CHAPTER 14 MULTIPLE INTEGRALS

14.1 Double Integrals (page 526)

The double integral $\iint_R f(x,y)dA$ gives the volume between R and **the surface $z = f(x,y)$**. The base is first cut into small **squares** of area ΔA. The volume above the ith piece is approximately $\mathbf{f(x_i, y_i)\Delta A}$. The limit of the sum $\sum \mathbf{f(x_i, y_i)\Delta A}$ is the volume integral. Three properties of double integrals are $\mathbf{\iint (f+g)dA = \iint f dA + \iint g dA}$ and $\mathbf{\iint cf dA = c\iint f dA}$ and $\mathbf{\iint_R f\,dA = \iint_S f\,dA + \iint_T f\,dA}$ **if R splits into S and T.**

If R is the rectangle $0 \le x \le 4, 4 \le y \le 6$, the integral $\iint x\,dA$ can be computed two ways. One is $\iint x\,dy\,dx$, when the inner integral is $\mathbf{xy]_4^6 = 2x}$. The outer integral gives $\mathbf{x^2]_0^4 = 16}$. When the x integral comes first it equals $\int x\,dx = \mathbf{\frac{1}{2}x^2]_0^4 = 8}$. Then the y integral equals $\mathbf{8y]_4^6 = 16}$. This is the volume between **the base rectangle and the plane $z = x$.**

The area R is $\iint \mathbf{1} dy\,dx$. When R is the triangle between $x = 0, y = 2x$, and $y = 1$, the inner limits on y are **$2x$ and 1.** This is the length of a **thin vertical** strip. The (outer) limits on x are **0 and $\frac{1}{2}$**. The area is $\mathbf{\frac{1}{4}}$. In the opposite order, the (inner) limits on x are **0 and $\frac{1}{2}y$**. Now the strip is **horizontal** and the outer integral is $\mathbf{\int_0^1 \frac{1}{2}y\,dy = \frac{1}{4}}$. When the density is $\rho(x,y)$, the total mass in the region R is $\iint \rho\,\mathbf{dx\,dy}$. The moments are $M_y = \iint \rho\mathbf{x\,dx\,dy}$ and $M_x = \iint \rho\mathbf{y\,dx\,dy}$. The centroid has $\overline{x} = M_y/M$.

1 $\frac{8}{3}; \frac{2}{3}$ **3** $1; \ln\frac{3}{2}$ **5** 2 **7** $\frac{1}{2}$ **9** $\frac{4}{3}$ **11** $\int_{y=1}^2 \int_{x=1}^2 dx\,dy + \int_{y=2}^4 \int_{y/2}^2 dx\,dy$

13 $\int_{y=0}^1 \int_{x=-\frac{1}{2}\ln y}^{-\ln y} dx\,dy$ **15** $\int_{x=0}^1 \int_{y=-\sqrt{x}}^{\sqrt{x}} dy\,dx$ **17** $\int_0^1 \int_0^{y/2} dx\,dy = \int_0^{1/2} \int_{2x}^1 dy\,dx = \frac{1}{4}$

19 $\int_0^3 \int_{-y}^y dx\,dy = \int_{-1}^0 \int_{-x}^3 dy\,dx + \int_0^1 \int_x^3 dy\,dx = 9$ **21** $\int_0^4 \int_{y/2}^y dx\,dy + \int_4^8 \int_{y/2}^4 dx\,dy = \int_0^4 \int_x^{2x} dy\,dx = 8$

23 $\int_0^1 \int_0^{bx} dy\,dx + \int_1^2 \int_0^{b(2-x)} dy\,dx = \int_0^b \int_{y/b}^{2-(y/b)} dx\,dy = b$ **25** $f(a,b) - f(a,0) - f(0,b) + f(0,0)$

27 $\int_0^1 \int_0^1 (2x - 3y + 1)dx\,dy = \frac{1}{2}$ **29** $\int_a^b f(x)dx = \int_a^b \int_0^{f(x)} 1dy\,dx$ **31** $50{,}000\pi$

33 $\int_1^3 \int_1^2 x^2\,dx\,dy = \frac{14}{3}$ **35** $2\int_0^{1/\sqrt{2}} \int_0^{\sqrt{1-y^2}} 1dx\,dy = \frac{\pi}{4}$

37 $\frac{1}{n^2}\Sigma_{j=1}^n \Sigma_{i=1}^n f(\frac{i-\frac{1}{2}}{n}, \frac{j-\frac{1}{2}}{n})$ is exact for $f = 1, x, y, xy$ **39** Volume 8.5 **41** Volumes $\ln 2, 2\ln(1+\sqrt{2})$

43 $\int_0^1 \int_0^1 x^y dx\,dy = \int_0^1 \frac{1}{y+1}dy = \ln 2; \int_0^1 \int_0^1 x^y dy\,dx = \int_0^1 \frac{x-1}{\ln x}dx = \ln 2$

45 With long rectangles $\sum y_i \Delta A = \sum \Delta A = 1$ but $\int\int y\,dA = \frac{1}{2}$

2 $\int_1^e 2xy\,dx = x^2y]_1^e = (e^2-1)y; \int_2^{2e}(e^2-1)y\,dy = (e^2-1)\frac{y^2}{2}]_2^{2e} = (e^2-1)(2e^2-2) = \mathbf{2(e^2-1)^2}$;
$\int_1^e \frac{dx}{xy} = \frac{\ln x}{y}]_1^e = \frac{1}{y}; \int_2^{2e} \frac{dy}{y} = \ln 2e - \ln 2 = \ln\frac{2e}{2} = \mathbf{1}$.

4 $\int_1^2 ye^{xy}dx = e^{xy}]_1^2 = e^{2y} - e^y; \int_0^1 (e^{2y} - e^y)dy = [\frac{1}{2}e^{2y} - e^y]_0^1 = \mathbf{\frac{1}{2}e^2 - e + \frac{1}{2}}; \int_0^3 \frac{dy}{\sqrt{3+2x+y}} = 2\sqrt{3+2x+y}]_0^3 =$
$2\sqrt{6+2x} - 2\sqrt{3+2x}$; the x integral is $[\frac{2}{3}(6+2x)^{3/2} - \frac{2}{3}(3+2x)^{3/2}]_{-1}^1 = \mathbf{\frac{2}{3}8^{3/2} - \frac{2}{3}5^{3/2} - \frac{2}{3}4^{3/2} + \frac{2}{3}}$.
Note! $3 + 2x + y$ is not zero in the region of integration.

6 The region is above $y = x^3$ and below $y = x$ (from 0 to 1). Area $= \int_0^1 (x - x^3)dx = [\frac{x^2}{2} - \frac{x^4}{4}]_0^1 = \mathbf{\frac{1}{4}}$.

8 The region is below the parabola $y = 1 - x^2$ and above its mirror image $y = x^2 - 1$.
Area $= \int_{-1}^1 (1 - x^2 - x^2 + 1)dx = [2x - \frac{2}{3}x^3]_{-1}^1 = \mathbf{\frac{8}{3}}$.

10 The area is all below the axis $y = 0$, where horizontal strips cross from $x = y$ to $x = |y|$ (which is $-y$). Note that the y integral stops at $y = 0$. Area $= \int_{-1}^{0} \int_{y}^{-y} dx\, dy = \int_{-1}^{0} -2y\, dy = [-y^2]_{-1}^{0} = \mathbf{1}$.

12 The strips in Problem 6 from $y = x^3$ up to x are changed to strips from $x = y$ across to $x = y^{1/3}$. The outer integral on y is by chance also from 0 to 1. Area $= \int_0^1 (y^{1/3} - y)dy = [\frac{3}{4}y^{4/3} - \frac{1}{2}y^2]_0^1 = \mathbf{\frac{1}{4}}$.

14 Between the upper parabola $y = 1 - x^2$ in Problem 8 and the x axis, the strips now cross from the left side $x = -\sqrt{1-y}$ to the right side $x = +\sqrt{1-y}$. This half of the area is $\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} dx\, dy =$ $\int_0^1 2\sqrt{1-y}\, dy = -\frac{4}{3}(1-y)^{3/2}]_0^1 = \frac{4}{3}$. The other half has strips from left side to right side of $y = x^2 - 1$ or $x = \pm\sqrt{1+y}$. This area is $\int_{-1}^{0} \int_{-\sqrt{1+y}}^{\sqrt{1+y}} dx\, dy$ (also $\frac{4}{3}$).

16 The triangle in Problem 10 had sides $x = y$, $x = -y$, and $y = -1$. Now the strips are vertical. They go from $y = -1$ up to $y = x$ on the left side: area $= \int_{-1}^{0} \int_{-1}^{x} dy\, dx = \int_{-1}^{0} (x+1)dx = \frac{1}{2}(x+1)^2]_{-1}^{0} = \frac{1}{2}$. The strips go from -1 up to $y = -x$ on the right side: area $= \int_0^1 \int_{-1}^{-x} dy\, dx = \int_0^1 (-x+1)dx = \frac{1}{2}$.
Check: $\frac{1}{2} + \frac{1}{2} = 1$.

18 The triangle has corners at (0,0) and $(-1,0)$ and $(-1,-1)$. Its area is $\int_{-1}^{0} \int_{0}^{-x} dy\, dx = \int_0^1 \int_{-1}^{-y} dx\, dy (= \mathbf{\frac{1}{2}})$.

20 The triangle has corners at (0,0) and (2,4) and (4,4). Horizontal strips go from $x = \frac{y}{2}$ to $x = y$: area $= \int_0^4 \int_{y/2}^{y} dx\, dy = \mathbf{4}$. Vertical strips are of two kinds: from $y = x$ up to $y = 2x$ or to $y = 4$.
Area $= \int_0^2 \int_x^{2x} dy\, dx + \int_2^4 \int_x^4 dy\, dx = 2 + 2 = \mathbf{4}$.

22 (*Hard Problem*) The boundary lines are $y = \frac{1}{2}x$ from $(-2,-1)$ to (0,0), and $y = -2x$ from (0,0) to $(1,-2)$, and $y = \frac{-x-5}{3}$ or $x = -3y - 5$ from $(-2,-1)$ to $(1,-2)$. (This is the hardest one: note first the slope $-\frac{1}{3}$.) Vertical strips go from the third line up to the first or second: area $= \int_{-2}^{0} \int_{(-x-5)/3}^{x/2} dy\, dx +$ $\int_0^1 \int_{(-x-5)/3}^{-2x} dy\, dx = \frac{5}{3} + \frac{5}{6} = \mathbf{\frac{5}{2}}$. Horizontal strips cross from the first or third lines to the second: area $= \int_{-2}^{-1} \int_{-3y-5}^{-y/2} dx\, dy + \int_{-1}^{0} \int_{2y}^{-y/2} dx\, dy = \frac{5}{4} + \frac{5}{4} = \mathbf{\frac{5}{2}}$.

24 The top of the triangle is (a,b). From $x = 0$ to a the vertical strips lead to $\int_0^a \int_{dx/c}^{bx/a} dy\, dx =$ $[\frac{bx^2}{2a} - \frac{dx^2}{2c}]_0^a = \frac{ba}{2} - \frac{da^2}{2c}$. From $x = a$ to c the strips go up to the third side:
$\int_a^c \int_{dx/c}^{b+(x-a)(d-b)/(c-a)} dy\, dx = [bx + \frac{(x-a)^2(d-b)}{2(c-a)} - \frac{dx^2}{2c}]_a^c = b(c-a) + \frac{(c-a)(d-b)}{2} - \frac{dc}{2} + \frac{da^2}{2c}$.
The sum is $\frac{ba}{2} + \frac{b(c-a)}{2} + \frac{d(c-a)}{2} - \frac{dc}{2} = \mathbf{\frac{bc-ad}{2}}$. This is half of a parallelogram.

26 $\int_0^b \int_0^a \frac{\partial f}{\partial x} dx\, dy = \int_0^b [f(a,y) - f(0,y)]dy$.

28 Over the square $\int_0^1 \int_0^1 (xe^y - ye^x)dy\, dx = \int_0^1 (xe - \frac{e^x}{2} - x)dx = [\frac{x^2 e}{2} - \frac{e^x}{2} - \frac{x^2}{2}]_0^1 = \frac{e}{2} - \frac{e}{2} - \frac{1}{2} + \frac{1}{2} = \mathbf{0}$. (Looking back: zero is not a surprise because of symmetry.) Over the triangle the integral up to $y = x$ is $\int_0^1 \int_0^x (xe^y - ye^x)dy\, dx$. Over the triangle across to $y = x$ the integral is $\int_0^1 \int_0^y (xe^y - ye^x)dx\, dy$. Exchange y and x in the second double integral to get *minus* the first double integral.

30 $\int_{-1}^{1} (1 - x^2)dx = [x - \frac{x^3}{3}]_{-1}^{1} = \frac{4}{3}$. With horizontal strips this is $\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} dx\, dy =$ $\int_0^1 2\sqrt{1-y}\, dy = -\frac{4}{3}(1-y)^{3/2}]_0^1 = \frac{4}{3}$.

32 The height is $z = \frac{1-ax-by}{c}$. Integrate over the triangular base ($z = 0$ gives the side $ax + by = 1$): volume $= \int_{x=0}^{1/a} \int_{y=0}^{(1-ax)/b} \frac{1-ax-by}{c} dy\, dx = \int_0^{1/a} \frac{1}{c}[y - axy - \frac{1}{2}by^2]_0^{(1-ax)/b} dx = \int_0^{1/a} \frac{1}{c} \frac{(1-ax)^2}{2b} dx =$ $-\frac{(1-ax)^3}{6abc}]_0^{1/a} = \mathbf{\frac{1}{6abc}}$.

34 From Problem 33 the mass is $\frac{14}{3}$. The moments are $\int_1^3 \int_1^2 x^3 dx\, dy = \int_1^3 \frac{2^4 - 1^4}{4} dy = \frac{15}{2}$ and $\int_1^3 \int_1^2 yx^2 dx\, dy =$ $\int_1^3 \frac{8-1}{3} y\, dy = \frac{28}{3}$. Then $\bar{x} = \frac{15/2}{14/3} = \mathbf{\frac{45}{28}}$ and $\bar{y} = \frac{28/3}{14/3} = \mathbf{2}$.

36 The area of the quarter-circle is $\frac{\pi}{4}$. The moment is zero around the axis $y = 0$ (by symmetry): $\bar{\mathbf{x}} = \mathbf{0}$. The other moment, with a factor 2 that accounts for symmetry of left and right, is $2\int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} y\, dy\, dx = 2\int_0^1 (\frac{1-x^2}{2} - \frac{x^2}{2})dx = 2[\frac{x}{2} - \frac{x^3}{3}]_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{3}$. Then $\bar{y} = \frac{\sqrt{2}/3}{\pi/4} = \mathbf{\frac{4\sqrt{2}}{3\pi}}$.

38 The integral $\int_0^1 \int_0^1 x^2 dx\, dy$ has the usual midpoint error $-\frac{(\Delta x)^2}{12}$ for the integral of x^2 (see Section 5.8). The y integral $\int_0^1 dy = 1$ is done exactly. So the error is $-\frac{1}{12n^2}$ (and the same for $\int\int y^2 dx\, dy$). The integral of xy is computed exactly. Errors decrease with exponent $\mathbf{p = 2}$, the order of accuracy.

40 The exact integral is $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{x^2+y^2}} = 2\int_0^{\pi/4}\int_0^{\sec\theta}\frac{r\,dr\,d\theta}{r} = 2\int_0^{\pi/4}\sec\theta\,d\theta = 2\ln(\sec\theta+\tan\theta)]_0^{\pi/4} =$ $\mathbf{2\ln(\sqrt{2}+1)}$.

42 The exact integral is $\int_0^1\int_0^1 e^x\sin\pi y\,dx\,dy = \int_0^1(e-1)\sin\pi y\,dy = \frac{e-1}{\pi}(-\cos\pi y)]_0^1 = \frac{\mathbf{2}}{\pi}(\mathbf{e}-\mathbf{1})$.

14.2 Change to Better Coordinates (page 534)

We change variables to improve the **limits** of integration. The disk $x^2+y^2 \le 9$ becomes the rectangle $0 \le r \le \mathbf{3}, 0 \le \theta \le \mathbf{2\pi}$. The inner limits of $\iint\ dy\,dx$ are $y = \pm\sqrt{\mathbf{9}-\mathbf{x^2}}$. In polar coordinates this area integral becomes $\iint \mathbf{r\,dr\,d\theta} = \mathbf{9\pi}$.

A polar rectangle has sides dr and $\mathbf{r}\,\mathbf{d\theta}$. Two sides are not **straight** but the angles are still **90°**. The area between the circles $r=1$ and $r=3$ and the rays $\theta=0$ and $\theta=\pi/4$ is $\frac{\mathbf{1}}{\mathbf{8}}(\mathbf{3^2}-\mathbf{1^2}) = \mathbf{1}$. The integral $\iint x\,dy\,dx$ changes to $\iint \mathbf{r^2\cos\theta\,dr\,d\theta}$. This is the **moment** around the **y** axis. Then $\bar{x}$ is the ratio $\mathbf{My/M}$. This is the x coordinate of the **centroid**, and it is the **average** value of x.

In a rotation through α, the point that reaches (u,v) starts at $x = u\cos\alpha - v\sin\alpha, y = \mathbf{u\sin\alpha + v\cos\alpha}$. A rectangle in the uv plane comes from a **rectangle** in xy. The areas are **equal** so the stretching factor is $J=\mathbf{1}$. This is the determinant of the matrix $\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$. The moment of inertia $\iint x^2dx\,dy$ changes to $\iint \mathbf{(u\cos\alpha - v\sin\alpha)^2}du\,dv$.

For single integrals dx changes to $\mathbf{(dx/du)}du$. For double integrals $dx\,dy$ changes to $J\,du\,dv$ with $J = \mathbf{\partial(x,y)/\partial(u,v)}$. The stretching factor J is the determinant of the 2 by 2 matrix $\begin{bmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{bmatrix}$. The functions $x(u,v)$ and $y(u,v)$ connect an xy region R to a uv region S, and $\iint_R dx\,dy = \iint_S \mathbf{J\,du\,dv}$ = area of **R**. For polar coordinates $x =$ **u cos v** and $y =$ **u sin v (or r sin θ)**. For $x=u, y=u+4v$ the 2 by 2 determinant is $J=\mathbf{4}$. A square in the uv plane comes from a **parallelogram** in xy. In the opposite direction the change has $u=x$ and $v=\frac{1}{4}(y-x)$ and a new $J = \frac{\mathbf{1}}{\mathbf{4}}$. This J is constant because this change of variables is **linear**.

1 $\int_{\pi/4}^{3\pi/4}\int_0^1 r\,dr\,d\theta = \frac{\pi}{4}$ **3** S = quarter-circle with $u\ge 0$ and $v \ge 0$; $\int_0^1\int_0^{\sqrt{1-v^2}} du\,dv$

5 R is symmetric across the y axis; $\int_0^1\int_0^{\sqrt{1-v^2}} u\,du\,dv = \frac{1}{3}$ divided by area gives $(\bar{u},\bar{v}) = (4/3\pi, 4/3\pi)$

7 $2\int_0^{1/\sqrt{2}}\int_{1+x}^{1+\sqrt{1-x^2}} dy\,dx$; xy region R^* becomes R in the x^*y^* plane; $dx\,dy = dx^*dy^*$ when region moves

9 $J = \begin{vmatrix}\partial x/\partial r^* & \partial x/\partial\theta^*\\ \partial y/\partial r^* & \partial y/\partial\theta^*\end{vmatrix} = \begin{vmatrix}\cos\theta^* & -r^*\sin\theta^*\\ \sin\theta^* & r^*\cos\theta^*\end{vmatrix} = r^*; \int_{\pi/4}^{3\pi/4}\int_0^1 r^*dr^*d\theta^*$

11 $I_y = \iint_R x^2dx\,dy = \int_{\pi/4}^{3\pi/4}\int_0^1 r^2\cos^2\theta\,r\,dr\,d\theta = \frac{\pi}{16}-\frac{1}{8}; I_x = \frac{\pi}{16}+\frac{1}{8}; I_0 = \frac{\pi}{8}$

13 (0,0), (1,2), (1,3), (0,1); area of parallelogram is 1

15 $x=u, y=u+3v+uv$; then $(u,v) = (1,0),(1,1),(0,1)$ give corners $(x,y) = (1,0),(1,5),(0,3)$

17 Corners (0,0), (2,1), (3,3), (1,2); sides $y=\frac{1}{2}x, y=2x-3, y=\frac{1}{2}x+\frac{3}{2}, y=2x$

19 Corners (1,1), $(e^2,e),(e^3,e^3),(e,e^2)$; sides $x=y^2, y=x^2/e^3, x=y^2/e^3, y=x^2$

21 Corners (0,0), (1,0), (1,2), (0,1); sides $y=0, x=1, y=1+x^2, x=0$

23 $J = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, area $\int_0^1\int_0^1 3du\,dv = 3$; $J = \begin{vmatrix} 2e^{2u+v} & e^{2u+v} \\ e^{u+2v} & 2e^{u+2v} \end{vmatrix} = 3e^{3u+3v}$, $\int_0^1\int_0^1 3e^{3u+3v}du\,dv = \int_0^1(e^{3+3v} - e^{3v})dv = \frac{1}{3}(e^6 - 2e^3 + 1)$

25 Corners $(x,y) = (0,0),(1,0),(1,f(1)),(0,f(0))$; $(\frac{1}{2},1)$ gives $x = \frac{1}{2}, y = f(\frac{1}{2})$; $J = \begin{vmatrix} 1 & 0 \\ vf'(u) & f(u) \end{vmatrix} = f(u)$

27 $B^2 = 2\int_0^{\pi/4}\int_0^{1/\sin\theta} e^{-r^2} r\,dr\,d\theta = \int_0^{\pi/4}(e^{-1/\sin^2\theta} - 1)d\theta$

29 $\bar{r} = \iint r^2 dr\,d\theta / \iint r\,dr\,d\theta = \int_0^\pi \frac{8}{3}a^3\sin^3\theta\,d\theta/\pi a^2 = \frac{32a}{9\pi}$ **31** $\int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta = \frac{\pi}{2}$

33 Along the right side; along the bottom; at the bottom right corner

35 $\iint xy\,dx\,dy = \int_0^1\int_0^1(u\cos\alpha - v\sin\alpha)(u\sin\alpha + v\cos\alpha)du\,dv = \frac{1}{4}(\cos^2\alpha - \sin^2\alpha)$

37 $\int_0^{2\pi}\int_4^5 r^2 r^2 r\,dr\,d\theta = \frac{2\pi}{6}(5^6 - 4^6)$ **39** $x = \cos\alpha - \sin\alpha, y = \sin\alpha + \cos\alpha$ goes to $u = 1, v = 1$

2 Area $= \int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{|x|}^{\sqrt{1-x^2}} dy\,dx$ splits into two equal parts left and right of $x = 0 : 2\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}} dy\,dx = 2\int_0^{\sqrt{2}/2}(\sqrt{1-x^2} - x)dx = [x\sqrt{1-x^2} + \sin^{-1}x - x^2]_0^{\sqrt{2}/2} = \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$. The limits on $\iint dx\,dy$ are $\int_0^{\sqrt{2}/2}\int_{-y}^{y} dx\,dy$ for the lower triangle plus $\int_{\sqrt{2}/2}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy$ for the circular top.

4 (See Problem 36 of Section 14.1) $\int_{\pi/4}^{3\pi/4}\int_0^1(r\ \sin\theta)r\,dr\,d\theta = [\frac{r^3}{3}]_0^1[-\cos\theta]_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{3}$; divide by area $\frac{\pi}{4}$ to reach $\bar{y} = \frac{\sqrt{2}/3}{\pi/4} = \frac{4\sqrt{2}}{3\pi}$.

6 Area of wedge $= \frac{b}{2\pi}(\pi a^2)$. Divide $\int_0^b\int_0^a(r\ \cos\theta)r\,dr\,d\theta = \frac{a^3}{3}\sin b$ by this area $\frac{ba^2}{2}$ to find $\bar{x} = \frac{2a}{3b}\sin b$. (*Interesting limit:* $\bar{x} \to \frac{2}{3}a$ as the wedge angle b approaches zero: like the centroid of a triangle.) For $\bar{y}$ divide $\int_0^b\int_0^a(r\ \sin\theta)r\,dr\,d\theta = \frac{a^3}{3}(1 - \cos b)$ by the area $\frac{ba^2}{2}$ to find $\bar{y} = \frac{2a}{3b}(1 - \cos b)$.

8 The limits on r,θ are extremely awkward for R^*. Contrast with the simple limits $0 \le r^* \le 1, \frac{\pi}{4} \le \theta^* \le \frac{3\pi}{4}$ when the coordinates are recentered at (0,1). (A point on the lower boundary of the wedge has $r = \frac{\sin\frac{3\pi}{4}}{\sin(\frac{\pi}{4}-\theta)}$ by the law of sines.)

10 The centroid $(0,\bar{y})$ of R moves up to the centroid $(0, \bar{y}+1)$ of R^*. The centroid of a circle is its **center (1,2)**. The centroid of the upper half is $(1, 2 + \frac{4}{\pi})$ because a half-circle has $\int_0^\pi\int_0^3(r\ \sin\theta)r\,dr\,d\theta = 18$ divided by its area $\frac{9\pi}{2}$ (which gives $\frac{4}{\pi}$).

12 $I_x = \int_{\pi/4}^{3\pi/4}\int_0^1(r\ \sin\theta + 1)^2 r\,dr\,d\theta = \frac{1}{4}\int\sin^2\theta\,d\theta + \frac{2}{3}\int\sin\theta d\theta + \frac{1}{2}\int d\theta = [\frac{\theta}{8} - \frac{\sin 2\theta}{16} - \frac{2}{3}\cos\theta + \frac{\theta}{2}]_{\pi/4}^{3\pi/4} = \frac{5\pi}{16} + \frac{2}{16} + \frac{4}{3}\frac{\sqrt{2}}{2}$; $I_y = \iint(r\ \cos\theta)^2 r\,dr\,d\theta = \frac{\pi}{16} - \frac{1}{8}$ (as in Problem 11); $I_0 = I_x + I_y = \frac{3\pi}{8} + \frac{4}{3}\frac{\sqrt{2}}{2}$.

14 The corner **(1,2)** should be **(a,c)**, when $u = 0$ and $v = 1$; the corner **(0,1)** should be **(b,d)**, when $u = 1$ and $v = 0$. Check at $u = v = 1$; there $x = au + bv = 1$ and $y = cu + dv = 3$ to give the correct corner (1,3). Then $J = ad - bc = (1)(1) - (0)(2) = \mathbf{1}$. The unit square has **area 1**; so does R.

16 A *linear* change takes the square S into a **parallelogram** R (with one corner at (0,0)). Reason: The vector sum of the two sides from (0,0) is still the vector to the far corner.

18 Corners when $u = 0$ or 1, $v = 0$ or 1: (0,0), (3,1), (5,2), (2,1). The sides have equations $y = \frac{1}{3}x, y = \frac{1}{2}x - \frac{1}{2}, y = \frac{1}{3}x + \frac{1}{3}, y = \frac{1}{2}x$.

20 Corners when $u = 0$ or 1, $v = 0$ or 1 : $(0,0), (0,-1), (1,0), (0,1)$. Actually (0,0) is not a corner because one side comes down the y axis. The side with $u = 1$ is $x = v, y = v^2 - 1$ or $y = x^2 - 1$. The side with $v = 1$ is $x = u, y = 1 - u^2$ or $y = 1 - x^2$.

22 Here $u = 0$ or 1, $v = 0$ or 1 gives the corners $(0,0),(1,0),(\cos 1, \sin 1)$. The side with $u = 1$ is a circular arc $x = \cos v, y = \sin v$ between the last two corners. The other sides are straight: the region is pie-shaped (a fraction $\frac{1}{2\pi}$ of the unit circle).

24 Problem 18 has $J = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1$. So the area of R is $1\times$ area of unit square $=$ **1**. Problem 20 has $J = \begin{vmatrix} v & u \\ -2u & 2v \end{vmatrix} = 2(u^2+v^2)$, and integration over the square gives area of $R =$ $\int_0^1\int_0^1 2(u^2+v^2)du\,dv = \mathbf{\frac{4}{3}}$. Check in x, y coordinates: area of $R = 2\int_0^1(1-x^2)dx = \frac{4}{3}$.

26 $\begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{1}{r}$. As in equation 12, this new J is $\frac{1}{\text{old}\,J}$.

28 $\int_{-\infty}^{\infty} x^2 e^{-x^2/2}dx = (u)(v) - \int v du = (x)(-e^{-x^2/2})]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}dx = 0 + \sqrt{2\pi}$ by Example 5. Divide by $\sqrt{2\pi}$ to find $\sigma^2 = 1$.

30 R is an infinite strip above the interval [0,1] on the x axis. Its boundary $x = 1$ is $r\cos\theta = 1$ or $r = \sec\theta$. The limits are $0 \le r \le \sec\theta$ and $0 \le \theta \le \frac{\pi}{2}$. The integral is $\int_0^{\pi/2}\int_0^{\sec\theta}\frac{r\,dr\,d\theta}{r^3} = \int_0^{\pi/2}(\infty)d\theta =$ infinite. For a finite example integrate $(x^2+y^2)^{-1/2} = \frac{1}{r}$.

32 Equation (3) with y instead of x has $\int\int y^2 dA = \int_0^1\int_0^1(u\sin\alpha + v\cos\alpha)^2 du\,dv = \sin^2\alpha\int\int u^2\,du\,dv +$ $\sin\alpha\cos\alpha\int\int 2uv\,du\,dv + \cos^2\alpha\int\int v^2 du\,dv = \mathbf{\frac{\sin^2\alpha}{3} + \frac{\sin\alpha\cos\alpha}{2} + \frac{\cos^2\alpha}{3}}$.

34 (a) **False** (forgot the stretching factor J) (b) **False** (x can be larger than x^2) (c) **False** (forgot to divide by the area) (d) **True** (odd function integrated over symmetric interval) (e) **False** (the straight-sided region is a trapezoid: angle from 0 to θ and radius from r_1 to r_2 yields area $\frac{1}{2}(r_2^2 - r_1^2)\sin\theta\cos\theta$).

36 $\int\int \rho dA = \int_0^{2\pi}\int_4^5 r^2(r\,dr\,d\theta) = \mathbf{2\pi\frac{5^4-4^4}{4}}$. This is the **polar moment** of inertia I_0 with density $\rho = 1$.

38 $\int\int f\,dA = f(P)\int\int dA$ is the **Mean Value Theorem for double integrals** (compare Property 7, Section 5.6). If $f = x$ or $f = y$, choose $P =$ **centroid** $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

14.3 Triple Integrals (page 540)

Six important solid shapes are **a box, prism, cone, cylinder, tetrahedron, and sphere.** The integral $\int\int\int dx\,dy\,dz$ adds the volume **dx dy dz** of small **boxes.** For computation it becomes **three** single integrals. The inner integral $\int dx$ is the **length** of a line through the solid. The variables **y** and **z** are held constant. The double integral $\int\int dx\,dy$ is the **area** of a slice, with **z** held constant. Then the z integral adds up the volumes of **slices.**

If the solid region V is bounded by the planes $x = 0, y = 0, z = 0$, and $x + 2y + 3z = 1$, the limits on the inner x integral are **0 and $1 - 2y - 3z$**. The limits on y are **0 and $\frac{1}{2}(1 - 3z)$**. The limits on z are **0 and $\frac{1}{3}$**. In the new variables $u = x, v = 2y, w = 3z$, the equation of the outer boundary is $\mathbf{u + v + w = 1}$. The volume of the tetrahedron in uvw space is $\mathbf{\frac{1}{6}}$. From $dx = du$ and $dy = dv/2$ and $dz = \mathbf{dw/3}$, the volume of an xyz box is $dx\,dy\,dz = \mathbf{\frac{1}{6}}du\,dv\,dw$. So the volume of V is $\mathbf{\frac{1}{36}}$.

To find the average height $\bar{z}$ in V we compute $\int\int\int \mathbf{z\,dV}/\int\int\int\mathbf{dV}$. To find the total mass if the density is $\rho = e^z$ we compute the integral $\int\int\int \mathbf{e^z\,dx\,dy\,dz}$. To find the average density we compute $\int\int\int \mathbf{e^z dV}/\int\int\int \mathbf{dV}$. In the order $\int\int\int dz\,dx\,dy$ the limits on the inner integral can depend on **x and y**. The limits on the middle integral can depend on **y**. The outer limits for the ellipsoid $x^2 + 2y^2 + 3z^2 \le 8$ are $\mathbf{-2 \le y \le 2}$.

1 $\int_0^1\int_0^z\int_0^y dx\,dy\,dz = \frac{1}{6}$

3 $0 \le y \le x \le z \le 1$ and all other orders xzy, yzx, zxy, zyx; all six contain $(0,0,0)$; to contain (1,0,1)

5 $\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 dx\, dy\, dz = 8$ **7** $\int_{-1}^1 \int_{-1}^z \int_{-1}^1 dx\, dy\, dz = 4$ **9** $\int_{-1}^1 \int_z^1 \int_1^z dx\, dy\, dz = \frac{4}{3}$

11 $\int_0^1 \int_0^{2-2z} \int_0^{2-y-2z} dx\, dy\, dz = \frac{2}{3}$ **13** $\int_0^{1/3} \int_0^{2-2z} \int_0^{2-y-2z} dx\, dy\, dz = \frac{7}{12}$

15 $\int_0^1 \int_0^{1-z} \int_0^{\sqrt{(1-z)^2-y^2}} dx\, dy\, dz = \frac{\pi}{3}$ **17** $\int_0^6 \int_0^1 \int_0^{\sqrt{1-y^2}} dx\, dy\, dz = 6\pi$ **19** $\int_0^1 \int_0^1 \int_0^{\sqrt{1-y^2}} dx\, dy\, dz = \pi$

21 Corner of cube at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$; sides $\frac{2}{\sqrt{3}}$; area $\frac{8}{3\sqrt{3}}$

23 Horizontal slices are circles of area $\pi r^2 = \pi(4-z)$; volume $= \int_0^4 \pi(4-z)dz = 8\pi$; centroid has $\bar{x} = 0, \bar{y} = 0, \bar{z} = \int_0^4 z\pi(4-z)dz/8\pi = \frac{4}{3}$

25 $I = \frac{z^2}{2}$ gives zeros; $\frac{\partial I}{\partial x} = \int_0^z \int_0^y f\, dy\, dz, \frac{\partial I}{\partial y} = \int_0^z \int_0^x f\, dx\, dz, \frac{\partial^2 I}{\partial y\, \partial z} = \int_0^x f\, dx$

27 $\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (y^2+z^2) dx\, dy\, dz = \frac{16}{3}; \iiint x^2 dV = \frac{8}{3}; 3\iiint (x - \frac{x+y+z}{3})^2 dV = \frac{16}{3}$

29 $\int_0^3 \int_0^2 \int_0^y dx\, dy\, dz = 6$ **31** Trapezoidal rule is second-order; correct for $1, x, y, z, xy, xz, yz, xyz$

2 The area of $0 \le x \le y \le z \le 1$ is $\int_0^1 \int_x^1 \int_y^1 dz\, dy\, dx$. The four faces are $x = 0, y = x, z = y, z = 1$.

4 $\int_0^1 \int_0^z \int_0^y x\, dx\, dy\, dz = \int_0^1 \int_0^z \frac{y^2}{2} dy\, dz = \int_0^1 \frac{z^3}{6} dz = \frac{1}{24}$. Divide by the volume $\frac{1}{6}$ to find $\bar{x} = \frac{1}{4}$; $\int_0^1 \int_0^z \int_0^y y\, dx\, dy\, dz = \int_0^1 \int_0^z y^2 dy\, dz = \int_0^1 \frac{z^3}{3} dz = \frac{1}{12}$ and $\bar{y} = \frac{1}{2}$; by symmetry $\bar{z} = \frac{3}{4}$.

6 Volume of half-cube $= \int_0^1 \int_{-1}^1 \int_{-1}^1 dx\, dy\, dz = 4$.

8 $\int_0^1 \int_{-1}^z \int_{-1}^1 dx\, dy\, dz = \int_0^1 2(z+1)dz = [(z+1)^2]_0^1 = 3$.

10 $\int_{-1}^1 \int_{-1}^z \int_{-1}^y dx\, dy\, dz = \int_{-1}^1 \int_{-1}^z (y+1)dy\, dz = \int_{-1}^1 \frac{(z+1)^2}{2} dz = [\frac{(z+1)^3}{6}]_{-1}^1 = \frac{4}{3}$ (tetrahedron).

12 The plane faces are $x = 0, y = 0, z = 0$, and $2x + y + z = 4$ (which goes through 3 points). The volume is $\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx = \int_0^2 \int_0^{4-2x} (4-2x-y)dy\, dx = \int_0^2 \frac{(4-2x)^2}{2} dx = [-\frac{(4-2x)^3}{12}]_0^2 = \frac{4^3}{12} = \frac{16}{3}$. Check: Multiply standard volume $\frac{1}{6}$ by $(4)(4)(2) = \frac{16}{3}$. Check: Double the volume in Problem 11.

14 Put dz last and stop at $z = 1$: $\int_0^1 \int_0^{4-z} \int_0^{(4-y-z)/2} dx\, dy\, dz = \int_0^1 \int_0^{4-z} \frac{4-y-z}{2} dy\, dz = \int_0^1 \frac{(4-z)^2}{4} dz = [-\frac{(4-z)^3}{12}]_0^1 = \frac{4^3-3^3}{12} = \frac{37}{12}$.

16 (Still tetrahedron of Problem 12: volume still $\frac{16}{3}$). Limits of integration: the top vertex falls from (0,0,4) onto the y axis at $(0,-4,0)$. The corner (2,0,0) stays on the x axis. The corner (0,4,0) swings up to (0,0,4). The volume integral is $\int_0^4 \int_{-4}^0 \int_0^2 dx\, dy\, dz = \frac{16}{3}$.

18 The plane $z = x$ cuts the circular base in half, leaving $x \ge 0$. Volume $= \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^x dz\, dy\, dx = \int_0^1 2x\sqrt{1-x^2} dx = [-\frac{2}{3}(1-x^2)^{3/2}]_0^1 = \frac{2}{3}$.

20 Lying along the x axis the cylinder goes from $x = 0$ to $x = 6$. Its slices are circular disks $y^2 + (z-1)^2 = 1$ resting on the x axis. Volume $= \int_0^6 \int_{-1}^1 \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} dz\, dy\, dx =$ still 6π.

22 Change variables to $X = \frac{x}{a}, Y = \frac{y}{b}, Z = \frac{z}{c}$; then $dX dY dZ = \frac{dx\, dy\, dz}{abc}$. Volume $= \iiint abc\, dX dY dZ = \frac{1}{6}abc$. Centroid $(\bar{x}, \bar{y}, \bar{z}) = (a\bar{X}, b\bar{Y}, c\bar{Z}) = (\frac{a}{4}, \frac{b}{4}, \frac{c}{4})$. (Recall volume $\frac{1}{6}$ and centroid $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ of standard tetrahedron: this is Example 2.)

24 (a) Change variables to $X = \frac{x}{4}, Y = \frac{y}{2}, Z = \frac{3z}{4}$. Then the solid is $X^2 + Y^2 + Z^2 = 1$, a unit sphere of volume $\frac{4\pi}{3}$. Therefore the original volume is $\frac{4\pi}{3}(4)(2)(\frac{4}{3}) = \frac{128\pi}{9}$. (b) The hypervolume in 4 dimensions is $\frac{1}{24}$, following the pattern of 1 for interval, $\frac{1}{2}$ for triangle, $\frac{1}{6}$ for tetrahedron.

26 Average of $f = \iint\int_V f(x,y,z)dV / \iint\int_V dV =$ **integral of f(x,y,z) divided by the volume.**

28 Volume of unit cube $= \sum_{i=1}^{1/\Delta x} \sum_{j=1}^{1/\Delta x} \sum_{k=1}^{1/\Delta x} (\Delta x)^3 = 1$.

30 In one variable, the midpoint rule is correct for the functions 1 and x. In three variables it is correct for **1, x, y, z, xy, xz, yz, xyz.**

32 Simpson's Rule has coefficients $\frac{1}{6}, \frac{4}{6}, \frac{1}{6}$ over a unit interval. In three dimensions the 8 corners of the cube will have coefficients $(\frac{1}{6})^3 = \frac{1}{216}$. The center will have $(\frac{4}{6})^3 = \frac{64}{216}$. The centers of the 12 edges will have $(\frac{1}{6})^2(\frac{4}{6}) = \frac{4}{216}$. The centers of the 6 faces have $(\frac{1}{6})(\frac{4}{6})^2 = \frac{16}{216}$. (Check: 8(1) + 64 + 12(4) + 6(16) = 216.) When N^3 cubes are stacked together, with N small cubes each way, there are only $2N+1$ meshpoints along each direction. This makes $(2N+1)^3$ points or about 8 per cube. (Visualize the 8 new points of the cube as having x, y, z equal to zero or $\frac{1}{2}$.)

14.4 Cylindrical and Spherical Coordinates (page 547)

The three **cylindrical** coordinates are $r\theta z$. The point at $x = y = z = 1$ has $r = \sqrt{2}, \theta = \pi/4, z = 1$. The volume integral is $\iiint \mathbf{r}\, \mathbf{dr}\, \mathbf{d\theta}\, \mathbf{dz}$. The solid region $1 \le r \le 2, 0 \le \theta \le 2\pi, 0 \le z \le 4$ is a **hollow cylinder (a pipe)**. Its volume is $\mathbf{12\pi}$. From the r and θ integrals the area of a **ring (or washer)** equals $\mathbf{3\pi}$. From the z and θ integrals the area of a **shell** equals $\mathbf{2\pi rz}$. In $r\theta z$ coordinates the shapes of **cylinders** are convenient, while **boxes** are not.

The three **spherical** coordinates are $\rho\phi\theta$. The point at $x = y = z = 1$ has $\rho = \sqrt{3}, \phi = \cos^{-1} 1/\sqrt{3}, \theta = \pi/4$. The angle ϕ is measured from **the z axis**. θ is measured from **the x axis**. ρ is the distance to **the origin**, where r was the distance to **the z axis**. If $\rho\phi\theta$ are known then $\mathbf{x} = \rho \sin\phi \cos\theta, \mathbf{y} = \rho \sin\phi \sin\theta, \mathbf{z} = \rho \cos\phi$. The stretching factor J is a 3 by 3 **determinant** and volume is $\iiint \mathbf{r^2} \sin\phi\, \mathbf{dr}\, \mathbf{d\phi}\, \mathbf{d\theta}$.

The solid region $1 \le \rho \le 2, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$ is a **hollow sphere**. Its volume is $4\pi(2^3 - 1^3)/3$. From the ϕ and θ integrals the area of a **spherical** shell at radius ρ equals $4\pi\rho^2$. Newton discovered that the outside gravitational attraction of a **sphere** is the same as for an equal mass located at **the center**.

1 $(r, \theta, z) = (D, 0, 0); (\rho, \phi, \theta) = (D, \frac{\pi}{2}, 0)$ **3** $(r, \theta, z) = (0, \text{any angle}, D); (\rho, \phi, \theta) = (D, 0, \text{any angle})$
5 $(x, y, z) = (2, -2, 2\sqrt{2}); (r, \theta, z) = (2\sqrt{2}, -\frac{\pi}{4}, 2\sqrt{2})$ **7** $(x, y, z) = (0, 0, -1); (r, \theta, z) = (0, \text{any angle}, -1)$
9 $\phi = \tan^{-1}(\frac{r}{z})$ **11** 45° cone in unit sphere: $\frac{2\pi}{3}(1 - \frac{1}{\sqrt{2}})$ **13** cone without top: $\frac{7\pi}{3}$
15 $\frac{1}{4}$ hemisphere: $\frac{\pi}{6}$ **17** $\frac{\pi^2}{8}$ **19** Hemisphere of radius π : $\frac{2}{3}\pi^4$ **21** $\pi(R^2 - z^2); 4\pi r\sqrt{R^2 - r^2}$
23 $\frac{2}{3}a^3 \tan\alpha$ (see 8.1.39) **27** $\frac{\partial q}{\partial D} = \frac{\rho - D\cos\phi}{q} = \frac{\text{near side}}{\text{hypotenuse}} = \cos\alpha$
31 Wedges are not exactly similar; the error is higher order $\Rightarrow$ proof is correct
33 Proportional to $1 + \frac{1}{h}(\sqrt{a^2 + (D-h)^2} - \sqrt{a^2 + D^2})$
35 $J = \begin{vmatrix} a & & \\ & b & \\ & & c \end{vmatrix} = abc$; straight edges at right angles **37** $\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$
39 $\frac{8\pi\rho^4}{3}; \frac{2}{3}$ **41** $\rho^3; \rho^2$; force $= 0$ inside hollow sphere

2 $(r, \theta, z) = (D, \frac{3\pi}{2}, 0); (\rho, \phi, \theta) = (D, \frac{\pi}{2}, \frac{3\pi}{2})$ **4** $(r, \theta, z) = (5, \cos^{-1}\frac{3}{5}, 5); (\rho, \phi, \theta) = (5\sqrt{2}, \frac{\pi}{4}, \cos^{-1}\frac{3}{5})$
6 $(x, y, z) = (\frac{3}{2}, \frac{\sqrt{3}}{2}, 1); (r, \theta, z) = (\sqrt{3}, \frac{\pi}{6}, 1)$ **8** $x = r$ on the **positive x axis** $(x \ge 0, y = 0 (= \theta), z = 0)$
10 $x = \cos t, y = \frac{\sqrt{2}}{2}\sin t, z = \frac{\sqrt{2}}{2}\sin t$. The unit sphere intersects the plane $y = z$.
12 The surface $z = 1 + r^2 = 1 + x^2 + y^2$ is a paraboloid (parabola rotated around the z axis). The region is above the half-disk $0 \le r \le 1, 0 \le \theta \le \pi$. The volume is $\frac{3}{4}\pi$.
14 This is the volume of a half-cylinder (because of $0 \le \theta \le \pi$) : height π, radius π, volume $\frac{1}{2}\pi^4$.
16 The upper surface $\rho = 2$ is the top of a sphere. The lower surface $\rho = \sec\phi$ is the plane $z = \rho\cos\phi = 1$. (The angle $\phi = \frac{\pi}{3}$ is the meeting of sphere and plane, where $\sec\phi = 2$.) The volume is $2\pi\int_0^{\pi/3}(\frac{8 - \sec^3\phi}{3})\sin\phi\, d\phi = 2\pi[-\frac{8}{3}\cos\phi - \frac{1}{6\cos^2\phi}]_0^{\pi/3} = 2\pi[-\frac{4}{3} - \frac{1}{6/4} + \frac{8}{3} + \frac{1}{6}] = \frac{5\pi}{3}$.

18 The region $1 \le \rho \le 3$ is a hollow sphere (spherical shell). The limits $0 \le \phi \le \frac{\pi}{4}$ keep the part that lies above a 45° cone. The volume is $\mathbf{\frac{52\pi}{3}(1 - \frac{\sqrt{2}}{2})}$.

20 From the unit ball $\rho \le 1$ keep the part above the cone $\phi = 1$ radian and inside the wedge $0 \le \theta \le 1$ radian. Volume $= \frac{1}{4}\int_0^1 \sin\phi d\phi = \mathbf{\frac{1}{4}(1 - \cos 1)}$.

22 The curve $\rho = 1 - \cos\phi$ is a **cardioid** in the xz plane (like $r = 1 - \cos\theta$ in the xy plane). So we have a **cardioid of revolution.** Its volume is $\mathbf{\frac{8\pi}{3}}$ as in Problem 9.3.35.

24 Mass $= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho\sin\phi(\rho+1)d\rho\, d\phi\, d\theta = \mathbf{\frac{4}{3}\pi R^3 + 2\pi R^2}$.

26 *Newton's achievement* The cosine law (see hint) gives $\cos\alpha = \frac{D^2+q^2-\rho^2}{2qD}$. Then integrate $\frac{\cos\alpha}{q^2}$: $\iiint(\frac{D^2-\rho^2}{2q^3D} + \frac{1}{2qD})dV$. The second integral is $\frac{1}{2D}\iiint\frac{dV}{q} = \frac{4\pi R^3/3}{2D^2}$. The first integral over ϕ uses the same $u = D^2 - 2\rho D\cos\phi + \rho^2 = q^2$ as in the text: $\int_0^{\pi}\frac{\sin\phi d\phi}{q^3} = \int\frac{du/2\rho D}{u^{3/2}} = [\frac{-1}{\rho D u^{1/2}}]_{\phi=0}^{\phi=\pi} = \frac{1}{\rho D}(\frac{1}{D-\rho} - \frac{1}{D+\rho}) = \frac{2}{D(D^2-\rho^2)}$. The θ integral gives 2π and then the ρ integral is $\int_0^R 2\pi\frac{2}{D(D^2-\rho^2)}\frac{D^2-\rho^2}{2D}\rho^2 d\rho = \frac{4\pi R^3/3}{2D^2}$. The two integrals give $\frac{4\pi R^3/3}{D^2}$ as Newton hoped and expected.

28 The small movement produces a right triangle with hypotenuse ΔD and almost the same angle α. So the new small side Δq is $\Delta D\cos\alpha$.

30 $\iint q\, dA = 4\pi\rho^2 D + \frac{4\pi}{3}\frac{\rho^4}{D}$. Divide by $4\pi\rho^2$ to find $\bar{q} = \mathbf{D + \frac{\rho^2}{3D}}$ for the shell. Then the integral over ρ gives $\iiint q\, dV = \frac{4\pi}{3}R^3 D + \frac{4\pi}{15}\frac{R^5}{D}$. Divide by the volume $\frac{4\pi}{3}R^3$ to find $\bar{q} = \mathbf{D + \frac{R^2}{5D}}$ for the solid ball.

32 *Yes.* First concentrate the Earth to a point at its center – this is OK for each point in the Sun. Then concentrate the Sun at its center – this does not change the force on the (concentrated) Earth.

34 $J = aei + bfg + cdh - ceg - afh - bdi$.

36 Column 1: $\sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi} = \mathbf{1}$; Column 2: $\sqrt{\rho^2\cos^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin^2\phi} = \rho$; Column 3: $\sqrt{\rho^2\sin^2\phi(\sin^2\theta + \cos^2\theta)} = \rho\,\mathbf{\sin}\,\phi$. These are the edge lengths of the box. The dot products of these columns are zero; so J = volume of box = $(1)\ (\rho)(\rho\sin\phi)$ as before.

38 Column 1: $\sqrt{\cos^2\theta + \sin^2\theta} = \mathbf{1}$; Column 2: $\sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = \mathbf{r}$; Column 3: $\sqrt{0^2+0^2+1^2} = \mathbf{1}$. Again the dot products of the columns are zero and $\mathbf{J}$ = volume of box = $(1)(r)(1) = \mathbf{r}$.

40 $I = \mathbf{\frac{8}{15}\pi R^5}$; $J = \mathbf{\frac{2}{5}}$; the mass is **closer to the axis.**

42 The ball comes to a stop at Australia and **returns** to its starting point. It continues to oscillate in harmonic motion $y = R\cos(\sqrt{c/m}\, t)$.