15.401 Finance Theory

MIT Sloan MBA Program

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Lecture 13–14: Risk Analytics and Portfolio Theory

Critical Concepts

- Motivation
- Measuring Risk and Reward
- Mean-Variance Analysis
- The Efficient Frontier
- The Tangency Portfolio

Readings:

- Brealey, Myers, and Allen Chapters 7 and 8.1

What Is A Portfolio and Why Is It Useful?

- A **portfolio** is simply a specific combination of securities, usually defined by **portfolio weights** that sum to 1:

$$\boldsymbol{\omega} = \{ \omega_1 , \omega_2 , \ldots , \omega_n \}$$

$$\omega_i = \frac{N_i P_i}{N_1 P_1 + \cdots + N_n P_n}$$

$$1 = \omega_1 + \omega_2 + \cdots + \omega_n$$

- Portfolio weights can sum to 0 (dollar-neutral portfolios), and weights can be positive (long positions) or negative (short positions).
- Assumption: Portfolio weights summarize all relevant information.

Motivation

Example:

- Your investment account of $100,000 consists of three stocks: 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	10%
B	1,000	$60	$60,000	60%
C	750	$40	$30,000	30%
Total			$100,000	100%

Motivation

Example (cont):

- Your broker informs you that you only need to keep $50,000 in your investment account to support the same portfolio of 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C; in other words, you can buy these stocks **on margin**. You withdraw $50,000 to use for other purposes, leaving $50,000 in the account. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	20%
B	1,000	$60	$60,000	120%
C	750	$40	$30,000	60%
Riskless Bond	–$50,000	$1	–$50,000	–100%
Total			$50,000	100%

Motivation

Example:

- You decide to purchase a home that costs $500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80% What are your portfolio weights for this investment?

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
Home	1	$500,000	$500,000	500%
Mortgage	1	–$400,000	–$400,000	–400%
Total			$100,000	100%

- What happens to your total assets if your home price declines by 15%?

Motivation

Example:

- You own 100 shares of stock A, and you have shorted 200 shares of stock B. Your portfolio is summarized by the following weights:

Stock	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	100	$50	$5,000	???
B	–200	$25	– $5,000	???

- Zero net-investment portfolios do not have portfolio weights in percentages (because the denominator is 0)—we simply use dollar amounts instead of portfolio weights to represent long and short positions

Motivation

Why Not Pick The Best Stock Instead of Forming a Portfolio?

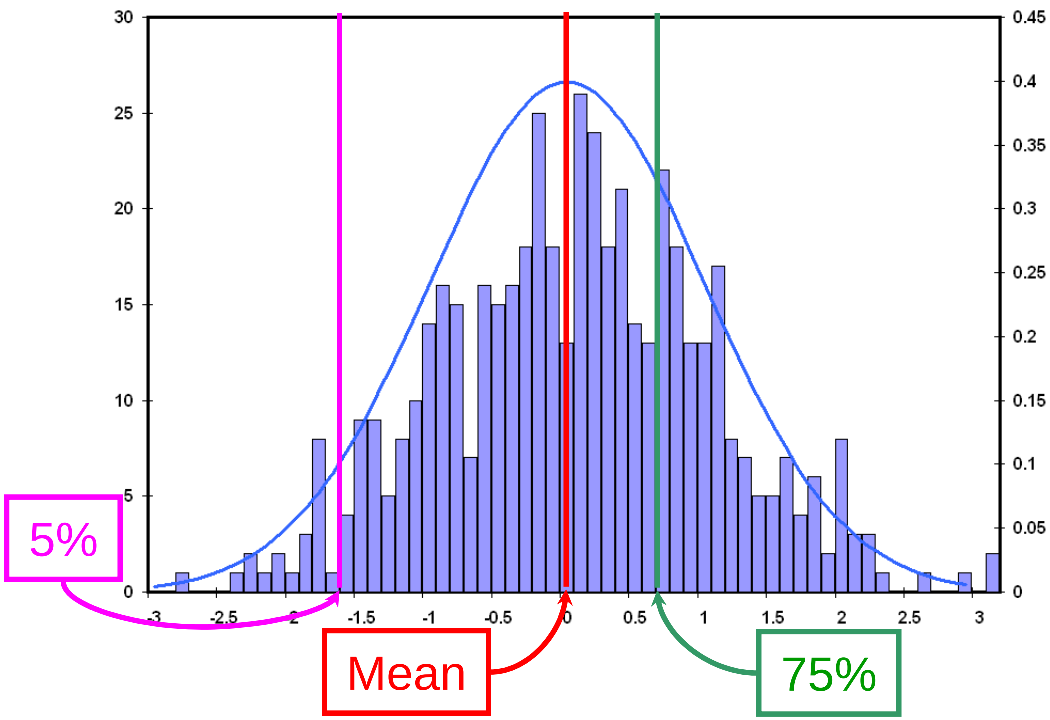
- We don't know which stock is best!
- Portfolios provide **diversification**, reducing unnecessary risks.
- Portfolios can enhance performance by focusing bets.
- Portfolios can customize and manage risk/reward trade-offs.

How Do We Construct a "Good" Portfolio?

- What does "good" mean?
- What characteristics do we care about for a given portfolio?
  - Risk and reward
- Investors like higher expected returns
- Investors dislike risk

Measuring Risk and Reward

- Reward is typically measured by return
- Higher returns are better than lower returns.
- But what if returns are unknown?
- Assume returns are random, and consider the **distribution** of returns.

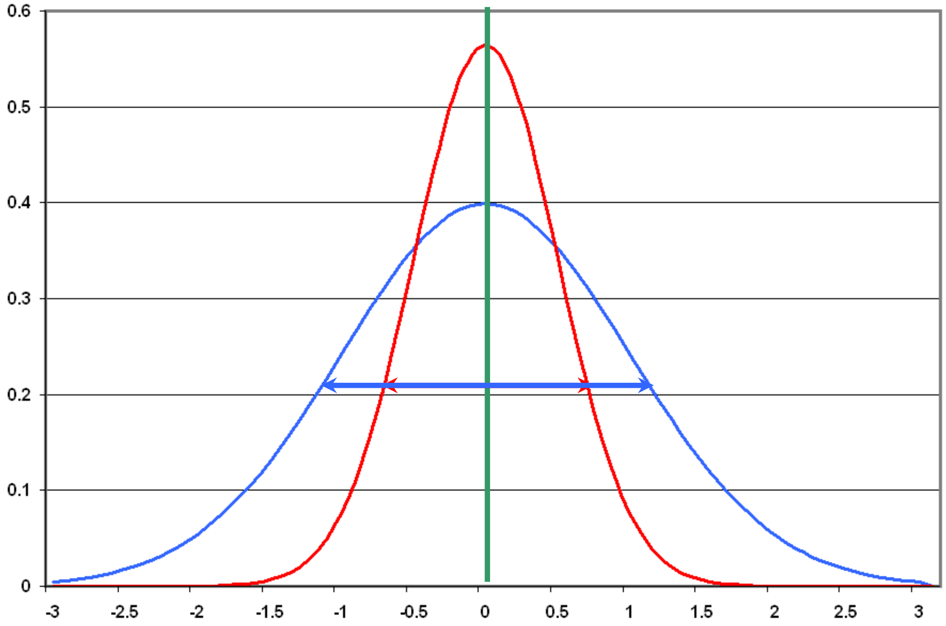


Several possible measures:

- Mean: central tendency.
- 75%: upper quartile.
- 5%: losses.

Measuring Risk and Reward

- How about risk?
- Likelihood of loss (negative return).
- But loss can come from positive return (e.g., short position).
- A **symmetric** measure of dispersion is **variance** or **standard deviation**.



Variance Measures Spread:

- **Blue** distribution is "riskier".
- Extreme outcomes more likely.
- This measure is symmetric.

Measuring Risk and Reward

Assumption

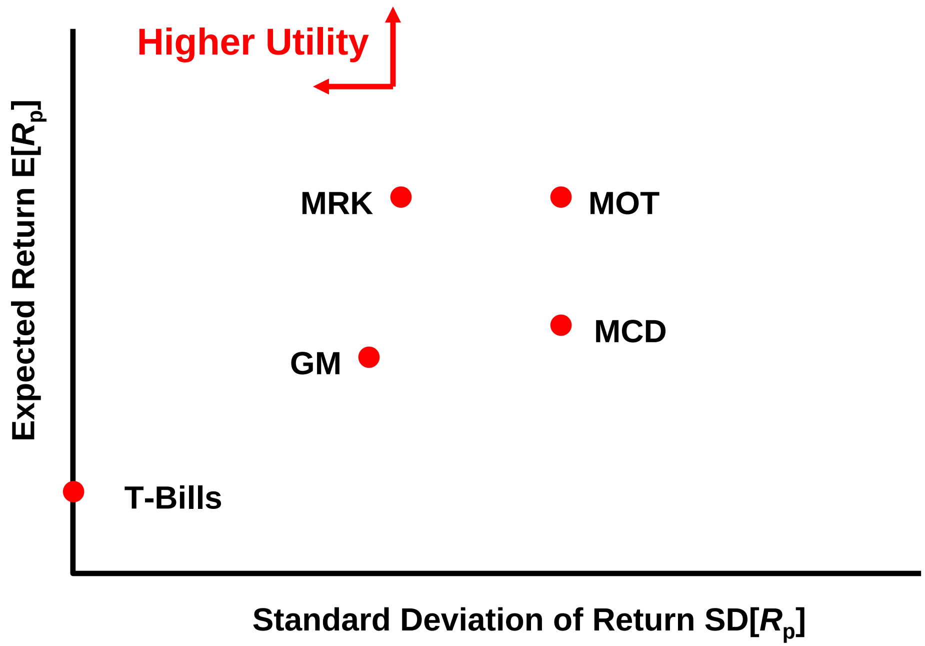
- Investors like high expected returns but dislike high volatility
- Investors care only about the expected return and volatility of their <u>overall portfolio</u>
  - Not individual stocks in the portfolio
  - Investors are generally assumed to be well-diversified

Key questions: How much does a stock contribute to the risk and return of a portfolio, and how can we choose portfolio weights to optimize the risk/reward characteristics of the overall portfolio?

Mean-Variance Analysis

Objective

- Assume investors focus only on the expected return and variance (or standard deviation) of their portfolios: higher expected return is good, higher variance is bad
- Develop a method for constructing optimal portfolios



Mean-Variance Analysis

Basic Properties of Mean and Variance For Individual Returns:

$$
\begin{aligned}
\text{Mean} &= \mathrm{E}[R_i] &&= \mu_i \\
\text{Variance} &= \mathrm{Var}[R_i] &&= \mathrm{E}[(R_i - \mu_i)^2] = \sigma_i^2 \\
\text{Standard Deviation} &= \sqrt{\mathrm{Var}[R_i]} &&= \sigma_i
\end{aligned}
$$

Basic Properties of Mean And Variance For Portfolio Returns:

$$
\begin{aligned}
R_p &= \omega_1 R_1 + \omega_2 R_2 + \cdots + \omega_n R_n \\
\mathrm{E}[R_p] &= \omega_1 \mu_1 + \omega_2 \mu_2 + \cdots + \omega_n \mu_n \\
&= \mu_p \quad \text{(Weighted Average)}
\end{aligned}
$$

Mean-Variance Analysis

Variance Is More Complicated:

$$
\begin{aligned}
\text{Var}[R_p] &= \text{E}[(R_p - \mu_p)^2] \\
&= \text{E}\Big[\ \Big(\omega_1(R_1 - \mu_1) + \omega_2(R_2 - \mu_2) + \cdots + \\
&\qquad \omega_n(R_n - \mu_n)\ \Big)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
\text{E}[\omega_i\omega_j(R_i - \mu_i)(R_j - \mu_j)] &= \omega_i\omega_j\text{Cov}[R_i, R_j] \\
&= \omega_i\omega_j\sigma_{ij} \\
&= \omega_i\omega_j\sigma_i\sigma_j\rho_{ij}
\end{aligned}
$$

Mean-Variance Analysis

Portfolio variance is the weighted sum of all the variances and covariances:

	$\omega_1(R_1-\mu_1)$	$\omega_2(R_2-\mu_2)$	$\cdots$	$\omega_n(R_n-\mu_n)$
$\omega_1(R_1-\mu_1)$	$\omega_1^2\sigma_1^2$	$\omega_1\omega_2\sigma_{12}$	$\cdots$	$\omega_1\omega_n\sigma_{1n}$
$\omega_2(R_2-\mu_2)$	$\omega_2\omega_1\sigma_{21}$	$\omega_2^2\sigma_2^2$	$\cdots$	$\omega_2\omega_n\sigma_{2n}$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\omega_n(R_n-\mu_n)$	$\omega_n\omega_1\sigma_{n1}$	$\omega_n\omega_2\sigma_{n2}$	$\cdots$	$\omega_n^2\sigma_n^2$

- There are n variances, and $n^2 - n$ covariances
- Covariances dominate portfolio variance
- Positive covariances increase portfolio variance; negative covariances decrease portfolio variance (diversification)

Mean-Variance Analysis

Consider The Special Case of Two Assets:

$$R_p = \omega_a R_a + \omega_b R_b$$

$$\mathrm{E}[R_p] = \omega_a \mu_a + \omega_b \mu_b$$

$$\begin{aligned} \mathrm{Var}[R_p] &= \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \mathrm{Cov}[R_a, R_b] \\ &= \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \sigma_a \sigma_b \rho_{ab} \end{aligned}$$

$$\text{Because} \quad \rho_{ab} \equiv \frac{\mathrm{Cov}[R_a, R_b]}{\sigma_a \sigma_b}$$

$$\mathrm{Cov}[R_a, R_b] = \sigma_a \sigma_b \rho_{ab}$$

- As correlation increases, overall portfolio variance increases

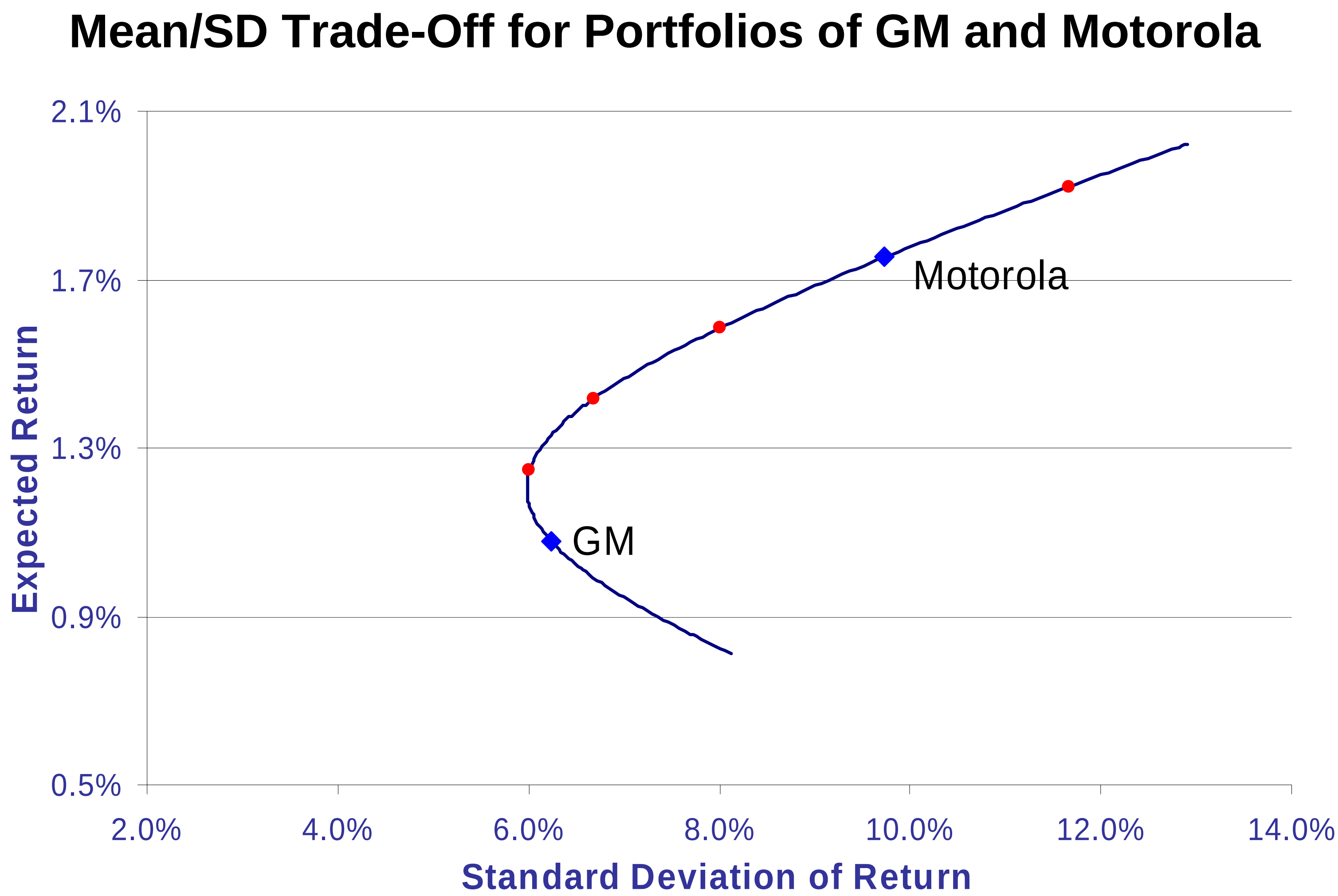
Mean-Variance Analysis

Example: From 1946 – 2001, Motorola had an average monthly return of 1.75% and a std dev of 9.73%. GM had an average return of 1.08% and a std dev of 6.23%. Their correlation is 0.37. How would a portfolio of the two stocks perform?

$$\begin{aligned} \mathrm{E}[R_p] &= \omega_{\mathrm{GM}}\, 1.08 + \omega_{\mathrm{MOT}}\, 1.75 \\ \mathrm{Var}[R_p] &= \omega_{\mathrm{GM}}^2\, 6.23^2 + \omega_{\mathrm{MOT}}^2\, 9.73^2 + \\ &\qquad 2\omega_{\mathrm{GM}}\omega_{\mathrm{MOT}}\,(0.37 \times 6.23 \times 9.73) \end{aligned}$$

w_{Mot}	w_{GM}	$E[R_P]$	$var(R_P)$	$stdev(R_P)$
0	**1**	1.08	38.8	6.23
0.25	**0.75**	1.25	36.2	6.01
0.50	**0.50**	1.42	44.6	6.68
0.75	**0.25**	1.58	64.1	8.00
1	**0**	1.75	94.6	9.73
1.25	**-0.25**	1.92	136.3	11.67

Mean-Variance Analysis

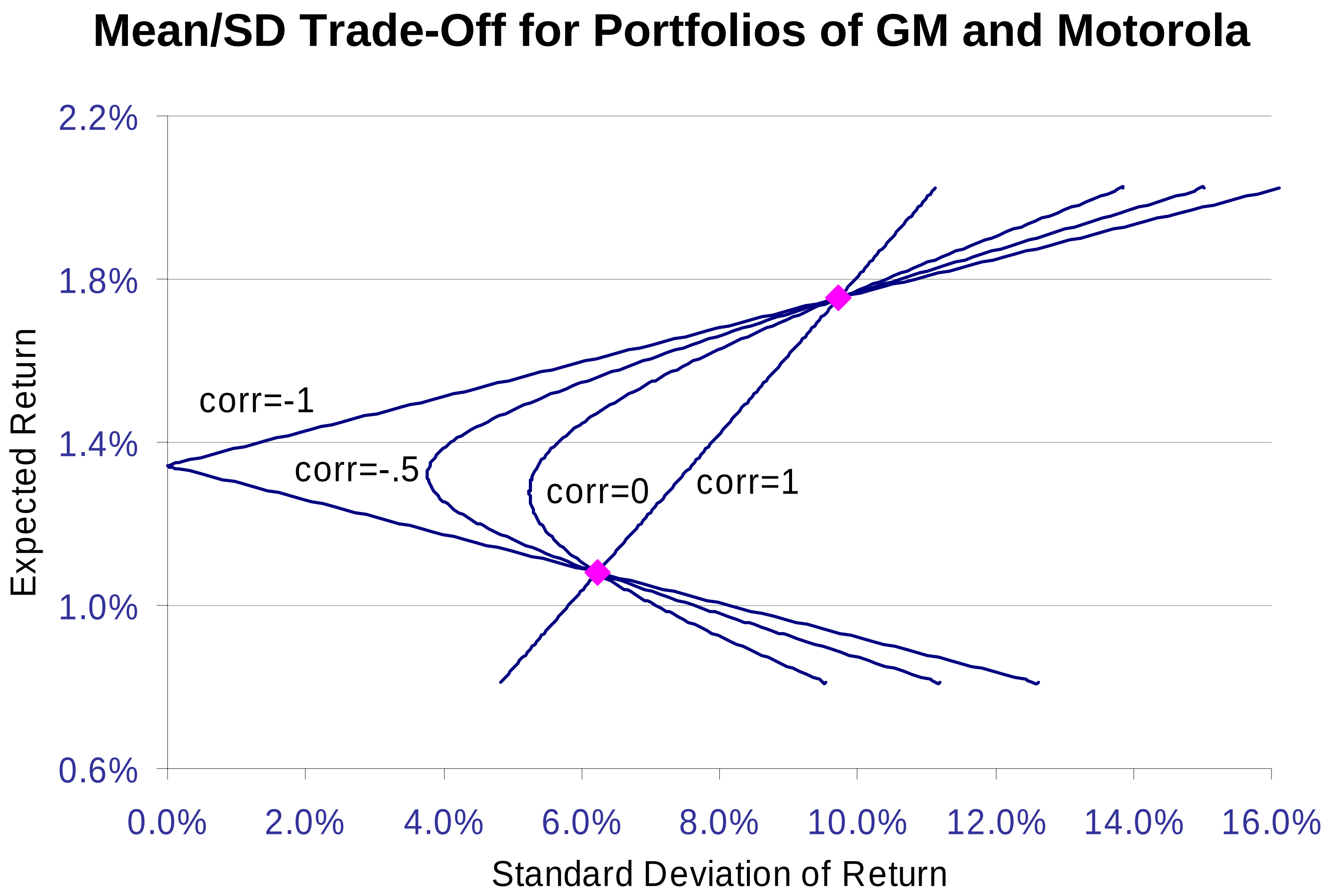


Example (cont): Suppose the correlation between GM and Motorola changes. What if it equals –1.0? 0.0? 1.0?

$$\begin{aligned} \mathrm{E}[R_p] &= \omega_{\mathrm{GM}}\,1.08 + \omega_{\mathrm{MOT}}\,1.75 \\ \mathrm{Var}[R_p] &= \omega_{\mathrm{GM}}^2\,6.23^2 + \omega_{\mathrm{MOT}}^2\,9.73^2 + \\ &\qquad 2\omega_{\mathrm{GM}}\omega_{\mathrm{MOT}}\left(\rho_{\mathrm{GM,MOT}} \times 6.23 \times 9.73\right) \end{aligned}$$

			Std dev of portfolio		
w_{Mot}	w_{GM}	$E[R_P]$	corr = -1	corr = 0	corr = 1
0	**1**	1.08%	6.23%	6.23%	6.23%
0.25	**0.75**	1.25	2.24	5.27	7.10
0.50	**0.50**	1.42	1.75	5.78	7.98
0.75	**0.25**	1.58	5.74	7.46	8.85
1	**0**	1.75	9.73	9.73	9.73

Mean-Variance Analysis



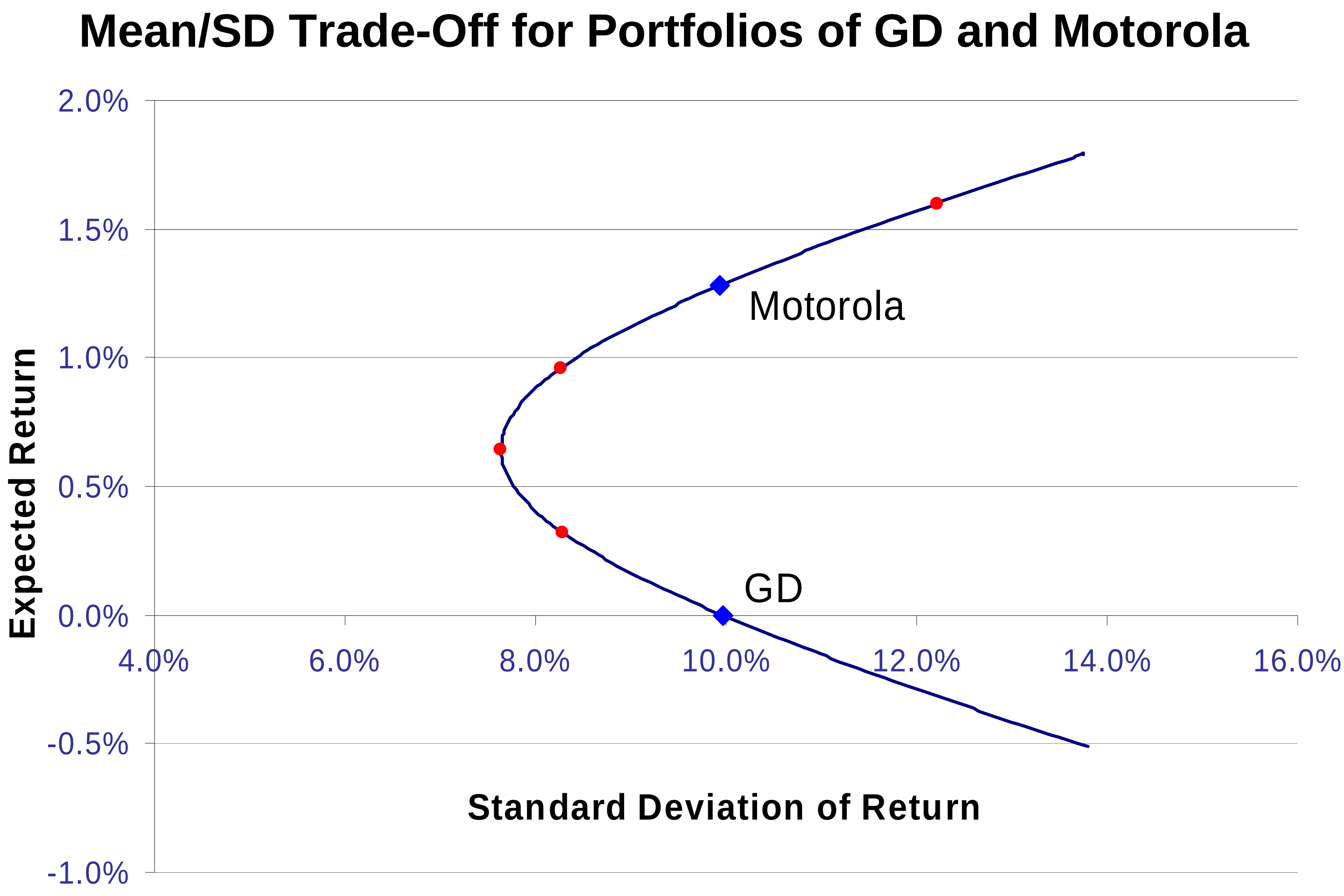
Mean-Variance Analysis

Example: In 1980, you were thinking about investing in GD. Over the subsequent 10 years, GD had an average monthly return of 0.00% and a std dev of 9.96%. Motorola had an average return of 1.28% and a std dev of 9.33%. Their correlation is 0.28. How would a portfolio of the two stocks perform?

$$\begin{aligned} \mathrm{E}[R_p] &= \omega_{\mathrm{GD}}\, 0.00 + \omega_{\mathrm{MOT}}\, 1.28 \\ \mathrm{Var}[R_p] &= \omega_{\mathrm{GD}}^2\, 9.96^2 + \omega_{\mathrm{MOT}}^2\, 9.93^2 + \\ &\qquad 2\omega_{\mathrm{GD}}\omega_{\mathrm{MOT}}\, (0.28 \times 9.96 \times 9.93) \end{aligned}$$

w_{Mot}	w_{GD}	$E[R_P]$	$var(R_P)$	$stdev(R_P)$
0	**1**	0.00	99.20	9.96
0.25	**0.75**	0.32	71.00	8.43
0.50	**0.50**	0.64	59.57	7.72
0.75	**0.25**	0.96	64.92	8.06
1	**0**	1.28	87.05	9.33

Mean-Variance Analysis



Mean-Variance Analysis

Example: You are trying to decide how to allocate your retirement savings between Treasury bills and the stock market. The T-Bill rate is 0.12% monthly. You expect the stock market to have a monthly return of 0.75% with a standard deviation of 4.25%.

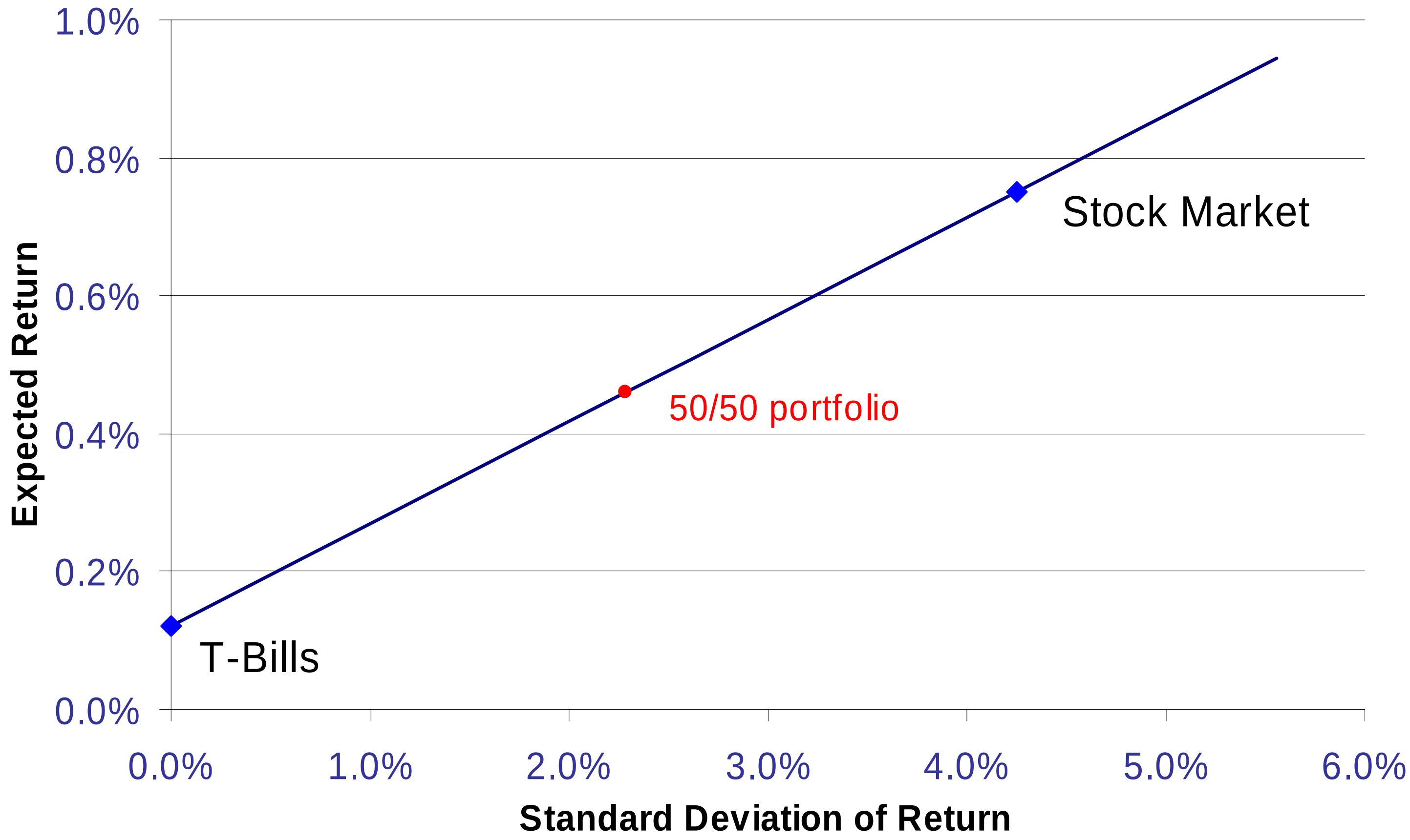
$$
\begin{aligned}
\mathrm{E}[R_p] &= \omega_{\mathrm{TBill}}\,0.12 + \omega_{\mathrm{STK}}\,0.75 \\
\mathrm{Var}[R_p] &= \omega^2_{\mathrm{TBill}}\,0.0^2 + \omega^2_{\mathrm{STK}}\,4.25^2 + \\
&\qquad 2\omega_{\mathrm{TBill}}\omega_{\mathrm{STK}}\,(0.00 \times 0.00 \times 4.25)
\end{aligned}
$$

$$
\sigma_p \equiv \sqrt{\mathrm{Var}[R_p]} = \omega_{\mathrm{STK}}\,4.25
$$

w_{Stk}	w_{Tbill}	$E[R_P]$	$var(R_P)$	$stdev(R_P)$
0	**1**	0.12	0.00	0.00
0.33	**0.67**	0.33	1.97	1.40
0.67	**0.33**	0.54	8.11	2.85
1	**0**	0.75	18.06	4.25

Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of T-Bills and The Stock Market



Mean-Variance Analysis

Summary

$$\text{E}[R_p] = \omega_a \mu_a + \omega_b \mu_b$$

$$\text{Var}[R_p] = \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \sigma_a \sigma_b \rho_{ab}$$

Observations

- $E[R_P]$ is a weighted average of stocks' expected returns
- $SD(R_P)$ is smaller if stocks' correlation is lower. It is **less than** a weighted average of the stocks' standard deviations (unless perfect correlation)
- The graph of portfolio mean/SD is nonlinear
- If we combine T-Bills with any risky stock, portfolios plot along a straight line

Mean-Variance Analysis

The General Case:

$$\mathrm{E}[R_p] = \omega_1\mu_1 + \cdots + \omega_n\mu_n$$

$$\mathrm{Var}[R_p] = \sum_{i=1}^{n} \omega_i^2\sigma_i^2 + \sum_{i\neq j} \omega_i\omega_j \mathrm{Cov}[R_i, R_j]$$

- Portfolio variance is the sum of weights times entries in the **covariance matrix**

	ω_1	ω_2	$\cdots$	ω_n
ω_1	σ_1^2	$\mathrm{Cov}[R_1, R_2]$	$\cdots$	$\mathrm{Cov}[R_1, R_n]$
ω_2	$\mathrm{Cov}[R_2, R_1]$	σ_2^2	$\cdots$	$\mathrm{Cov}[R_2, R_n]$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω_n	$\mathrm{Cov}[R_n, R_1]$	$\mathrm{Cov}[R_n, R_2]$	$\cdots$	σ_n^2

Mean-Variance Analysis

The General Case:

- Covariance matrix contains n^2 terms
  - n terms are variances
  - $n^2 - n$ terms are covariances
- In a well-diversified portfolio, covariances are more important than variances
- A stock's covariance with other stocks determines its contribution to the portfolio's overall variance
- Investors should care more about the risk that is common to many stocks; risks that are unique to each stock can be diversified away

Mean-Variance Analysis

Special Case:

- Consider an equally weighted portfolio:

$$
\begin{aligned}
\omega_i &= \frac{1}{n} \\
\text{Var}[R_p] &= \sum_{i=1}^{n} \frac{\sigma_i^2}{n^2} + \frac{1}{n^2} \sum_{i \neq j} \text{Cov}[R_i, R_j] \\
&= \frac{1}{n} \times \text{Average Variance} + \frac{n-1}{n} \times \text{Average Covariance} \\
&\approx \text{Average Covariance}
\end{aligned}
$$

- For portfolios with many stocks, the variance is determined by the average covariance among the stocks

Mean-Variance Analysis

Example: The average stock has a monthly standard deviation of 10% and the average correlation between stocks is 0.40. If you invest the same amount in each stock, what is variance of the portfolio? What if the correlation is 0.0? 1.0?

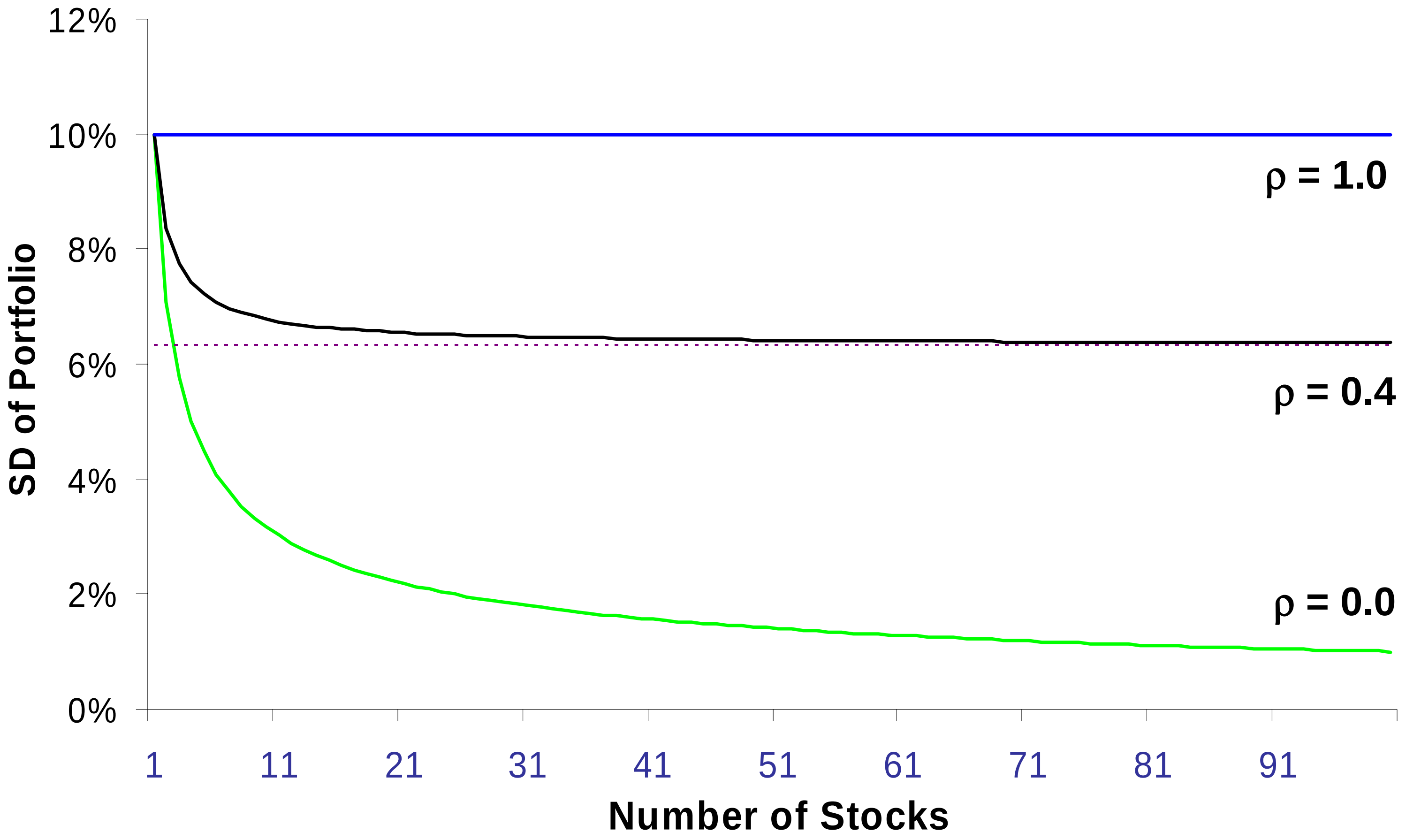
$$\text{Cov}[R_i, R_j] \;=\; \rho_{ij} \times \sigma_i \sigma_j \;=\; 0.40 \times 0.10 \times 0.10 \;=\; 0.004$$

$$\text{Var}[R_p] \;=\; \frac{1}{n} 0.10^2 + \frac{n-1}{n} 0.004 \;\approx\; 0.004 \quad \text{if } n \text{ large}$$

$$\sigma_p \;\approx\; \sqrt{0.004} \;=\; 6.3\%$$

Mean-Variance Analysis

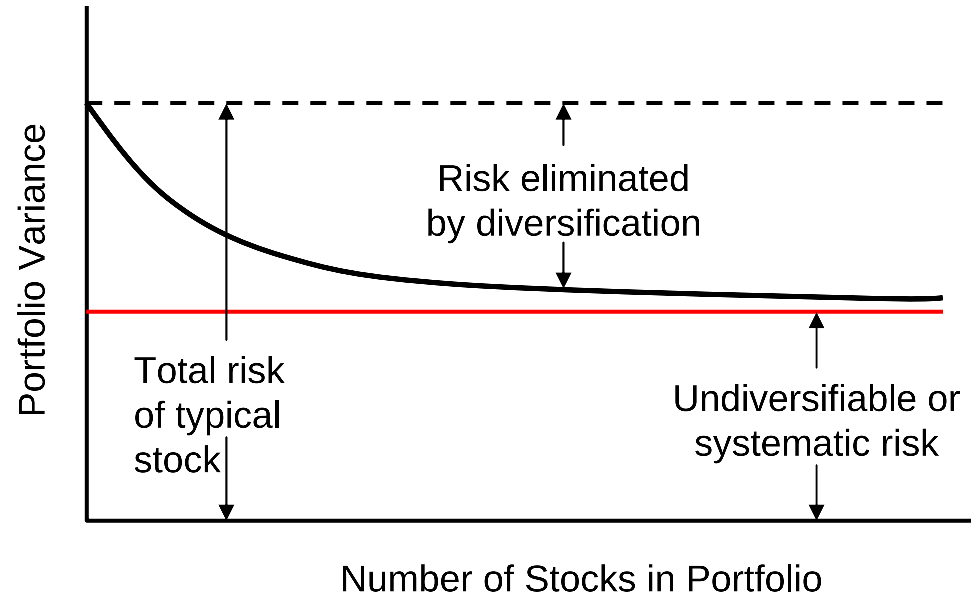
Example (cont):



Mean-Variance Analysis

Eventually, Diversification Benefits Reach A Limit:

- Remaining risk known as **systematic** or **market risk**
- Due to common factors that cannot be diversified
- Example: S&P 500
- Other sources of systematic risk may exist:
  - Credit
  - Liquidity
  - Volatility
  - Business Cycle
  - Value/Growth
- Provides motivation for **linear factor models**



The Efficient Frontier

Given Portfolio Expected Returns and Variances:

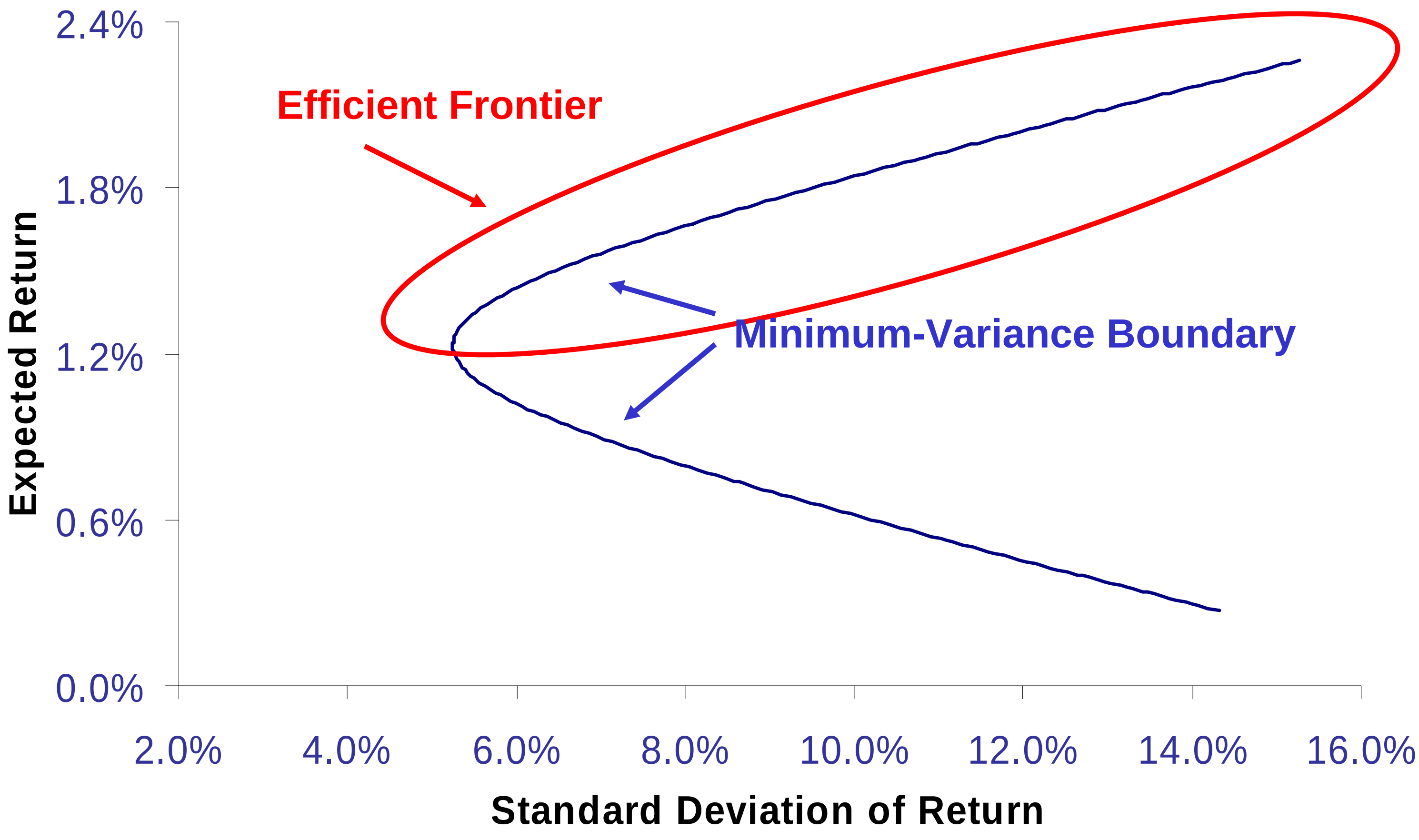
$$\mathsf{E}[R_p] = \omega_1\mu_1 + \cdots + \omega_n\mu_n$$

$$\mathsf{Var}[R_p] = \sum_{i=1}^{n} \omega_i^2\sigma_i^2 + \sum_{i \neq j} \omega_i\omega_j \mathsf{Cov}[R_i, R_j]$$

How Should We Choose The Best Weights?

- All feasible portfolios lie inside a bullet-shaped region, called the **minimum-variance boundary or frontier**
- The **efficient frontier** is the top half of the minimum-variance boundary (why?)
- Rational investors should select portfolios from the efficient frontier

The Efficient Frontier



The Efficient Frontier

Example: You can invest in any combination of GM, IBM, and MOT. What portfolio would you choose?

			Variance / covariance		
Stock	Mean	Std dev	GM	IBM	Motorola
GM	1.08	6.23	38.80	16.13	22.43
IBM	1.32	6.34	16.13	40.21	23.99
Motorola	1.75	9.73	22.43	23.99	94.63

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$$\mathbf{E[R_P]} = (w_{GM} \times 1.08) + (w_{IBM} \times 1.32) + (w_{Mot} \times 1.75)$$

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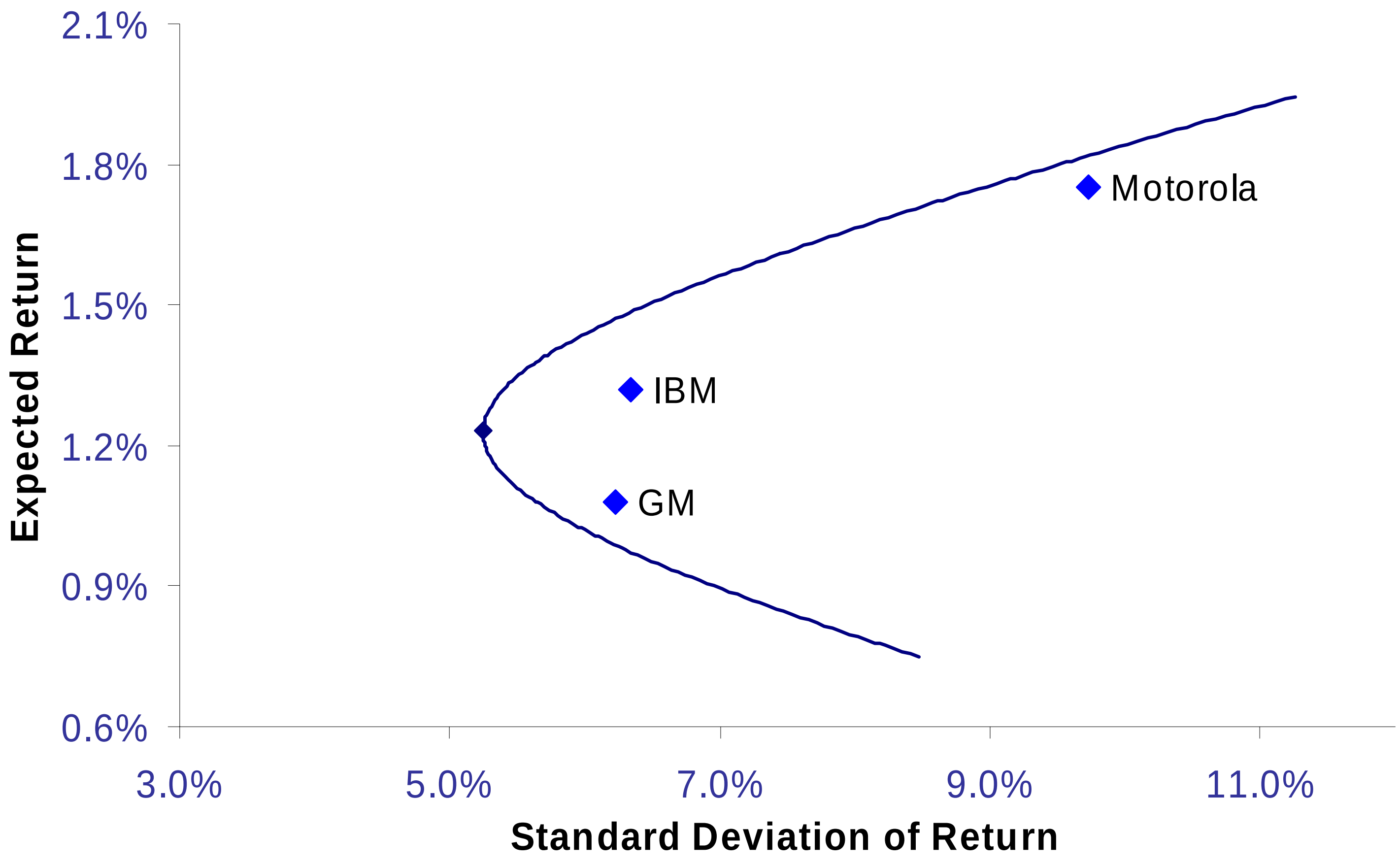
The Efficient Frontier

Example (cont): Feasible Portfolios



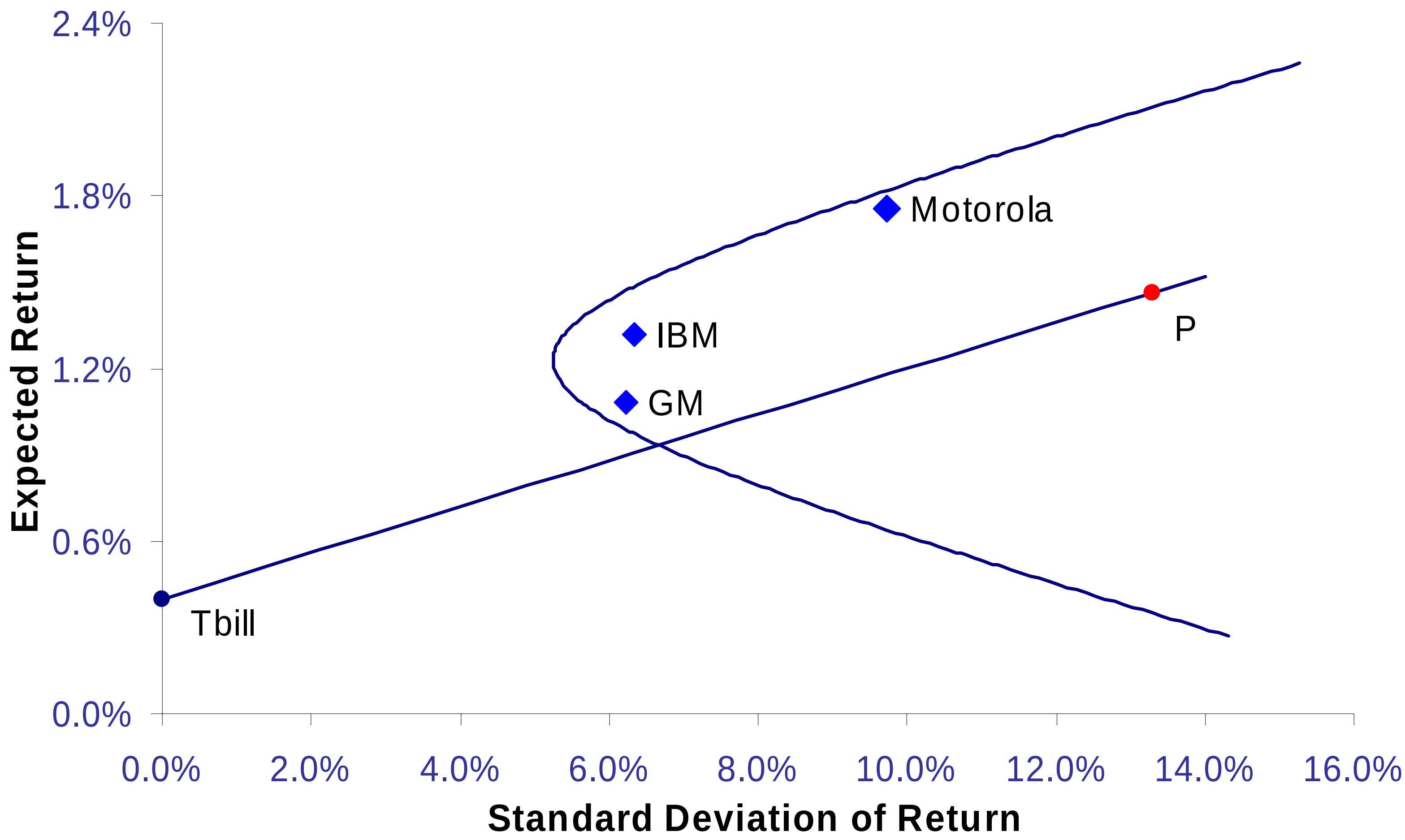
The Tangency Portfolio

The Tangency Portfolio

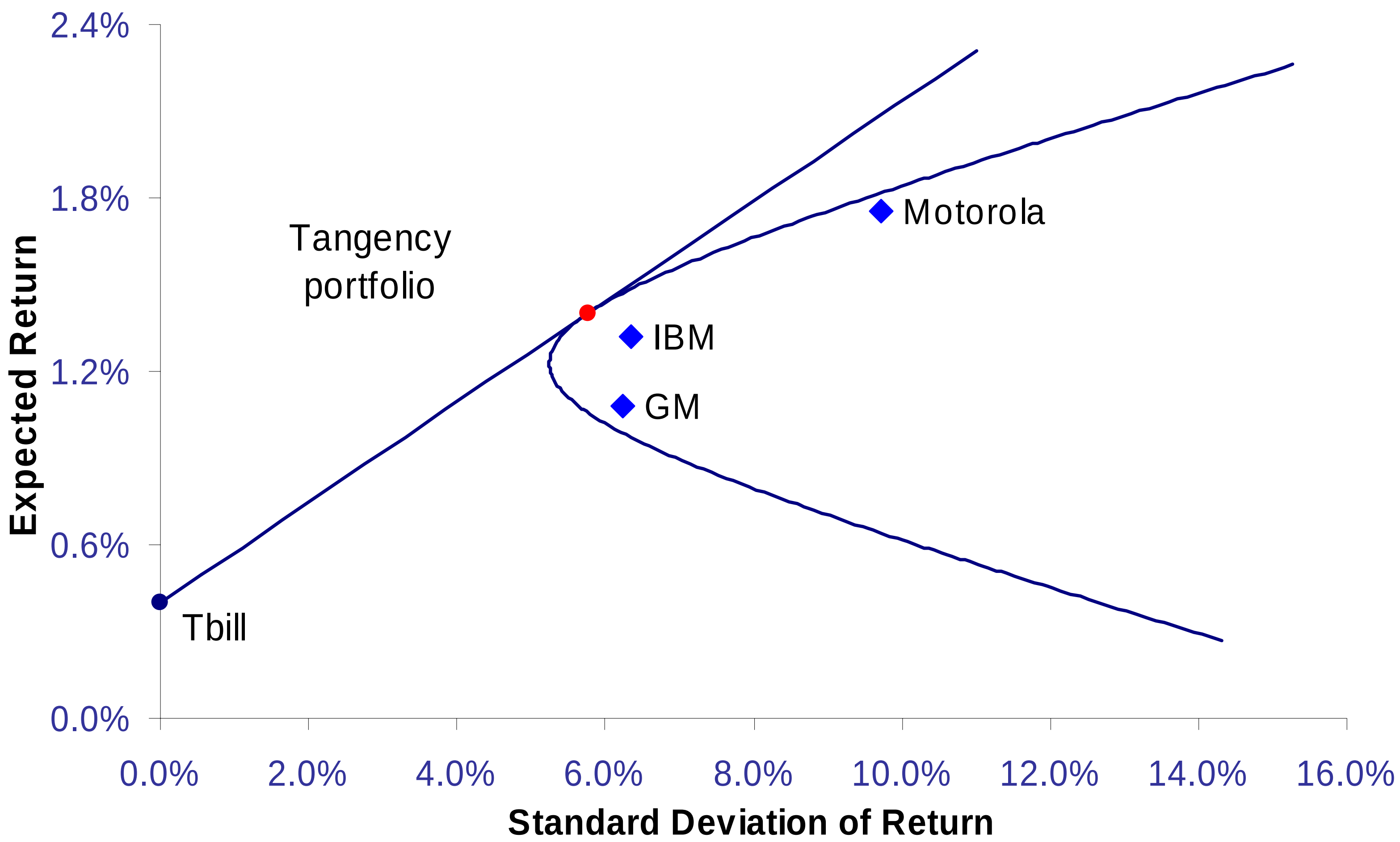
- If there is also a riskless asset (T-Bills), all investors should hold exactly the same stock portfolio!
- All efficient portfolios are combinations of the riskless asset and a unique portfolio of stocks, called the tangency portfolio.*
  - In this case, efficient frontier becomes straight line

* Harry Markowitz, Nobel Laureate

The Tangency Portfolio



The Tangency Portfolio



The Tangency Portfolio

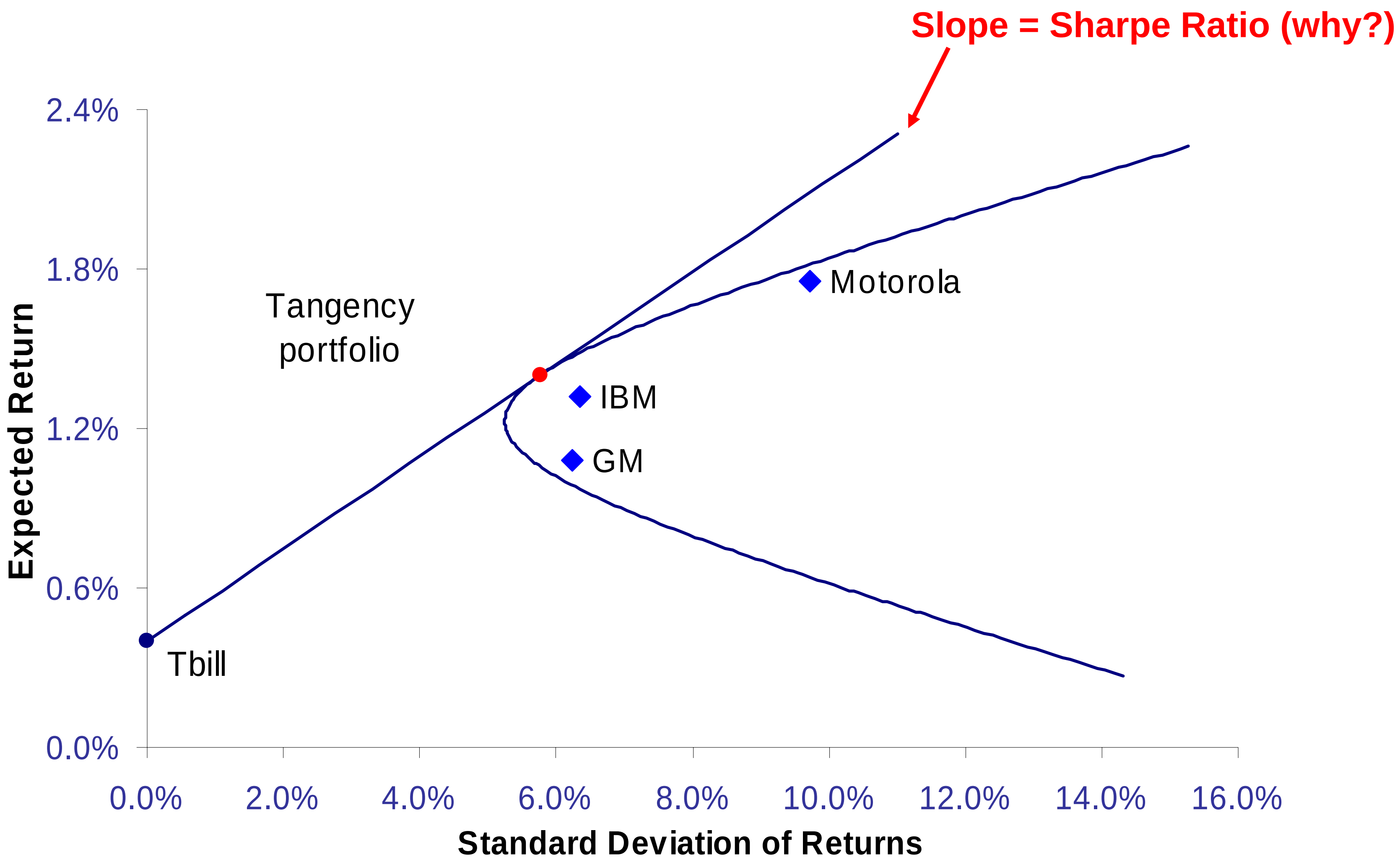
Sharpe ratio

A measure of a portfolio's risk-return trade-off, equal to the portfolio's risk premium divided by its volatility:

$$\text{Sharpe Ratio} \equiv \frac{\mathrm{E}[R_p] - r_f}{\sigma_p} \quad \text{(higher is better!)}$$

- **The tangency portfolio has the highest possible Sharpe ratio of any portfolio**
- Aside: **Alpha** is a measure of a mutual fund's risk-adjusted performance. The tangency portfolio also maximizes the fund's alpha.

The Tangency Portfolio



Key Points

- **Diversification reduces risk.** The standard deviation of a portfolio is always less than the average standard deviation of the individual stocks in the portfolio.
- **In diversified portfolios, covariances among stocks are more important than individual variances.** Only systematic risk matters.
- **Investors should try to hold portfolios on the efficient frontier.** These portfolios maximize expected return for a given level of risk.
- **With a riskless asset, all investors should hold the tangency portfolio.** This portfolio maximizes the trade-off between risk and expected return.

Additional References

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