CHAPTER 6 EXPONENTIALS AND LOGARITHMS

6.1 An Overview (page 234)

In $10^4 = 10{,}000$, the exponent 4 is the **logarithm** of 10,000. The base is $b = \mathbf{10}$. The logarithm of 10^m times 10^n is $\mathbf{m+n}$. The logarithm of $10^m/10^n$ is $\mathbf{m-n}$. The logarithm of $10{,}000^x$ is $\mathbf{4x}$. If $y = b^x$ then $x = \mathbf{\log_b y}$. Here x is any number, and y is always **positive**.

A base change gives $b = a^{\mathbf{\log_a b}}$ and $b^x = a^{\mathbf{x \log_a b}}$. Then 8^5 is 2^{15}. In other words $\log_2 y$ is $\mathbf{\log_2 8}$ times $\log_8 y$. When $y = 2$ it follows that $\log_2 8$ times $\log_8 2$ equals **1**.

On ordinary paper the graph of $y = \mathbf{mx + b}$ is a straight line. Its slope is $\mathbf{m}$. On semilog paper the graph of $y = \mathbf{Ab^x}$ is a straight line. Its slope is $\mathbf{\log b}$. On log-log paper the graph of $y = \mathbf{Ax^k}$ is a straight line. Its slope is $\mathbf{k}$.

The slope of $y = b^x$ is $dy/dx = \mathbf{cb^x}$, where c depends on b. The number c is the limit as $h \to 0$ of $\mathbf{\frac{b^h-1}{h}}$. Since $x = \log_b y$ is the inverse, $(dx/dy)(dy/dx) = \mathbf{1}$. Knowing $dy/dx = cb^x$ yields $dx/dy = \mathbf{1/cb^x}$. Substituting b^x for y, the slope of $\log_b y$ is $\mathbf{1/cy}$. With a change of letters, the slope of $\log_b x$ is $\mathbf{1/cx}$.

1 $5; -5; -1; \frac{1}{5}; \frac{3}{2}; 2$ **5** $1; -10; 80; 1; 4; -1$ **7** $n \log_b x$ **9** $\frac{10}{3}; \frac{3}{10}$ **13** 10^5
15 0; $I_{SF} = 10^7 I_0$; $8.3 + \log_{10} 4$ **17** $A = 7, b = 2.5$ **19** $A = 4, k = 1.5$
21 $\frac{1}{cx}; \frac{2}{cx}; \log 2$ **23** $y - 1 = cx; y - 10 = c(x-1)$ **25** $(.1^{-h} - 1)/(-h) = (10^h - 1)/(-h)$
27 $y'' = c^2 b^x; x'' = -1/cy^2$ **29** Logarithm

2 (a) 5 (b) 25 (c) 1 (d) 2 (e) 10^4 (f) 3
4 The graph of 2^{-x} goes through $(0,1), (1, \frac{1}{2}), (2, \frac{1}{4})$. The mirror image is $x = \log_{\frac{1}{2}} y$ (y is now horizontal): $\log_{1/2} 2 = \mathbf{-1}$ and $\log_{1/2} 4 = \mathbf{-2}$.
6 (a) 7 (b) 3 (c) $\sqrt{10}$ (d) $\frac{1}{4}$ (e) $\sqrt{8}$ (f) 5
8 $\log_b a = (\log_b d)(\log_d a)$ and $(\log_b d)(\log_d c) = \log_b c$. Multiply left sides, multiply right sides, cancel $\log_b d$.
10 Number of decimal digits $\approx$ logarithm to base 10. For 2^{1000} this logarithm is $1000 \log_{10} 2 \approx 1000(.3) = 300$.
12 $y = \log_{10} x$ is a straight line on "inverse" semilog paper: y axis normal, x axis scaled logarithmically (so $x = 1, 10, 100$ are equally spaced). Any equation $y = \log_b x + C$ will have a straight line graph.
14 $y = 10^{1-x}$ drops from 10 to 1 to .1 with slope -1 on semilog paper; $y = \frac{1}{2}\sqrt{10}^x$ increases with slope $\frac{1}{2}$ from $y = \frac{1}{2}$ at $x = 0$ to $y = 5$ at $x = 2$.
16 If 440/second is the frequency of middle A, then the next A is **880/second.** The 12 steps from A to A are approximately multiples of $2^{1/12}$. So 7 steps multiplies by $2^{7/12} \approx 1.5$ to give $(1.5)(440) = 660$. The seventh note from A is **E.**
18 $\log y = 2 \log x$ is a straight line with slope 2; $\log y = \frac{1}{2} \log x$ has slope $\frac{1}{2}$.
20 $g(f(y)) = y$ gives $g'(f(y))\frac{df}{dy} = 1$ or $cg(f(y))\frac{df}{dy} = 1$ or $cy\frac{df}{dy} = 1$ or $\frac{df}{dy} = \frac{1}{cy}$.
22 The slope of $y = 10^x$ is $\frac{dy}{dx} = c10^x$ (later we find that $c = \ln 10$). At $x = 0$ and $x = 1$ the slope is c and $10c$. So the tangent lines are $y - 1 = c(x - 0)$ and $y - 10 = 10c(x - 1)$.

24 $h = 1$ gives $c = 9$; $h = .1$ gives $c = 2.6$; $h = .01$ gives $c = 2.339$; $h = .001$ gives $c = 2.305$; the limit is $c = \ln 10 = 2.3026$.

26 (The right base is $b = e$.) With $h = \frac{1}{4}$ we pick the base so that $\frac{b^{1/4}-1}{1/4} = 1$ or $b^{1/4} = (1 + \frac{1}{4})$ or $b = (1+\frac{1}{4})^4 = \frac{625}{256}$. Generally $b = (1+h)^{1/h}$ which approaches e as $h \to 0$.

28 $c = \lim_{h\to 0} \frac{10^h - 1}{h} = \lim_{h \to 0} \frac{10^{2h}-1}{2h} = \frac{1}{2} \lim_{h\to 0} \frac{100^h - 1}{h} = \frac{1}{2}C$.

6.2 The Exponential e^x (page 241)

The number e is approximately **2.78**. It is the limit of $(1+h)$ to the power **1/h**. This gives 1.01^{100} when $h = .01$. An equivalent form is $e = \lim (\mathbf{1} + \frac{\mathbf{1}}{\mathbf{n}})^{\mathbf{n}}$.

When the base is $b = e$, the constant c in Section 6.1 is **1**. Therefore the derivative of $y = e^x$ is $dy/dx = \mathbf{e^x}$. The derivative of $x = \log_e y$ is $dx/dy = \mathbf{1/y}$. The slopes at $x = 0$ and $y = 1$ are both **1**. The notation for $\log_e y$ is **ln y**, which is the **natural** logarithm of y.

The constant c in the slope of b^x is $c = \mathbf{\ln b}$. The function b^x can be rewritten as $\mathbf{e^{x \ln b}}$. Its derivative is $(\mathbf{\ln b})\mathbf{e^{x\ln b}} = (\mathbf{\ln b})\mathbf{b^x}$. The derivative of $e^{u(x)}$ is $\mathbf{e^{u(x)}\frac{du}{dx}}$. The derivative of $e^{\sin x}$ is $\mathbf{e^{\sin x}\cos x}$. The derivative of e^{cx} brings down a factor **c**.

The integral of e^x is $\mathbf{e^x + C}$. The integral of e^{cx} is $\mathbf{\frac{1}{c}e^{cx} + C}$. The integral of $e^{u(x)}du/dx$ is $\mathbf{e^{u(x)} + C}$. In general the integral of $e^{u(x)}$ by itself is **impossible** to find.

1 $49e^{7x}$ **3** $8e^{8x}$ **5** $3^x \ln 3$ **7** $(\frac{2}{3})^x \ln \frac{2}{3}$ **9** $\frac{-e^x}{(1+e^x)^2}$ **11** 2 **13** xe^x **15** $\frac{4}{(e^x+e^{-x})^2}$

17 $e^{\sin x}\cos x + e^x \sin e^x$ **19** .1246, .0135, .0014 are close to $\frac{e}{2n}$ **21** $\frac{1}{e}; \frac{1}{e}$

23 $Y(h) = 1 + \frac{1}{10}; Y(1) = (1 + \frac{1}{10})^{10} = 2.59$ **25** $(1+\frac{1}{x})^x < e < e^x < e^{3x/2} < e^{2x} < 10^x < x^x$

27 $\frac{e^{3x}}{3} + \frac{e^{7x}}{7}$ **29** $x + \frac{2^x}{\ln 2} + \frac{3^x}{\ln 3}$ **31** $\frac{(2e)^x}{\ln(2e)} + 2e^x$ **33** $\frac{e^{x^2}}{2} - \frac{e^{-x^2}}{2}$

35 $2e^{x/2} + \frac{e^{2x}}{2}$ **37** e^{-x} drops faster at $x = 0$ (slope -1); meet at $x = 1; e^{-x^2}/e^{-x} < e^{-9}/e^{-3} < \frac{1}{100}$ for $x > 3$

39 $y - e^a = e^a(x-a)$; need $-e^a = -ae^a$ or $a = 1$

41 $y' = x^x(\ln x + 1) = 0$ at $x_{\min} = \frac{1}{e}; y'' = x^x[(\ln x + 1)^2 + \frac{1}{x}] > 0$

43 $\frac{d}{dx}(e^{-x}y) = e^{-x}\frac{dy}{dx} - e^{-x}y = 0$ so $e^{-x}y =$ Constant or $y = Ce^x$

45 $\frac{e^{2x}}{2}]_0^1 = \frac{e^2-1}{2}$ **47** $\frac{2^x}{\ln 2}]_{-1}^1 = \frac{2-\frac{1}{2}}{\ln 2} = \frac{3}{2\ln 2}$ **49** $-e^{-x}]_0^\infty = 1$ **51** $e^{1+x}]_0^1 = e^2 - e$ **53** $2^{\sin x}]_0^\pi = 0$

55 $\int \frac{du/dx}{e^u}dx = -e^{-u} + C$; $\int (e^u)^2 \frac{du}{dx}dx = \frac{1}{2}e^{2u} + C$ **57** $yy' = 1$ gives $\frac{1}{2}y^2 = x + C$ or $y = \sqrt{2x + 2C}$

59 $\frac{dF}{dx} = (n-x)x^{n-1}/e^x < 0$ for $x > n; F(2x) < \frac{\text{constant}}{e^x} \to 0$ **61** $\frac{6!}{\sqrt{12\pi}} \approx 117; (\frac{6}{e})^6 \approx 116$; 7 digits

2 $49e^{-7x}$ **4** $8e^{8x}$ **6** $(\ln 3)e^{x\ln 3} = (\ln 3)3^x$ **8** $4(\ln 4)4^{4x}$ **10** $\frac{-1}{(1+x)^2}e^{1/(1+x)}$ **12** $(-\frac{1}{x} + 1)e^{1/x}$ **14** x^2e^x

16 $x^2 + x^2$ has derivative $4x$ **18** $x^{-1/x} = e^{-(\ln x)/x}$ has derivative $(-\frac{1}{x^2} + \frac{\ln x}{x^2})e^{-(\ln x)/x} = (\frac{\ln x - 1}{x^2})x^{-1/x}$

20 $(1+\frac{1}{n})^{2n} \to \mathbf{e^2} \approx 7.7$ and $(1+\frac{1}{n})^{\sqrt{n}} \to \mathbf{1}$. Note that $(1+\frac{1}{n})^{\sqrt{n}}$ is squeezed between 1 and $e^{1/\sqrt{n}}$ which approaches 1.

22 $(1.001)^{1000} = 2.717$ and $(1.0001)^{10000} = 2.7181$ have 3 and 4 correct decimals. $(1.00001)^{100000} = 2.71827$ has one more correct decimal. The difference between $(1+\frac{1}{n})^n$ and e is proportional to $\frac{1}{n}$.

24 $y = \mathbf{e^{-x}}$ solves $\frac{dy}{dx} = -y$. The difference equation $Y(x + \frac{1}{4}) = Y(x) - \frac{1}{4}Y(x)$ with $Y(0) = 1$ gives $Y(\frac{1}{4}) = \mathbf{\frac{3}{4}}$ and $Y(1) = (\mathbf{\frac{3}{4}})^{\mathbf{4}}$. (Compare $e^{-1} = .37$ with $(\frac{3}{4})^4 = .32$. See the end of Section 6.6.)

26 $\sqrt{e^x}$ is the same as $\mathbf{e^{x/2}}$. Its graph at $x = -2, 0, 2$ has the same heights $\frac{1}{e}, 1, e$ as the graph of e^x at $x = -1, 0, 1$.

28 $(e^{3x})(e^{7x}) = e^{10x}$ which is the derivative of $\mathbf{\frac{1}{10}e^{10x}}$

30 $2^{-x} = e^{-x\ln 2}$ which has antiderivative $\mathbf{\frac{-1}{\ln 2}e^{-x\ln 2}} = \mathbf{\frac{-1}{\ln 2}2^{-x}}$.

32 $e^{-x} + x^{-e}$ has antiderivative $-e^{-x} + \frac{x^{1-e}}{1-e}$ **34** $-e^{\cos x} + e^{\sin x}$ **36** $xe^x - e^x$

38 e^x meets x^e at $x = e$. Their slopes are e^x and $x^x(1 + \ln x)$ by Example 6. At $x = e$ those slopes are e^e and $2e^e$. The ratio $\frac{x^x}{e^x} = (\frac{x}{e})^x$ approaches infinity.

40 At $x = 0$ equality holds: $e^0 = 1 + 0$ and $e^{-0} = 1 - 0$. (a) Beyond $x = 0$ the slope of e^x exceeds the slope of $1 + x$ (this means $e^x > 1$). So e^x increases faster than $1 + x$. (b) Beyond $x = 0$ the slope of e^{-x} is larger than the slope of $1 - x$ (this means $-e^{-x} > -1$). Since they start together, e^{-x} is larger than $1 - x$.

42 $x^{1/x} = e^{(\ln x)/x}$ has slope $e^{(\ln x)/x}(\frac{1}{x^2} - \frac{\ln x}{x^2}) = \mathbf{x^{1/x}(\frac{1-\ln x}{x^2})}$. This slope is zero at $\mathbf{x = e}$, when $\ln x = 1$. The second derivative is *negative* so the maximum of $x^{1/x}$ is $e^{1/e}$. Check: $\frac{d}{dx}e^{(\ln x)/x}(\frac{1-\ln x}{x^2}) = e^{(\ln x)/x}[(\frac{1-\ln x}{x^2})^2 + \frac{(-2-1+2\ln x)}{x^3}] = -\frac{1}{e^3}e^{1/e}$ at $x = e$.

44 $x^e = e^x$ at $x = e$. This is the only point where $x^e e^{-x} = 1$ because the derivative is $x^e(-e^{-x}) + ex^{e-1}e^{-x} = (\frac{e}{x} - 1)x^e e^{-x}$. This derivative is positive for $x < e$ and negative for $x > e$. So the function $x^e e^{-x}$ increases to 1 at $x = e$ and then decreases: it never equals 1 again.

46 $\int_0^\pi \sin x\, e^{\cos x} dx = [-e^{\cos x}]_0^\pi = \mathbf{-e^{-1} + e}$.

48 $\int_{-1}^1 2^{-x} dx =$ (by Problem 30) $[\frac{-1}{\ln 2}2^{-x}]_{-1}^1 = \frac{-1}{\ln 2}(\frac{1}{2} - 2) = \mathbf{\frac{3}{2\ln 2}}$.

50 $\int_0^\infty xe^{-x^2} dx = \int_0^\infty e^{-u}\frac{du}{2} = [-\frac{e^{-u}}{2}]_0^\infty = \mathbf{\frac{1}{2}}$. **52** $\int_0^1 e^{1+x^2} x\, dx = [\frac{1}{2}e^{1+x^2}]_0^1 = \mathbf{\frac{1}{2}(e^2 - e)}$

54 $\int_0^1 (1 - e^x)^{10} e^x dx = [-\frac{(1-e^x)^{11}}{11}]_0^1 = \mathbf{-\frac{(1-e)^{11}}{11}}$.

56 $y'(x) = 5y(x)$ is solved by $y = Ae^{5x}$ (A is any constant). Choose $A = 2$ so that $\mathbf{y(x) = 2e^{5x}}$ has $y(0) = 2$.

58 The asymptotes of $(1 + \frac{1}{x})^x = (\frac{x+1}{x})^x = (\frac{x}{x+1})^{-x}$ are $x = -1$ (from the last formula) and $y = e$ (from the first formula).

60 The maximum of $x^6 e^{-x}$ occurs when its derivative $(6x^5 - x^6)e^{-x}$ is zero. Then $\mathbf{x = 6}$ (note that $x = 0$ is a minimum).

62 $\lim \frac{x^6}{e^x} = \lim \frac{6x^5}{e^x} = \lim \frac{30x^4}{e^x} = \lim \frac{120x^3}{e^x} = \lim \frac{360x^2}{e^x} = \lim \frac{720x}{e^x} = \lim \frac{720}{e^x} = 0$.

6.3 Growth and Decay in Science and Economics (page 250)

If $y' = cy$ then $y(t) = \mathbf{y_0 e^{ct}}$. If $dy/dt = 7y$ and $y_0 = 4$ then $y(t) = \mathbf{4e^{7t}}$. This solution reaches 8 at $t = \mathbf{\frac{\ln 2}{7}}$. If the doubling time is T then $c = \mathbf{\frac{\ln 2}{T}}$. If $y' = 3y$ and $y(1) = 9$ then y_0 was $\mathbf{9e^{-3}}$. When c is negative, the solution approaches **zero** as $t \to \infty$.

The constant solution to $dy/dt = y + 6$ is $y = \mathbf{-6}$. The general solution is $y = Ae^t - 6$. If $y_0 = 4$ then $A = \mathbf{10}$. The solution of $dy/dt = cy + s$ starting from y_0 is $y = Ae^{ct} + B = \mathbf{(y_0 + \frac{s}{c})e^{ct} - \frac{s}{c}}$. The output from the source is $\mathbf{\frac{s}{c}(e^{ct} - 1)}$. An input at time T grows by the factor $\mathbf{e^{c(t-T)}}$ at time t.

At $c = 10\%$, the interest in time dt is $dy = \mathbf{.01\, y\, dt}$. This equation yields $y(t) = \mathbf{y_0 e^{.01t}}$. With a source term instead of y_0, a continuous deposit of $s = 4000$/year yields $y = \mathbf{40{,}000(e - 1)}$ after ten years. The deposit required to produce 10,000 in 10 years is $s = \mathbf{yc/(e^{ct} - 1) = 1000/(e - 1)}$. An income of 4000/year forever (!) comes from $y_0 = \mathbf{40,000}$. The deposit to give 4000/year for 20 years is $y_0 = \mathbf{40,000(1 - e^{-2})}$. The payment rate s to clear a loan of 10,000 in 10 years is $\mathbf{1000e/(e - 1)}$ per year.

The solution to $y' = -3y + s$ approaches $y_\infty = \mathbf{s/3}$.

1 $t^2 + y_0$ **3** $y_0 e^{2t}$ **5** $10\,e^{4t}; t = \frac{\ln 10}{4}$ **7** $\frac{1}{4}e^{4t} + 9.75;\ t = \frac{\ln 361}{4}$ **11** $c = \frac{\ln 2}{2}; t = \frac{\ln 10}{c}$
13 $\frac{5568}{-.7}\ln(\frac{1}{5})$ **15** $c = \frac{\ln 2}{20}; t = \frac{1}{c}\ln(\frac{8}{5})$ **17** $t = \frac{\ln(1/240)}{\ln(.98)}$ **19** $e^c = 3$ so $y_0 = e^{-3c}1000 = \frac{1000}{27}$
21 $p = 1013\,e^{ch};\ 50 = 1013\,e^{20c};\ c = \frac{1}{20}\ln(\frac{50}{1013});\ p(10) = 1013\,e^{10c} = 1013\sqrt{\frac{50}{1013}} = \sqrt{(1013)(50)}$
23 $c = \frac{\ln 2}{3}; (\frac{1}{2})^3 = \frac{1}{8}$ **25** $y = y_0 - at$ reaches y_1 at $t = \frac{y_0 - y_1}{a}$; then $y = Ae^{-at/y_1}$ **27** F; F; T; T
29 $A = \frac{1}{3}, B = -\frac{1}{3}$ **31** $e^t - 1$ **33** $1 - e^{-t}$ **35** $6; 6 + Ae^{-2t}; 6 - 6e^{-2t}, 6 + 4e^{-2t}; 6$
37 $4; 4 - \frac{1}{e}; 4$ **39** $ye^{-t}; y(t) = te^t$ **41** $A = 1, B = -1, C = -1$ **43** $e^{.0725} > .075$ **45** $s(e-1); \frac{s(e-1)}{e}$
47 $(1.02)(1.03) \to 5.06\%; 5\%$ by Problem 27 **49** $20,000\,e^{(20-T)(.5)} = 34,400$ (it grows for $20 - T$ years)
51 $s = -cy_0e^{ct}/(e^{ct} - 1) = -(.01)(1000)e^{.60}/(e^{.60} - 1)$ **53** $y_0 = \frac{100}{.005}(1 - e^{-.005(48)})$
55 $e^{4c} = 1.20$ so $c = \frac{\ln 1.20}{4}$ **57** $24e^{36.5} = ?$ **59** To $-\infty$; constant; to $+\infty$
61 $\frac{dY}{dT} = 60cY; \frac{dY}{dT} = 60(-Y + 5)$; still $Y_\infty = 5$
63 $y = 60e^{ct} + 20, 60 = 60e^{12c} + 20, c = \frac{1}{12}\ln(\frac{40}{60}); 100 = 60e^{ct} + 20$ at $t = \frac{1}{c}\ln(\frac{80}{60})$ **65** 0

2 $\frac{dy}{dt} = -t$ gives $dy = -t\,dt$ and $y = -\frac{1}{2}t^2 + C$. Then $\mathbf{y = -\frac{1}{2}t^2 + 1}$ and $\mathbf{y = -\frac{1}{2}t^2 - 1}$ start from 1 and -1.
4 $\frac{dy}{dt} = -y$ gives $\frac{dy}{y} = -dt$ and $\ln y = -t + C$ and $y = Ae^{-t}$ (where $A = e^C$). (*Question:* How does a negative y appear, since e^C is positive? *Answer:* $\int \frac{dy}{y} = \ln|y|$ leads to $|y| = Ae^{-t}$ and allows $y < 0$.) To start from 1 and -1 choose $\mathbf{y = e^{-t}}$ and $\mathbf{y = -e^{-t}}$.
6 $\frac{dy}{dt} = 4t$ gives $dy = 4t\,dt$ and $y = 2t^2 + C = \mathbf{2t^2 + 10}$. This equals 100 when $2t^2 = 90$ or $\mathbf{t = \sqrt{45}}$.
8 $\frac{dy}{dt} = e^{-4t}$ has $y(t) = \frac{e^{-4t}}{-4} + C = \mathbf{\frac{e^{-4t}}{-4} + 10\frac{1}{4}}$. This only increases from 10 to $10\frac{1}{4}$ as $t \to \infty$. Before $t = 0$ we find $y(t) = 1$ when $\frac{e^{-4t}}{4} = 9\frac{1}{4}$ or $e^{-4t} = 37$ or $\mathbf{t = \frac{\ln 37}{-4}}$.
10 The solutions of $y' = y - 1$ (which is also $(y-1)' = y - 1$) are $y - 1 = Ae^x$ or $y = Ae^x + 1$. Figure 6.7b is raised by 1 unit. (The solution that was $y = e^x$ is lifted to $y = e^x + 1$. The solution that was $y = 0$ is lifted to $y = 1$.)
12 To multiply again by 10 takes ten more hours, a total of **20 hours**. If $e^{10c} = 10$ (and $e^{20c} = 100$) then $10c = \ln 10$ and $\mathbf{c = \frac{\ln 10}{10} \approx .23}$.
14 Following Example 2, the ratio e^{ct} would be 90% or .9. Then $t = \frac{\ln .9}{c} = (\frac{\ln .9}{\ln \frac{1}{2}})5568 = (\ln 1.8)5568 = 3273$ years. So the material is dated earlier than the year 0.
16 $8e^{.01t} = 6e^{.014t}$ gives $\frac{8}{6} = e^{.004t}$ and $t = \frac{1}{.004}\ln\frac{8}{6} = 250\ln\frac{4}{3} = \mathbf{72}$ **years**.
18 At $t = 3$ days, $e^{3c} = 40\% = .4$ and $c = \frac{\ln .4}{3} = -.3$. At T days, 20% remember: $e^{-.3T} = 20\% = .2$ at $T = \frac{\ln .2}{(-.3)} = \mathbf{5.36}$ **days**. (Check after 6 days: $(.4)^2 = 16\%$ will remember.)
20 If y is divided by 10 in 4 time units, it will be divided by 10 again in 4 more units. Thus $y = 1$ at $\mathbf{t = 12}$. Returning to $t = 0$ multiplies by 10 so $\mathbf{y_0 = 1000}$.
22 Exponential decay is $y = Ae^{ct}$. Then $y(0) = A$ and $y(2t) = Ae^{2ct}$. The square root of $y(0)y(2t) = A^2e^{2ct}$ is $y(t) = Ae^{ct}$. One way to find $y(3t) = Ae^{3ct}$ is $y(0)(\frac{y(2t)}{y(0)})^{3/2}$. (A better question is to find $y(4t) = Ae^{4ct} = y(0)(\frac{y(2t)}{y(0)})^2 = \frac{(y(2t))^2}{y(0)}$.)

24 Go from 4 mg back down to 1 mg in T hours. Then $e^{-.01T} = \frac{1}{4}$ and $-.01T = \ln\frac{1}{4}$ and $T = \frac{\ln\frac{1}{4}}{-.01} = 139$ hours (not so realistic).

26 The second-order equation is $(\frac{d}{dt}-c)(\frac{d}{dt}-C)y = \frac{d^2y}{dt^2}-(c+C)\frac{dy}{dt}+cCy = 0$. Check the solution $y = Ae^{ct}+Be^{Ct}$ by substituting into the equation: $c^2Ae^{ct} + C^2Be^{Ct} - (c+C)(cAe^{ct} + CBe^{Ct}) + cC(Ae^{ct} + Be^{Ct})$ does equal zero.

28 Given $mv = mv - v\Delta m + m\Delta v - (\Delta m)\Delta v + \Delta m(v-7)$; cancel terms to leave $m\Delta v - (\Delta m)\Delta v = 7\Delta m$; divide by Δm and approach the limit $\mathbf{m\frac{dv}{dm} = 7}$. Then $v = 7\ln m + C$. At $t = 0$ this is $20 = 7\ln 4 + C$ so that $v = 7\ln m + 20 - 7\ln 4 = \mathbf{7\ln\frac{m}{4} + 20}$.

30 Substitute $y = Ae^{-t} + B$ into $y' = 8 - y$ to find $-Ae^{-t} = 8 - Ae^{-t} - B$. Then $\mathbf{B = 8}$. At the start $y_0 = A + B = A + 8$ so $A = y_0 - 8$. Then $\mathbf{y = (y_0 - 8)e^{-t} + 8}$ or $y = y_0e^{-t} + 8(1-e^{-t})$.

32 Apply formula (8) to $\frac{dy}{dt} = y - 1$ with $y_0 = 0$. Then $y(t) = \frac{-1}{1}(e^t - 1) = \mathbf{1 - e^t}$.

34 Formula (8) applied to $\frac{dy}{dt} = -y - 1$ with $y_0 = 0$ gives $y = \frac{-1}{-1}(e^{-t} - 1) = \mathbf{e^{-t} - 1}$.

36 (a) $\frac{dy}{dt} = 3y + 6$ gives $y \to \infty$ (b) $\frac{dy}{dt} = -3y + 6$ gives $y \to 2$ (c) $\frac{dy}{dt} = -3y - 6$ gives $y \to -2$ (d) $\frac{dy}{dt} = 3y - 6$ gives $y \to -\infty$.

38 Solve $y' = y + e^t$ by adding inputs at all times T times growth factors e^{t-T} : $y(t) = \int_0^t e^{t-T}e^T dT = \int_0^t e^t dT = \mathbf{te^t}$. Substitute in the equation to check: $(te^t)' = te^t + e^t$.

40 Solve $y' + y = 1$ by multiplying to give $e^ty' + e^ty = e^t$. The left side is the derivative of ye^t (by the product rule). Integrate both sides: $ye^t - y_0e^0 = e^t - e^0$ or $ye^t = y_0 + e^t - 1$ or $y = \mathbf{y_0e^{-t} + 1 - e^{-t}}$.

42 \$1000 changes by (\$1000) $(-.04dt)$, a decrease of $40dt$ dollars in time dt. The printing rate should be $s = 40$.

44 First answer: With continuous interest at $c = .09$ the multiplier after a year is $e^{.09} = 1.094$ and the effective rate is 9.4%. Second answer: The continuous rate c that gives an effective annual rate of 9% is $e^c = 1.09$ or $c = \ln 1.09 = .086$ or 8.6%.

46 y_0 grows to $y_0e^{(.1)(20)} = 50{,}000$ so the grandparent gives $y_0 = 50{,}000e^{-2} \approx$ **\$6767**. A continuous deposit s grows to $\frac{s}{.1}(e^{(.1)(20} - 1) = 50{,}000$ so the parent deposits $s = \frac{(.1)50{,}000}{e^2-1} =$ **\$783 per year**.
Saving $s = \$1000/\text{yr}$ grows to $\frac{1000}{.1}(e^{.1t} - 1) = 50{,}000$ when $e^{.1t} = 1 + \frac{5000}{1000}$ or $.1t = \ln 6$ or $t =$ **17.9 years**.

48 The deposit of $4dT$ grows with factor c from time T to time t, and reaches $e^{c(t-T)}4dT$. With $t = 2$ add deposits from $T = 0$ to $T = 1$: $\int_0^1 e^{c(2-T)}4dT = [\frac{4e^{c(2-T)}}{-c}]_0^1 = \mathbf{\frac{4e^c - 4e^{2c}}{-c}}$.

50 $y(t) = (5000 - \frac{500}{.08})e^{.08t} + \frac{500}{.08}$ is zero when $e^{.08t} = \frac{-\frac{500}{.08}}{5000-\frac{500}{.08}} = 5$. Then $.08t = \ln 5$ and $t = \frac{\ln 5}{.08} \approx$ **20 years**. (Remember the deposit grows until it is withdrawn.)

52 After 365 days the value is $y = e^{(.01)365} = e^{3.65} =$ **\$38**.

54 (a) Income = expense when $I_0e^{2ct} = E_0e^{ct}$ or $e^{ct} = \frac{E_0}{I_0}$ or $t = \frac{\ln(E_0/I_0)}{c}$. (b) Integrate $E_0e^{ct} - I_0e^{2ct}$ until $e^{ct} = \frac{E_0}{I_0}$. At the upper limit the integral is $\frac{E_0}{c}e^{ct} - \frac{I_0}{2c}e^{2ct} = \frac{1}{c}(\frac{E_0^2}{I_0} - \frac{I_0}{2}\frac{E_0^2}{I_0^2}) = \frac{E_0^2}{2cI_0}$. Lower limit is $t = 0$ so subtract $\frac{E_0}{c} - \frac{I_0}{2c}$: Borrow $\frac{E_0^2}{2cI_0} - \frac{E_0}{c} + \frac{I_0}{2c}$.

56 After 10 years (halfway through the mortgage) the variable rate $.09 + .001(10)$ equals the fixed rate $10\% = .1$. Since the variable was lower early, and therefore longer, the **variable rate is preferred.**

58 If $\frac{dy}{dt} = -y + 7$ then $\frac{dy}{dt}$ is zero at $y_\infty = \mathbf{7}$ (this is $-\frac{s}{c} = \frac{7}{1}$). The derivative of $y - y_\infty$ is $\frac{dy}{dt}$, so the derivative of $y - 7$ is $-(y-7)$. The decay rate is $c = -1$, and $\mathbf{y - 7 = e^{-t}(y_0 - 7)}$.

60 All solutions to $\frac{dy}{dt} = c(y - 12)$ converge to $\mathbf{y = 12}$ provided c is **negative**.

62 (a) **False** because $(y_1 + y_2)' = cy_1 + s + cy_2 + s$. We have $2s$ not s. (b) **True** because $(\frac{1}{2}y_1 + \frac{1}{2}y_2)' = \frac{1}{2}cy_1 + \frac{1}{2}s + \frac{1}{2}cy_2 + \frac{1}{2}s$. (c) **False** because the derivative of $y' = cy + s$ is $(y')' = c(y')$ and s is gone.

64 The solution is $y = Ae^{ct} + B$. Substitute $t = 0, 1, 2$ and move B to the left side: $100 - B = A$, $90 - B = Ae^c, 84 - B = Ae^{2c}$. Then $(100 - B)(84 - B) = (90 - B)(90 - B)$; both sides are A^2e^{2c}. Solve for $B : 8400 - 184B + B^2 = 8100 - 180B + B^2$ or $300 = 4B$. The steady state is $\mathbf{B = 75}$. (This problem is a good challenge and was meant to have a star.)

66 (a) The white coffee cools to $y_\infty + (y_0 - y_\infty)e^{ct} = \mathbf{20 + 40e^{ct}}$. (b) The black coffee cools to $\mathbf{20 + 50e^{ct}}$. The milk warms to $\mathbf{20 - 10e^{ct}}$. The mixture $\frac{5(\text{black coffee})+1(\text{milk})}{6}$ has $20 + \frac{250-10}{6}e^{ct} = \mathbf{20 + 40e^{ct}}$. So it doesn't matter when you add the milk!

6.4 Logarithms (page 258)

The natural logarithm of x is $\int_1^x \frac{\mathbf{dt}}{\mathbf{t}}$ (or $\int_1^x \frac{\mathbf{dx}}{\mathbf{x}}$). This definition leads to $\ln xy = \mathbf{\ln x + \ln y}$ and $\ln x^n = \mathbf{n \ln x}$. Then e is the number whose logarithm (area under $1/x$ curve) is **1**. Similarly e^x is now defined as the number whose natural logarithm is **x**. As $x \to \infty, \ln x$ approaches **infinity**. But the ratio $(\ln x)/\sqrt{x}$ approaches **zero**. The domain and range of $\ln x$ are $\mathbf{0 < x < \infty, -\infty < \ln x < \infty}$.

The derivative of $\ln x$ is $\frac{\mathbf{1}}{\mathbf{x}}$. The derivative of $\ln(1+x)$ is $\frac{\mathbf{1}}{\mathbf{1+x}}$. The tangent approximation to $\ln(1+x)$ at $x = 0$ is **x**. The quadratic approximation is $\mathbf{x - \frac{1}{2}x^2}$. The quadratic approximation to e^x is $\mathbf{1 + x + \frac{1}{2}x^2}$.

The derivative of $\ln u(x)$ by the chain rule is $\frac{\mathbf{1}}{\mathbf{u(x)}}\frac{\mathbf{du}}{\mathbf{dx}}$. Thus $(\ln \cos x)' = \mathbf{-\frac{\sin x}{\cos x} = -\tan x}$. An antiderivative of $\tan x$ is $\mathbf{-\ln \cos x}$. The product $p = xe^{5x}$ has $\ln p = \mathbf{5x + \ln x}$. The derivative of this equation is $\mathbf{p'/p = 5 + \frac{1}{x}}$. Multiplying by p gives $p' = \mathbf{xe^{5x}(5 + \frac{1}{x}) = 5xe^{5x} + e^{5x}}$, which is **LD** or logarithmic differentiation.

The integral of $u'(x)/u(x)$ is $\mathbf{\ln u(x)}$. The integral of $2x/(x^2+4)$ is $\mathbf{\ln(x^2+4)}$. The integral of $1/cx$ is $\frac{\mathbf{\ln x}}{\mathbf{c}}$. The integral of $1/(ct+s)$ is $\frac{\mathbf{\ln(ct+s)}}{\mathbf{c}}$. The integral of $1/\cos x$, after a trick, is $\mathbf{\ln(\sec x + \tan x)}$. We should write $\ln|x|$ for the antiderivative of $1/x$, since this allows $\mathbf{x < 0}$. Similarly $\int du/u$ should be written $\mathbf{\ln|u|}$.

1 $\frac{1}{x}$ **3** $\frac{-1}{x(\ln x)^2}$ **5** $\ln x$ **7** $\frac{\cos x}{\sin x} = \cot x$ **9** $\frac{7}{x}$ **11** $\frac{1}{3}\ln t + C$ **13** $\ln\frac{4}{3}$
15 $\frac{1}{2}\ln 5$ **17** $-\ln(\ln 2)$ **19** $\ln(\sin x) + C$ **21** $-\frac{1}{3}\ln(\cos 3x) + C$ **23** $\frac{1}{3}(\ln x)^3 + C$
27 $\ln y = \frac{1}{2}\ln(x^2+1)$; $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$ **29** $\frac{dy}{dx} = e^{\sin x}\cos x$
31 $\frac{dy}{dx} = e^x e^{e^x}$ **33** $\ln y = e^x \ln x$; $\frac{dy}{dx} = ye^x(\ln x + \frac{1}{x})$ **35** $\ln y = -1$ so $y = \frac{1}{e}, \frac{dy}{dx} = 0$ **37** 0
39 $-\frac{1}{x}$ **41** $\sec x$ **47** $.1; .095; .095310179$ **49** $-.01; -.01005; -.010050335$
51 l'Hôpital: 1 **53** $\frac{1}{\ln b}$ **55** $3 - 2\ln 2$ **57** Rectangular area $\frac{1}{2} + \cdots + \frac{1}{n} < \int_1^n \frac{dt}{t} = \ln n$
59 Maximum at e **61** 0 **63** $\log_{10} e$ or $\frac{1}{\ln 10}$ **65** $1 - x; 1 + x\ln 2$
67 Fraction is $y = 1$ when $\ln(T+2) - \ln 2 = 1$ or $T = 2e - 2$ **69** $y' = \frac{2}{(t+2)^2} \to y = 1 - \frac{2}{t+2}$ never equals 1
71 $\ln p = x\ln 2$; **LD** $2^x \ln 2$; **ED** $p = e^{x\ln 2}, p' = \ln 2\, e^{x\ln 2}$
75 $2^4 = 4^2$; $y\ln x = x\ln y \to \frac{\ln x}{x} = \frac{\ln y}{y}$; $\frac{\ln x}{x}$ decreases after $x = e$, and the only integers before e are 1 and 2.

2 $\frac{\mathbf{2}}{\mathbf{2x+1}}$ **4** $\frac{x(\frac{1}{x})-(\ln x)}{x^2} = \frac{\mathbf{1-\ln x}}{\mathbf{x^2}}$ **6** Use $(\log_e 10)(\log_{10} x) = \log_e x$. Then $\frac{d}{dx}(\log_{10} x) = \frac{1}{\log_e 10} \cdot \frac{1}{x} = \frac{\mathbf{1}}{\mathbf{x \ln 10}}$.
8 $y = \ln u$ so $\frac{dy}{dx} = \frac{du/dx}{u} = \frac{1/x}{\ln x} = \frac{\mathbf{1}}{\mathbf{x \ln x}}$. **10** $y = 7\ln 4x = 7\ln 4 + 7\ln x$ so $\frac{dy}{dx} = \frac{\mathbf{7}}{\mathbf{x}}$.
12 $\mathbf{\ln(1+x)}$ from $\int \frac{du}{u}$. **14** $\frac{1}{2}\ln(3+2t)]_0^1 = \frac{1}{2}(\ln 5 - \ln 3) = \mathbf{\frac{1}{2}\ln\frac{5}{3}}$.
16 $y = \frac{x^3}{x^2+1}$ equals $x - \frac{x}{x^2+1}$. Its integral is $[\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1)]_0^2 = \mathbf{2 - \frac{1}{2}\ln 5}$.
18 $\int \frac{du}{u^2} = -\frac{1}{u} = [-\frac{1}{\ln x}]_2^e = \mathbf{-1 + \frac{1}{\ln 2}}$.

20 $\int \frac{\sin x}{\cos x}dx = \int \frac{-du}{u} = -\ln u = -\ln(\cos x)]_0^{\pi/4} = -\ln\frac{1}{\sqrt{2}} + 0 = \mathbf{\frac{1}{2}\ln 2}$.

22 $\int \frac{\cos 3x}{\sin 3x}dx = \mathbf{\frac{1}{3}\ln(\sin 3x)} + C$.

24 Set $u = \ln\ln x$. By the chain rule $\frac{du}{dx} = \frac{1}{\ln x}\frac{1}{x}$. Our integral is $\int \frac{du}{u} = \ln u = \mathbf{\ln(\ln(\ln x))} + C$.

26 The graph starts at $-\infty$ when $x = 0$. It reaches zero when $x = \frac{\pi}{2}$ and goes down again. At $x = \pi$ it stops.

28 $\ln y = \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln(x^2-1)$. Then $\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+1} + \frac{x}{x^2-1} = \frac{2x^3}{x^4-1}$. Then $\frac{dy}{dx} = \frac{2x^3y}{x^4-1} = \frac{2x^3}{\sqrt{x^4-1}}$.

30 $\ln y = -\frac{1}{x}\ln x$ and $\frac{1}{y}\frac{dy}{dx} = \frac{\ln x - 1}{x^2}$ so $\frac{dy}{dx} = \mathbf{(\frac{\ln x-1}{x^2})x^{-1/x}}$.

32 $\ln y = e\ln x$ and $\frac{1}{y}\frac{dy}{dx} = \frac{e}{x}$ so $\frac{dy}{dx} = \frac{e}{x}x^e = \mathbf{ex^{e-1}}$.

34 $\ln y = \frac{1}{2}\ln x + \frac{1}{3}\ln x + \frac{1}{6}\ln x = \ln x$ and eventually $\frac{dy}{dx} = 1$.

36 $\ln y = -\ln x$ so $\frac{1}{y}\frac{dy}{dx} = \frac{-1}{x}$ and $\frac{dy}{dx} = -\frac{e^{-\ln x}}{x}$. Alternatively we have $y = \frac{1}{x}$ and $\frac{dy}{dx} = \mathbf{-\frac{1}{x^2}}$.

38 $[\ln x]_1^{e^\pi} + [\ln|x|]_{-2}^{-1} = (\pi - 0) + (0 - \ln|-2|) = \pi - \mathbf{\ln 2}$.

40 $\frac{d}{dx}\ln x = \mathbf{\frac{1}{x}}$. Alternatively use $\frac{1}{x^2}\frac{d}{dx}(x^2) - \frac{1}{x}\frac{d}{dx}(x) = \frac{1}{x}$.

42 This is $\int \frac{du}{u}$ with $u = \sec x + \tan x$ so the integral is $\mathbf{\ln(\sec x + \tan x)}$. See Problem 41!

44 $\frac{d}{dx}(\ln(x-a) - \ln(x+a)) = \mathbf{\frac{1}{x-a}} - \mathbf{\frac{1}{x+a}} = \frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{2a}{x^2-a^2}$.

46 Misprint! $\frac{1+\frac{x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}\frac{\sqrt{x^2+a^2}+x}{x+\sqrt{x^2+a^2}} = \mathbf{\frac{1}{\sqrt{x^2+a^2}}}$.

48 Linear: $e^{.1} \approx 1 + .1 = 1.1$. Quadratic: $e^{.1} \approx 1 + .1 + \frac{1}{2}(.1)^2 = 1.105$. Calculator: $e^{.1} = 1.105170918$.

50 Linear: $e^2 \approx 1 + 2 = 3$. Quadratic: $e^2 \approx 1 + 2 + \frac{1}{2}(2^2) = 5$. Calculator: $e^2 = 7.389$.

52 Use l'Hôpital's Rule: $\lim_{x\to 0}\frac{e^x}{1} = \mathbf{1}$.

54 Use l'Hôpital's Rule: $\lim_{x\to 0}\frac{b^x\ln b}{1} = \mathbf{\ln b}$. We have redone the derivative of b^x at $x = 0$.

56 Upper rectangles $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \approx \mathbf{.7595}$. Lower rectangles: $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \approx \mathbf{.6345}$. Exact area $\ln 2 \approx .693$.

58 $\frac{1}{t}$ is smaller than $\frac{1}{\sqrt{t}}$ when $1 < t < x$. Therefore $\int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{\sqrt{t}}$ or $\ln x < 2\sqrt{x} - 2$. (In Problem 59 this leads to $\frac{\ln x}{x} \to 0$. Another approach is from $\frac{x}{e^x} \to 0$ in Problem 6.2.59. If x is much smaller than e^x then $\ln x$ is much smaller than x.)

60 From $\frac{\ln x}{x} \to 0$ we know $\frac{\ln x^{1/n}}{x^{1/n}} \to 0$. This is $\frac{1}{n}\frac{\ln x}{x^{1/n}} \to 0$. Since n is fixed we have $\frac{\ln x}{x^{1/n}} \to 0$.

62 $\frac{1}{x}\ln\frac{1}{x} = -\frac{\ln x}{x} \to 0$ as $x \to \infty$. This means $\mathbf{y\ln y \to 0}$ as $y = \frac{1}{x} \to 0$. (Emphasize: The factor $y \to 0$ is "stronger" than the factor $\ln y \to -\infty$.)

64 From $\int_1^x t^{h-1}dt = \frac{x^h-1}{h}$ we find $\int_1^x t^{-1}dt = \lim_{h\to 0}\frac{x^h-1}{h}$. The left side is recognized as $\mathbf{\ln x}$. (The right side is the "mysterious constant c" when the base is $b = x$. We discovered earlier that $c = \ln b$.)

66 $.01 - \frac{1}{2}(.01)^2 + \frac{1}{3}(.01)^3 = .00995033\cdots$ Also $\ln 1.02 \approx .02 - \frac{1}{2}(.02)^2 + \frac{1}{3}(.02)^3 = \mathbf{.01980266}\cdots$

68 To emphasize: If the ant didn't crawl, the fraction y would be constant (the ant would move as the band stretches). By crawling $v\,dt$ the fraction y increases by $\frac{v\,dt}{\text{band length}}$. So $\frac{dy}{dt} = \frac{v}{\ell} = \frac{1}{8t+2}$. Then $y = \frac{1}{8}\ln(8t+2) + C = \frac{1}{8}(\ln(8t+2) - \ln 2)$. This equals 1 when $8 = \ln\frac{8t+2}{2}$ or $4t + 1 = e^8$ or $\mathbf{t = \frac{1}{4}(e^8 - 1)}$

70 LD: $\ln p = x\ln x$ so $\frac{1}{p}\frac{dp}{dx} = 1 + \ln x$ and $\frac{dp}{dx} = p(1 + \ln x) = x^x(1 + \ln x)$. Now find the same answer by **ED:** $\frac{d}{dx}(e^{x\ln x}) = e^{x\ln x}\frac{d}{dx}(x\ln x) = x^x(1 + \ln x)$.

72 To compute $\int_1^2 \frac{dx}{x} = \ln 2$ with error $\approx 10^{-5}$ the trapezoidal rule needs $\Delta x \approx 10^{-2}$. *Six Simpson steps:* $S_6 = \frac{1}{36}[\frac{1}{1} + \frac{4}{13/12} + \frac{2}{7/6} + \frac{4}{15/12} + \frac{2}{8/6} + \frac{4}{17/12} + \frac{2}{9/6} + \frac{4}{19/12} + \frac{2}{10/6} + \frac{4}{21/12} + \frac{2}{11/6} + \frac{4}{23/12} + \frac{1}{12/6}] =$ .693149 compared to $\ln 2 = .693147$. Predicted error $\frac{1}{2880}(\frac{1}{6})^4(6 - \frac{6}{2^4}) = 1.6 \times 10^{-6}$, actual error 1.5×10^{-6}.

74 $\frac{1}{\ln 90{,}000} = .0877$ says that about 877 of the next 10,000 numbers are prime: close to the actual count 879.

76 $\frac{\ln x}{x} = \frac{t\ln(\frac{t+1}{t})}{(\frac{t+1}{t})^t}$. This equals $\frac{\ln y}{y} = \frac{(t+1)\ln(\frac{t+1}{t})}{(\frac{t+1}{t})^{t+1}} = \frac{t+1}{\frac{t+1}{t}}\frac{\ln(\frac{t+1}{t})}{(\frac{t+1}{t})^t}$. The curve $x^y = y^x$ is asymptotic to $x = 1$, for t near zero. It approaches $x = e, y = e$ as $t \to \infty$. It is symmetric across the 45°

line (no change by reversing x and y), roughly like the hyperbola $(x-1)(y-1)=(e-1)^2$.

6.5 Separable Equations Including the Logistic Equation (page 266)

The equations $dy/dt = cy$ and $dy/dt = cy + s$ and $dy/dt = u(y)v(t)$ are called **separable** because we can separate y from t. Integration of $\int dy/y = \int c\,dt$ gives $\mathbf{\ln y = ct + constant}$. Integration of $\int dy/(y+s/c) = \int c\,dt$ gives $\mathbf{\ln(y + \frac{s}{c}) = ct + C}$. The equation $dy/dx = -x/y$ leads to $\int \mathbf{y\,dy} = -\int \mathbf{x\,dx}$. Then $y^2 + x^2 =$ **constant** and the solution stays on a circle.

The logistic equation is $dy/dt = \mathbf{cy - by^2}$. The new term $-by^2$ represents **competition** when cy represents growth. Separation gives $\int dy/(cy - by^2) = \int dt$, and the y-integral is $1/c$ times $\ln \frac{\mathbf{y}}{\mathbf{c-by}}$. Substituting y_0 at $t = 0$ and taking exponentials produces $y/(c - by) = e^{ct}\mathbf{y_0/(c - by_0)}$. As $t \to \infty$,
y approaches $\frac{\mathbf{c}}{\mathbf{b}}$. That is the steady state where $cy - by^2 = \mathbf{0}$. The graph of y looks like an **S**, because it has an inflection point at $\frac{1}{2}\frac{\mathbf{c}}{\mathbf{b}}$.

In biology and chemistry, concentrations y and z react at a rate proportional to y times z. This is the Law of **Mass Action.** In a model equation $dy/dt = c(y)y$, the rate c depends on **y**. The MM equation is $dy/dt = \mathbf{-cy/(y+K)}$. Separating variables yields $\int \frac{\mathbf{y+K}}{\mathbf{y}} dy = \int \mathbf{-c\,dt} = -ct + C$.

1 $7e^t - 5$ **3** $(\frac{3}{2}x^2+1)^{1/3}$ **5** x **7** $e^{1-\cos t}$ **9** $(\frac{ct}{2} + \sqrt{y_0})^2$ **11** $y_\infty = 0; t = \frac{1}{by_0}$
15 $z = 1 + e^{-t}, y$ is in **13** **17** $ct = \ln 3, ct = \ln 9$
19 $b = 10^{-9}, c = 13 \cdot 10^{-3}; y_\infty = 13 \cdot 10^6$; at $y = \frac{c}{2b}$ (10) gives $\ln \frac{1}{b} = ct + \ln \frac{10^6}{c - 10^6 b}$ so $t = 1900 + \frac{\ln 12}{c} = 2091$
21 y^2 dips down and up (a valley) **23** $sc = 1 = sbr$ so $s = \frac{1}{c}, r = \frac{c}{b}$
25 $y = \frac{N}{1+e^{-Nt}(N-1)}; T = \frac{\ln(N-1)}{N} \to 0$ **27** Dividing cy by $y + K > 1$ slows down y'
29 $\frac{dR}{dy} = \frac{cK}{(y+K)^2} > 0, \frac{cy}{y+K} \to c$
31 $\frac{dY}{dT} = \frac{-Y}{Y+1}$; multiply $e^{y/K}\frac{y}{K} = e^{-ct/K}e^{y_0/K}(\frac{y_0}{K})$ by K and take the Kth power to reach (19)
33 $y' = (3-y)^2; \frac{1}{3-y} = t + \frac{1}{3}; y = 2$ at $t = \frac{2}{3}$
35 $Ae^t + D = Ae^t + B + Dt + t \to D = -1, B = -1; y_0 = A + B$ gives $A = 1$
37 $y \to 1$ from $y_0 > 0$, $y \to -\infty$ from $y_0 < 0; y \to 1$ from $y_0 > 0, y \to -1$ from $y_0 < 0$
39 $\int \frac{\cos y\,dy}{\sin y} = \int dt \to \ln(\sin y) = t + C = t + \ln\frac{1}{2}$. Then $\sin y = \frac{1}{2}e^t$ stops at 1 when $t = \ln 2$

2 $y\,dy = dt$ gives $\frac{1}{2}y^2 = t + C$. Then $C = \frac{1}{2}$ at $t = 0$. So $y^2 = 2t + 1$ and $y = \mathbf{\sqrt{2t+1}}$.
4 $\frac{dy}{y^2+1} = dx$ gives $\tan^{-1} y = x + C$. Then $C = 0$ at $x = 0$. So $y = \mathbf{\tan x}$.
6 $\frac{dy}{\tan y} = \cos x dx$ gives $\ln(\sin y) = \sin x + C$. Then $C = \ln(\sin 1)$ at $x = 0$. After taking exponentials $\mathbf{\sin y = (\sin 1)e^{\sin x}}$. No solution after $\sin y$ reaches 1 (at the point where $(\sin 1)e^{\sin x} = 1$).
8 $e^y dy = e^t dt$ so $e^y = e^t + C$. Then $C = e^e - 1$ at $t = 0$. After taking logarithms $y = \mathbf{\ln(e^t + e^e - 1)}$.
10 $\frac{d(\ln y)}{d(\ln x)} = \frac{dy/y}{dx/x} = n$. Therefore $\ln y = n \ln x + C$. Therefore $y = (x^n)(e^C) =$ **constant times $\mathbf{x^n}$**.

12 $y' = by^2$ gives $y^{-2}dy = b\,dt$ and $-\frac{1}{y} = bt + C$. Then $C = -\frac{1}{2}$ at $t = 0$. Therefore $y = \frac{-1}{bt-\frac{1}{2}}$ which becomes infinite when $bt = \frac{1}{2}$ or $t = \frac{1}{2b}$.

14 (a) Compare $\frac{2}{1+e^{-t}}$ with $\frac{c}{b+de^{-ct}}$. In the exponent $c = 1$. Then $b = d = \frac{1}{2}$. Thus $y' = y - \frac{1}{2}y^2$ with $y_0 = 1$. (b) For $\frac{1}{1+e^{-3t}}$ the exponent gives $c = 3$. Then also $b = d = 3$. Thus $y' = 3y - 3y^2$ with $y_0 = \frac{1}{2}$.

16 Equation (14) is $z = \frac{1}{c}(b + \frac{c-by_0}{y_0}e^{-ct})$. Turned upside down this is $y = \frac{c}{b+de^{-ct}}$ with $d = \frac{c-by_0}{y_0}$.

18 Correction: $u = \frac{y}{c-by}$. Then $\frac{du}{dt} = \frac{d}{dt}(\frac{y}{c-by}) = \frac{(c-by)\frac{dy}{dt} - y(-b\frac{dy}{dt})}{(c-by)^2} = \frac{c}{(c-by)^2}\frac{dy}{dt}$. Substitute $\frac{dy}{dt} = y(c-by)$ to obtain $\frac{du}{dt} = \frac{cy}{c-by} = cu$. So $u = u_0e^{ct}$.

20 $y' = y + y^2$ has $c = 1$ and $b = -1$ with $y_0 = 1$. Then $y(t) = \frac{1}{-1+2e^{-t}}$ by formula (12). The denominator is zero and y blows up when $2e^{-t} = 1$ or $t = \ln 2$.

22 If $u = \frac{1}{y^2}$ then $\frac{du}{dt} = \frac{-2y'}{y^3} = \frac{-2(cy-by^3)}{y^3} = -2cu + 2b$. The solution is $u = (u_0 - \frac{2b}{2c})e^{-2ct} + \frac{2b}{2c}$. Then $y = [(\frac{1}{y_0^2} - \frac{b}{c})e^{-2ct} + \frac{b}{c}]^{-1/2}$ solves the equation $y' = cy - by^3$ with "cubic competition". Another S-curve!

24 $y_0 = rY_0$ and $\frac{dY}{dT} = \frac{dy/r}{dt/s}$ so $(\frac{dY}{dT})_0 = \frac{s}{r}y_0'$.

26 At the middle of the S-curve $y = \frac{c}{2b}$ and $\frac{dy}{dt} = c(\frac{c}{2b}) - b(\frac{c}{2b})^2 = \frac{c^2}{4b}$. If b and c are multiplied by 10 then so is this slope $\frac{c^2}{4b}$, which becomes steeper.

28 If $\frac{cy}{y+K} = d$ then $cy = dy + dK$ and $y = \frac{dK}{c-d}$. At this steady state the maintenance dose replaces the aspirin being eliminated.

30 The rate $R = \frac{cy}{y+K}$ is a decreasing function of K because $\frac{dR}{dK} = \frac{-cy}{(y+K)^2}$.

34 $\frac{d[A]}{dt} = -r[A][B] = -r[A](b_0 - \frac{n}{m}(a_0 - [A]))$. The changes $a_0 - [A]$ and $b_0 - [B]$ are in the proportion m to n; we solved for $[B]$.

36 To change $cy - by^2$ (with linear term) to $a^2 - x^2$ (no linear term), set $x = \sqrt{b}y - \frac{c}{2\sqrt{b}}$ and $a = \frac{c}{2\sqrt{b}}$. (We completed the square in $cy - by^2$.) Now match integrals: The factor $\frac{1}{2a}$ is $\frac{1}{c}$ times $\sqrt{b}$ (from $dx = \sqrt{b}\,dy$). The ratio $\frac{a+x}{a-x} = \frac{\sqrt{b}y}{\frac{c}{\sqrt{b}} - \sqrt{b}y}$ is $\frac{y}{c-by}$.

38 The y line shows where y increases (by $y' = f(y)$) and where y decreases. Then the points where $f(y) = 0$ are either approached or left behind.

40 $y' = cy(1 - \frac{y}{K})$ agrees with $y' = cy - by^2$ if $K = \frac{c}{b}$. Then $y = K$ is the steady state where $y' = 0$ (this agrees with $y_\infty = \frac{c}{b}$). The inflection point is halfway: $y = \frac{K}{2}$ where $y' = c\frac{K}{2}(1 - \frac{1}{2}) = \frac{c}{4}K$ and $y'' = 0$.

6.6 Powers Instead of Exponentials (page 276)

The infinite series for e^x is $\mathbf{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3} + \cdots$. Its derivative is $\mathbf{e^x}$. The denominator $n!$ is called "**n factorial**" and is equal to $\mathbf{n(n-1)\cdots(1)}$. At $x = 1$ the series for e is $\mathbf{1 + 1 + \frac{1}{2} + \frac{1}{6}} + \cdots$.

To match the original definition of e, multiply out $(1 + 1/n)^n = \mathbf{1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}(\frac{1}{n})^2}$ (first three terms). As $n \to \infty$ those terms approach $\mathbf{1 + 1 + \frac{1}{2}}$ in agreement with e. The first three terms of $(1 + x/n)^n$ are $\mathbf{1 + n(\frac{x}{n}) + \frac{n(n-1)}{2}(\frac{x}{n})^2}$. As $n \to \infty$ they approach $\mathbf{1 + x + \frac{1}{2}x^2}$ in agreement with e^x. Thus $(1 + x/n)^n$ approaches $\mathbf{e^x}$. A quicker method computes $\ln(1 + x/n)^n \approx \mathbf{x}$ (first term only) and takes the exponential.

Compound interest (n times in one year at annual rate x) multiplies by $\mathbf{(1 + \frac{x}{n})^n}$. As $n \to \infty$, continuous

compounding multiplies by $\mathbf{e^x}$. At $x = 10\%$ with continuous compounding, \$1 grows to $\mathbf{e^{.1} \approx \$1.105}$ in a year.

The difference equation $y(t+1) = ay(t)$ yields $y(t) = \mathbf{a^t}$ times y_0. The equation $y(t+1) = ay(t) + s$ is solved by $y = a^t y_0 + s[1 + a + \cdots + a^{t-1}]$. The sum in brackets is $\mathbf{\frac{1-a^t}{1-a}}$ or $\mathbf{\frac{a^t-1}{a-1}}$. When $a = 1.08$ and $y_0 = 0$, annual deposits of $s = 1$ produce $y = \mathbf{\frac{1.08^t-1}{.08}}$ after t years. If $a = \frac{1}{2}$ and $y_0 = 0$, annual deposits of $s = 6$ leave $\mathbf{12(1 - \frac{1}{2^t})}$ after t years, approaching $y_\infty = \mathbf{12}$. The steady equation $y_\infty = ay_\infty + s$ gives $y_\infty = \mathbf{s/(1-a)}$.

When i = interest rate per period, the value of $y_0 = \$1$ after N periods is $y(N) = \mathbf{(1+i)^N}$. The deposit to produce $y(N) = 1$ is $y_0 = \mathbf{(1+i)^{-N}}$. The value of $s = \$1$ deposited after each period grows to $y(N) = \mathbf{\frac{1}{i}((1+i)^N - 1)}$. The deposit to reach $y(N) = 1$ is $s = \mathbf{\frac{1}{i}(1 - (1+i)^{-N})}$.

Euler's method replaces $y' = cy$ by $\Delta y = cy\Delta t$. Each step multiplies y by $\mathbf{1 + c\Delta t}$. Therefore y at $t = 1$ is $(1 + c\Delta t)^{1/\Delta t} y_0$, which converges to $\mathbf{y_0 e^c}$ as $\Delta t \to 0$. The error is proportional to $\mathbf{\Delta t}$, which is too large for scientific computing.

1 $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$ **3** $1 \pm x + \frac{x^2}{2} \pm \frac{x^3}{6} + \cdots$ **5** 1050.62; 1050.95; 1051.25
7 $1 + n(\frac{-1}{n}) + \frac{n(n+1)}{2}(\frac{-1}{n})^2 \to 1 - 1 + \frac{1}{2}$ **9** square of $(1 + \frac{1}{n})^n$; set $N = 2n$
11 Increases; $\ln(1 + \frac{1}{x}) - \frac{1}{x+1} > 0$ **13** $y(3) = 8$ **15** $y(t) = 4(3^t)$ **17** $y(t) = t$
19 $y(t) = \frac{1}{2}(3^t - 1)$ **21** $s(\frac{a^t-1}{a-1})$ if $a \neq 1$; st if $a = 1$ **23** $y_0 = 6$ **25** $y_0 = 3$
27 $-2, -10, -26 \to -\infty; -5, -\frac{17}{2}, -\frac{41}{4} \to -12$ **29** $P = \frac{b}{c+d}$ **31** 10.38% **33** $100(1.1)^{20} = \$673$
35 $\frac{100{,}000(.1/12)}{1-(1+.1/12)^{-240}} = 965$ **37** $\frac{1000}{.1}(1.1^{20} - 1) = 57{,}275$ **39** $y_\infty = 1500$ **41** $2; (\frac{53}{52})^{52} = 2.69; e$
43 $1.0142^{12} = 1.184 \to$ Visa charges 18.4%

2 $y = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \cdots$ Integrate each term and multiply by 2 to find the next term.
4 A larger series is $1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \mathbf{3}$. This is greater than $1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = e$.
6 $\ln(1 - \frac{1}{n})^n = n \ln(1 - \frac{1}{n}) \approx n(-\frac{1}{n}) = -1$. Take exponentials: $\mathbf{(1 - \frac{1}{n})^n \approx e^{-1}}$. Similarly $\ln(1 + \frac{2}{n})^n = n\ln(1 + \frac{2}{n}) \approx n(\frac{2}{n}) = 2$. Take exponentials: $\mathbf{(1 + \frac{2}{n})^n \approx e^2}$.
8 The exact sum is $e^{-1} \approx .37$ (Problem 6). After five terms $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = .375$.
10 By the quick method $\ln(1 + \frac{1}{n^2})^n \approx n(\frac{1}{n^2}) \to 0$. So $(1 + \frac{1}{n^2})^n \to e^0 = \mathbf{1}$. Similarly $\ln(1 + \frac{1}{n})^{n^2} \approx n^2(\frac{1}{n}) \to \infty$ so $(1 + \frac{1}{n})^{n^2} \to \infty$.
12 Under the graph of $\frac{1}{t}$, the area from 1 to $1 + \frac{1}{x}$ is $\ln(1 + \frac{1}{x})$. The rectangle inside this area has base $\frac{1}{x}$ and height $\frac{1}{1+\frac{1}{x}}$. **Its area is $\mathbf{\frac{1}{x+1}}$** so this is below $\ln(1 + \frac{1}{x})$.
14 $y(0) = 0, y(1) = 1, y(2) = 3, y(3) = 7$ (and $y(n) = \mathbf{2^n - 1}$). **16** $y(t) = \mathbf{(\frac{1}{2})^t}$.
18 $y(t) = \mathbf{t}$ (Notice that $a = 1$). **20** $y(t) = \mathbf{3^t} + \mathbf{s}[\mathbf{\frac{3^t-1}{2}}]$. **22** $y(t) = 5a^t + s[\frac{a^t-1}{a-1}]$.
24 Ask for $\frac{1}{2}y(0) - 6 = y(0)$. Then $y(0) = \mathbf{-12}$. **26** Ask for $-\frac{1}{2}y(0) + 6 = y(0)$. Then $y(0) = \mathbf{4}$.
28 If $\mathbf{-1 < a < 1}$ then $\frac{1-a^t}{1-a}$ approaches $\mathbf{\frac{1}{1-a}}$.
30 The equation $-dP(t+1) + b = cP(t)$ becomes $-2P(t+1) + 8 = P(t)$ or $P(t+1) = -\frac{1}{2}P(t) + 4$. Starting from $P(0) = 0$ the solution is $P(t) = 4[\frac{(-\frac{1}{2})^t-1}{-\frac{1}{2}-1}] = \mathbf{\frac{8}{3}(1 - (-\frac{1}{2})^t)} \to \frac{8}{3}$.
32 $(1 + \frac{.10}{365})^{365} = 1.105156\cdots$ (Compare with $e^{.1} \approx 1 + .1 + \frac{1}{2}(.1)^2 = 1.105$.) The effective rate is **5.156%.**
34 Present value $= \$1{,}000\ (1.1)^{-20} \approx \mathbf{\$148.64}$.
36 **Correction** to formulas **5** and **6** on page 273: Change $.05n$ to $.05/n$. In this problem $n = 12$ and $N = 6(12) = 72$ months and .05 becomes .1 in the loan formula: $s = \$10{,}000\ (.1)/12[1 - (1 + \frac{.1}{12})^{-72}] \approx \mathbf{\$185}$.

38 Solve $\$1000 = \$8000\,[\frac{.1}{1-(1.1)^{-n}}]$ for n. Then $1-(1.1)^{-n} = .8$ or $(1.1)^{-n} = .2$. Thus $1.1^n = 5$ and $n = \frac{\ln 5}{\ln 1.1} \approx$ **17 years.**

40 The interest is $(.05)1000 = \$50$ in the first month. You pay \$60. So your debt is now $\$1000 - \$10 =$ **\$990**. Suppose you owe $y(t)$ after month t, so $y(0) = \$1000$. The next month's interest is $.05y(t)$. You pay \$60. So $y(t+1) = 1.05y(t) - 60$. After 12 months $y(12) = (1.05)^{12}1000 - 60[\frac{(1.05)^{12}-1}{1.05-1}]$. This is also $\frac{60}{.05} + (1000 - \frac{60}{.05})(1.05)^{12} \approx \$841$.

42 Compounding n times in a year at 100% per year gives $(1+\frac{1}{n})^n$. Its logarithm is $n\ln(1+\frac{1}{n}) \approx n[\frac{1}{n} - \frac{1}{2n^2}] = 1 - \frac{1}{2n}$. Therefore $(1+\frac{1}{n})^n \approx e(e^{-1/2n}) \approx e(1-\frac{1}{2n})$.

44 Use the loan formula with $.09/n$ not $.09n$: payments $s = 80{,}000\frac{.09/12}{[1-(1+\frac{.09}{12})^{-360}]} \approx \$643.70$. Then 360 payments equal \$231,732.

6.7 Hyperbolic Functions (page 280)

Cosh $x = \mathbf{\frac{1}{2}(e^x + e^{-x})}$ and sinh $x = \mathbf{\frac{1}{2}(e^x - e^{-x})}$ and $\cosh^2 x - \sinh^2 x = \mathbf{1}$. Their derivatives are $\mathbf{\sinh x}$ and $\mathbf{\cosh x}$ and **zero**. The point $(x, y) = (\cosh t, \sinh t)$ travels on the hyperbola $\mathbf{x^2 - y^2 = 1}$. A cable hangs in the shape of a catenary $y = \mathbf{a \cosh \frac{x}{a}}$.

The inverse functions $\sinh^{-1} x$ and $\tanh^{-1} x$ are equal to $\ln[x + \sqrt{x^2+1}]$ and $\frac{1}{2}\ln\mathbf{\frac{1+x}{1-x}}$. Their derivatives are $\mathbf{1/\sqrt{x^2+1}}$ and $\frac{1}{1-x^2}$. So we have two ways to write the **antiderivative**. The parallel to cosh x + sinh $x = e^x$ is Euler's formula $\mathbf{\cos x + i \sin x = e^{ix}}$. The formula $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ involves **imaginary** exponents. The parallel formula for $\sin x$ is $\mathbf{\frac{1}{2i}(e^{ix} - e^{-ix})}$.

1 $e^x, e^{-x}, \frac{e^{2x}-e^{-2x}}{4} = \frac{1}{2}\sinh 2x$ **7** $\sinh\ nx$ **9** $3\sinh(3x+1)$ **11** $\frac{-\sinh x}{\cosh^2 x} = -\tanh x\,\text{sech}\,x$

13 $4\cosh x \sinh x$ **15** $\frac{x}{\sqrt{x^2+1}}(\text{sech}\sqrt{x^2+1})^2$ **17** $6\sinh^5 x \cosh x$

19 $\cosh(\ln x) = \frac{1}{2}(x + \frac{1}{x}) = 1$ at $x = 1$ **21** $\frac{5}{13}, \frac{13}{5}, -\frac{12}{5}, -\frac{13}{12}, -\frac{5}{12}$ **23** $0, 0, 1, \infty, \infty$

25 $\frac{1}{2}\sinh(2x+1)$ **27** $\frac{1}{3}\cosh^3 x$ **29** $\ln(1+\cosh x)$ **31** e^x

33 $\int y\,dx = \int \sinh\ t(\sinh\ t\,dt); A = \frac{1}{2}\sinh\ t\cosh t - \int y\,dx; A' = \frac{1}{2}; A = 0$ at $t = 0$ so $A = \frac{1}{2}t$.

41 $e^y = x + \sqrt{x^2+1}, y = \ln[x + \sqrt{x^2+1}]$ **47** $\frac{1}{4}\ln|\frac{2+x}{2-x}|$ **49** $\sinh^{-1} x$ (see 41) **51** $-\text{sech}^{-1}x$

53 $\frac{1}{2}\ln 3; \infty$ **55** $y(x) = \frac{1}{c}\cosh cx; \frac{1}{c}\cosh cL - \frac{1}{c}$

57 $y'' = y - 3y^2; \frac{1}{2}(y')^2 = \frac{1}{2}y^2 - y^3$ is satisfied by $y = \frac{1}{2}\text{sech}^2\frac{x}{2}$

2 $\frac{d}{dx}(\frac{e^x+e^{-x}}{2}) = \frac{e^x-e^{-x}}{2} = \mathbf{\sinh x}$; $\frac{d}{dx}(\frac{e^x-e^{-x}}{2}) = \frac{e^x+e^{-x}}{2} = \mathbf{\cosh x}$.

4 $\frac{d}{dx}(\frac{\sinh x}{\cosh x}) = \frac{(\cosh x)^2-(\sinh x)^2}{(\cosh x)^2} = \mathbf{\frac{1}{(\cosh x)^2} = \text{sech}^2\, x}$.

6 The factor $\frac{1}{2}$ should be removed from Problem 5 Then the derivative of Problem 5 is $2\cosh x \sinh x + 2\sinh x \cosh x = 2\sinh 2x$. Therefore $\sinh 2x = 2\sinh x\cosh x$ (similar to $\sin 2x$).

8 $(\frac{e^x-e^{-x}}{2})(\frac{e^y+e^{-y}}{2}) + (\frac{e^x+e^{-x}}{2})(\frac{e^y-e^{-y}}{2}) = \frac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \sinh(x+y)$. The x derivative gives $\cosh(x+y) = \mathbf{\cosh x \cosh y + \sinh x \sinh y}$.

10 $2x\cosh x^2$ **12** $\sinh(\ln x) = \frac{1}{2}(e^{\ln x} - e^{-\ln x}) = \frac{1}{2}(x - \frac{1}{x})$ with derivative $\mathbf{\frac{1}{2}(1 + \frac{1}{x^2})}$.

14 $\cosh^2 x - \sinh^2 x = 1$ with derivative **zero.**

16 $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$ by the equation following (4). Its derivative is $\mathbf{2e^{2x}}$. More directly the quotient rule gives

$\frac{(1-\tanh x)\text{sech}^2 x+(1+\tanh x)\text{sech}^2 x}{(1-\tanh x)^2} = \frac{2\,\text{sech}^2 x}{(1-\tanh x)^2} = \frac{2}{(\cosh x-\sinh x)^2} = \frac{2}{e^{-2x}} = \mathbf{2e^{2x}}$.

18 $\frac{d}{dx}\ln u = \frac{du/dx}{u} = \frac{\text{sech}\,x\tanh x-\text{sech}^2 x}{\text{sech}\,x+\tanh x}$. Because of the minus sign we do not get sech x. The integral of sech x is $\sin^{-1}(\tanh x) + C$.

20 $\text{sech}\,x = \sqrt{1-(\frac{3}{5})^2} = \frac{4}{5}, \cosh x = \frac{5}{4}, \sinh x = \sqrt{(\frac{5}{4})^2 - 1} = \frac{3}{4}, \coth x = \frac{\sinh x}{\cosh x} = \frac{5}{3}, \text{csch}\,x = \frac{4}{3}$.

22 $\cosh x = \sqrt{(2)^2+1} = \sqrt{5}, \tanh x = \frac{2}{\sqrt{5}}, \text{csch}\,x = \frac{1}{2}, \text{sech}\,x = \frac{1}{\sqrt{5}}, \coth x = \frac{\sqrt{5}}{2}$.

24 $\sinh(\ln 5) = \frac{e^{\ln 5}-e^{-\ln 5}}{2} = \frac{5-\frac{1}{5}}{2} = \mathbf{\frac{12}{5}}; \tanh(2\ln 4) = \frac{e^{2\ln 4}-e^{-2\ln 4}}{e^{2\ln 4}+e^{-2\ln 4}} = \frac{16-\frac{1}{16}}{16+\frac{1}{16}} = \mathbf{\frac{255}{257}}$.

26 $\int x\cosh(x^2)dx = \mathbf{\frac{1}{2}\sinh(x^2)} + C$. **28** $\mathbf{\frac{1}{3}(\tanh x)^3} + C$.

30 $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}dx = \mathbf{\ln(\sinh x)} + C$. **32** $\sinh x + \cosh x = e^x$ and $\int e^{nx}dx = \mathbf{\frac{1}{n}e^{nx}} + C$.

34 $y = \tanh x$ is an odd function, with asymptote $y = -1$ as $x \to -\infty$ and $y = +1$ as $x \to +\infty$. The inflection point is (0,0).

36 $y = \text{sech} x$ looks like a bell-shaped curve with $y_{\max} = 1$ at $x = 0$. The x axis is the asymptote. But note that y decays like $2e^{-x}$ and not like e^{-x^2}.

38 To define $y = \cosh^{-1} x$ we require $x \geq 1$. Select the *positive* y (there are two y's so strictly there is no inverse). For large values, $\cosh y$ is close to $\frac{1}{2}e^y$ so $\cosh^{-1} x$ is close to $\ln 2x$.

40 $\frac{1}{2}\ln(\frac{1+x}{1-x})$ approaches $+\infty$ as $x \to 1$ and $-\infty$ as $x \to -1$. The function is *odd* (so is the tanh function). The graph is an S curve rotated by 90°.

42 The quadratic equation for e^y has solution $e^y = x \pm \sqrt{x^2-1}$. Choose the plus sign so $y \to \infty$ as $x \to \infty$. Then $y = \ln(x+\sqrt{x^2-1})$ is another form of $y = \cosh^{-1} x$.

44 The x derivative of $x = \sinh y$ is $1 = \cosh y\frac{dy}{dx}$. Then $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \mathbf{\frac{1}{\sqrt{1+x^2}}}$ = slope of $\sinh^{-1} x$.

46 The x derivative of $x = \text{sech}\,y$ is $1 = -\text{sech}\,y\tanh y\frac{dy}{dx}$. Then $\frac{dy}{dx} = \frac{-1}{\text{sech}\,y\tanh y} = \mathbf{\frac{-1}{x\sqrt{1-x^2}}}$.

48 Set $x = au$ and $dx = a\,du$ to reach $\int \frac{a\,du}{a^2(1-u^2)} = \frac{1}{a}\tanh^{-1} u = \mathbf{\frac{1}{a}\tanh^{-1}\frac{x}{a}} + C$.

50 Not hyperbolic! Just $\int(x^2+1)^{-1/2}x\,dx = \mathbf{(x^2+1)^{1/2}} + C$.

52 Not hyperbolic! $\int \frac{dx}{\sqrt{1-x^2}} = \mathbf{\sin^{-1}x} + C$.

54 (a) $\frac{dv}{dt} = (\sqrt{g})^2\text{sech}^2\sqrt{g}\,t = g(1-\tanh^2\sqrt{g}t) = g - v^2$. (b) $\int \frac{dv}{g-v^2} = \int dt$ gives (by Problem 48) $\frac{1}{\sqrt{g}}\tanh^{-1}\frac{v}{\sqrt{g}} = t$ or $\tanh^{-1}\frac{v}{\sqrt{g}} = \sqrt{g}t$ or $\frac{v}{\sqrt{g}} = \tanh\sqrt{g}t$. (c) $f(t) = \int \sqrt{g}\tanh\sqrt{g}t\,dt = \int \frac{\sinh\sqrt{g}\,t}{\cosh\sqrt{g}t}\sqrt{g}\,dt = \ln(\cosh\sqrt{g}\,t) + C$.

56 **Change to** $dx = \frac{dW}{\frac{1}{2}W^2-W} = -\frac{dW}{2-W} - \frac{dW}{W}$ and integrate: $x = \ln(2-W) - \ln W = \ln(\frac{2-W}{W})$. Then $\frac{2-W}{W} = e^x$ and $W = \mathbf{\frac{2}{1+e^x}}$. (Note: The text suggests $W - 2$ but that is negative. Writing $\frac{2}{1+e^x}$ as $e^{-x/2}\text{sech}\frac{x}{2}$ is not simpler.)

58 $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x}+e^x) = \cosh x$. Then $\cos i = \cosh 1 = \mathbf{\frac{e+e^{-1}}{2}}$ (real!).

60 The derivative of $e^{ix} = \cos x + i\sin x$ is $ie^{ix} = i(\cos x + i\sin x)$ on the left side and $\frac{d}{dx}\cos x + i\frac{d}{dx}\sin x$ on the right side. Comparing we again find $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = i^2\sin x$.

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Gilbert Strang

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