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Sloan School of Management

15.450-Fall 2010
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Problem Set 3

1. (To be solved in a group) Consider the Black-Scholes model for stock returns, with constant interest rate r and constant drift and diffusion of stock returns, μ and σ. Your objective is to price and replicate an exotic European option with the payoff equal to the square of the terminal stock price, $(S_T)^2$.

   (a) Describe the price of the option, $f(t, S_t)$, as a solution of a PDE with appropriate boundary conditions.

   (b) Solve the PDE in item (1a) to obtain the option price. Use a guess for the solution $f(t, S_t) = a(t)(S_t)^2$.

   (c) Show how to replicate the payoff of the option by trading in the stock and the bond.

   (d) Using the risk-neutral pricing approach, express the option price as an expectation.

   (e) Compute the option price using the risk-neutral valuation formula and show that you obtain the same expression for the price as in item (1b).

   (f) Compute the instantaneous Sharpe ratio of returns on the option, defined as

   $$\frac{\text{expected return} - \text{risk-free rate}}{|\text{return volatility}|}$$

   Show that this is the same as the Sharpe ratio of stock returns. Explain why this result is intuitive.

   (g) Explain why in the Black-Scholes model all options must exhibit the same instantaneous Sharpe ratio of returns (up to the sign).

2. (To be solved individually) This problem illustrates that the strike price of the variance swap can be derived from prices of European options without explicitly specifying the stock return volatility process. It's a model-free approach to variance swap pricing.

   Consider a model of stock return with stochastic volatility

   $$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dZ_t$$

where μ_t and σ_t are Ito processes. Assume that the interest rate is constant, r. A time-0 variance swap delivers a time-T payment of

$$\left[\int_0^T (d\ln S_t)^2\right] - K_0^2$$

(a) Using Ito's lemma, show that under the risk-neutral probability measure $\mathbf{Q}$ the log of the stock price satisfies

$$d\ln S_t = \left(r - \frac{\sigma_t^2}{2}\right) dt + \sigma_t \, dZ_t^{\mathbf{Q}}$$

(b) Show that under the risk-neutral probability measure

$$\mathrm{E}_0^{\mathbf{Q}}\left[\ln S_T - \ln S_0 - rT\right] = -\frac{1}{2}\mathrm{E}_0^{\mathbf{Q}}\left[\int_0^T \sigma_t^2 \, dt\right]$$

and derive the time-0 strike price of a variance swap, K_0, using a risk-neutral expectation of $\ln S_T$.

3. (To be solved individually) Consider the Black-Scholes model with stock return parameters μ and σ and interest rate r. You need to analyze a digital option which matures at time T and pays \$1 if $S_T > K$ and nothing otherwise. Strike price $K > 0$.

(a) Express the price of the digital option using the risk-neutral pricing approach. You do not need to compute the expectation at this point.

(b) Characterize the option price as a solution of a PDE. Write down the boundary conditions.

(c) Suppose the function $P(t, S_t)$ is the option price. Describe the replicating portfolio for this option using derivatives of function P.

(d) Compute the expected instantaneous excess return on the digital option as a function of time and the stock price. You should express the answer in terms of derivatives of P without computing P.

(e) Derive the analytical formula for $P(t, S_t)$.

4. (To be solved in a group) Consider an interest rate model. The short-term interest rate follows the following process under the risk-neutral probability measure $\mathbf{Q}$:

$$dr_t = -\theta(r_t - \bar{r})\, dt + \sigma\sqrt{r_t}\, dZ_t^{\mathbf{Q}}$$

(a) Derive the PDE on the price of a zero-coupon bond maturing at time T. Specify the terminal condition on the bond price at time T.

(b) Show that the bond price can be expressed in the form $P(t) = \exp(a(t) + b(t)r_t)$.

(c) Compute the instantaneous volatility of bond returns as a function of t and r_t.

(d) Taking the bond price above as given, assume that the price of risk associated with the Brownian motion $Z_t^{\mathbf{Q}}$ is constant, η. Derive the expected excess return on the bond as a function of the interest rate and time.

(e) Under the above assumptions, derive the process for the interest rate under the physical probability measure $\mathbf{P}$.

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