- D24** Show that $y = \sum_{n=0}^\infty \frac{x^{2n}}{n!}$ solves the differential equation $y' = 2xy$ with $y(0) = 1$. Explain $y = e^{x^2}$.
- D25** Estimate the error when $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{10}$ approximates $\sum \frac{(-1)^{n+1}}{n} = \underline{\hspace{2cm}}$.
- D26** We know that $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots$. How many terms to give six correct decimals?
- D27** Without a calculator $\int_0^\pi \sin x \, dx$ equals 2. With a calculator $\int_0^\pi (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}) dx = \underline{\hspace{2cm}}?$

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Resource: Calculus Online Textbook
Gilbert Strang

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