Introduction to Semidefinite Programming (SDP)

Robert M. Freund and Brian Anthony
with assistance from David Craft

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1 Outline

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- Alternate View of Linear Programming
- Facts about Symmetric and Semidefinite Matrices
- SDP
- SDP Duality
- Examples of SDP
  - Combinatorial Optimization: MAXCUT
  - Convex Optimization: Quadratic Constraints, Eigenvalue Problems, $\log\det(X)$ problems
- Interior-Point Methods for SDP
- Application: Truss Vibration Dynamics via SDP

2 Linear Programming

2.1 Alternative Perspective

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$$\begin{array}{llll} LP: & \text{minimize} & c \cdot x & \\ & \text{s.t.} & a_i \cdot x = b_i, & i = 1, \ldots, m \\ & & x \in \Re^n_+. & \end{array}$$

"$c \cdot x$" means the linear function "$\sum_{j=1}^n c_j x_j$"
$\Re^n_+ := \{x \in \Re^n \mid x \geq 0\}$ is the nonnegative orthant.
$\Re^n_+$ is a *convex cone.*
K is convex cone if $x, w \in K$ and $\alpha, \beta \geq 0 \quad \Rightarrow \quad \alpha x + \beta w \in K$.

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$$\begin{array}{llll} LP: & \text{minimize} & c \cdot x & \\ & \text{s.t.} & a_i \cdot x = b_i, & i = 1, \ldots, m \\ & & x \in \Re^n_+. & \end{array}$$

"Minimize the linear function $c \cdot x$, subject to the condition that x must solve m given equations $a_i \cdot x = b_i, i = 1, \ldots, m$, and that x must lie in the convex cone $K = \Re^n_+$."

2.1.1 LP Dual Problem

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$$\begin{array}{lll} LD: & \text{maximize} & \displaystyle\sum_{i=1}^m y_i b_i \\ & \text{s.t.} & \displaystyle\sum_{i=1}^m y_i a_i + s = c \\ & & s \in \Re^n_+. \end{array}$$

For feasible solutions x of LP and (y,s) of LD, the duality gap is simply

$$c \cdot x - \sum_{i=1}^{m} y_i b_i = \left(c - \sum_{i=1}^{m} y_i a_i \right) \cdot x = s \cdot x \geq 0$$

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If LP and LD are feasible, then there exists x^* and (y^*, s^*) feasible for the primal and dual, respectively, for which

$$c \cdot x^* - \sum_{i=1}^{m} y_i^* b_i = s^* \cdot x^* = 0$$

3 Facts about the Semidefinite Cone

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If X is an $n \times n$ matrix, then X is a symmetric positive semidefinite (SPSD) matrix if $X = X^T$ and

$$v^T X v \geq 0 \quad \text{for any} \quad v \in \Re^n$$

If X is an $n \times n$ matrix, then X is a symmetric positive definite (SPD) matrix if $X = X^T$ and

$$v^T X v > 0 \quad \text{for any} \quad v \in \Re^n, v \neq 0$$

4 Facts about the Semidefinite Cone

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S^n denotes the set of symmetric $n \times n$ matrices
S^n_+ denotes the set of (SPSD) $n \times n$ matrices.
S^n_{++} denotes the set of (SPD) $n \times n$ matrices. Let $X, Y \in S^n$. SLIDE 8
"$X \succeq 0$" denotes that X is SPSD
"$X \succeq Y$" denotes that $X - Y \succeq 0$
"$X \succ 0$" to denote that X is SPD, etc.
Remark: $S^n_+ = \{X \in S^n \mid X \succeq 0\}$ is a convex cone.

5 Facts about Eigenvalues and Eigenvectors

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If M is a square $n \times n$ matrix, then λ is an eigenvalue of M with corresponding eigenvector q if

$$Mq = \lambda q \quad \text{and} \quad q \neq 0 \ .$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ enumerate the eigenvalues of M.

6 Facts about Eigenvalues and Eigenvectors

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The corresponding eigenvectors $q^1, q^2, \ldots, q^n$ of M can be chosen so that they are orthonormal, namely

$$\left(q^i\right)^T \left(q^j\right) = 0 \text{ for } i \neq j, \quad \text{and} \quad \left(q^i\right)^T \left(q^i\right) = 1$$

Define:

$$Q := \begin{bmatrix} q^1 & q^2 & \cdots & q^n \end{bmatrix}$$

Then Q is an *orthonormal* matrix:

$$Q^T Q = I, \quad \text{equivalently} \quad Q^T = Q^{-1}$$

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$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of M
$q^1, q^2, \ldots, q^n$ are the corresponding orthonormal eigenvectors of M

$$Q := \begin{bmatrix} q^1 & q^2 & \cdots & q^n \end{bmatrix}$$

$$Q^T Q = I, \quad \text{equivalently} \quad Q^T = Q^{-1}$$

Define D:

$$D := \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Property: $M = QDQ^T$. SLIDE 12

The decomposition of M into $M = QDQ^T$ is called its *eigendecomposition.*

7 Facts about Symmetric Matrices

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- If $X \in S^n$, then $X = QDQ^T$ for some orthonormal matrix Q and some diagonal matrix D. The columns of Q form a set of n orthogonal eigenvectors of X, whose eigenvalues are the corresponding entries of the diagonal matrix D.
- $X \succeq 0$ if and only if $X = QDQ^T$ where the eigenvalues (i.e., the diagonal entries of D) are all nonnegative.
- $X \succ 0$ if and only if $X = QDQ^T$ where the eigenvalues (i.e., the diagonal entries of D) are all positive.

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- If M is symmetric, then

$$\det(M) = \prod_{j=1}^{n} \lambda_j$$

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- Consider the matrix M defined as follows:

$$M = \begin{pmatrix} P & v \\ v^T & d \end{pmatrix},$$

  where $P \succ 0$, v is a vector, and d is a scalar. Then $M \succeq 0$ if and only if $d - v^T P^{-1} v \geq 0$.
- For a given column vector a, the matrix $X := aa^T$ is SPSD, i.e., $X = aa^T \succeq 0$.
- If $M \succeq 0$, then there is a matrix N for which $M = N^T N$. To see this, simply take $N = D^{\frac{1}{2}} Q^T$.

8 SDP

8.1 Semidefinite Programming

8.1.1 Think about X

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Let $X \in S^n$. Think of X as:

- a matrix
- an array of n^2 components of the form $(x_{11}, \ldots, x_{nn})$
- an object (a vector) in the space S^n.

All three different equivalent ways of looking at X will be useful.

8.1.2 Linear Function of X

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Let $X \in S^n$. What will a linear function of X look like?
If $C(X)$ is a linear function of X, then $C(X)$ can be written as $C \bullet X$, where

$$C \bullet X := \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}.$$

There is no loss of generality in assuming that the matrix C is also symmetric.

8.1.3 Definition of SDP

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$$\begin{array}{lll} SDP: & \text{minimize} & C \bullet X \\ & \text{s.t.} & A_i \bullet X = b_i \quad, i = 1, \ldots, m, \\ & & X \succeq 0, \end{array}$$

"$X \succeq 0$" is the same as "$X \in S^n_+$"
The data for SDP consists of the symmetric matrix C (which is the data for the objective function) and the m symmetric matrices $A_1, \ldots, A_m$, and the $m-$vector b, which form the m linear equations.

8.1.4 Example

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$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix}, \; b = \begin{pmatrix} 11 \\ 19 \end{pmatrix}, \; \text{and} \; C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix},$$

The variable X will be the 3×3 symmetric matrix:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

$$\begin{array}{llll} SDP: & \text{minimize} & x_{11}+4x_{12}+6x_{13}+9x_{22}+0x_{23}+7x_{33} & \\ & \text{s.t.} & x_{11}+0x_{12}+2x_{13}+3x_{22}+14x_{23}+5x_{33} & = 11 \\ & & 0x_{11}+4x_{12}+16x_{13}+6x_{22}+0x_{23}+4x_{33} & = 19 \end{array}$$

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \succeq 0.$$

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$$\begin{array}{llll} SDP: & \text{minimize} & x_{11}+4x_{12}+6x_{13}+9x_{22}+0x_{23}+7x_{33} & \\ & \text{s.t.} & x_{11}+0x_{12}+2x_{13}+3x_{22}+14x_{23}+5x_{33} & = 11 \\ & & 0x_{11}+4x_{12}+16x_{13}+6x_{22}+0x_{23}+4x_{33} & = 19 \end{array}$$

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \succeq 0.$$

It may be helpful to think of "$X \succeq 0$" as stating that each of the n *eigenvalues* of X must be nonnegative.

8.1.5 $LP \subset SDP$

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$$\begin{array}{lll} LP: & \text{minimize} & c \cdot x \\ & \text{s.t.} & a_i \cdot x = b_i, \quad i = 1, \ldots, m \\ & & x \in \Re^n_+. \end{array}$$

Define:

$$A_i = \begin{pmatrix} a_{i1} & 0 & \ldots & 0 \\ 0 & a_{i2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{in} \end{pmatrix}, \quad i = 1, \ldots, m, \quad \text{and} \quad C = \begin{pmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_n \end{pmatrix}.$$

$$\begin{array}{lll} SDP: & \text{minimize} & C \bullet X \\ & \text{s.t.} & A_i \bullet X = b_i \quad, i = 1, \ldots, m, \\ & & X_{ij} = 0, \quad i = 1, \ldots, n, \quad j = i+1, \ldots, n, \\ & & X = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_n \end{pmatrix} \succeq 0, \end{array}$$

9 SDP Duality

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$$\begin{array}{lll} SDD: & \text{maximize} & \sum_{i=1}^{m} y_i b_i \\ & \text{s.t.} & \sum_{i=1}^{m} y_i A_i + S = C \\ & & S \succeq 0. \end{array}$$

Notice

$$S = C - \sum_{i=1}^{m} y_i A_i \succeq 0$$

10 SDP Duality

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and so equivalently:

$$\begin{array}{lll} SDD: & \text{maximize} & \sum_{i=1}^{m} y_i b_i \\ & \text{s.t.} & C - \sum_{i=1}^{m} y_i A_i \succeq 0 \end{array}$$

10.1 Example

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$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix}, \; b = \begin{pmatrix} 11 \\ 19 \end{pmatrix}, \quad \text{and } C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix},$$

$$\begin{array}{lll} SDD: & \text{maximize} & 11y_1 + 19y_2 \\ & \text{s.t.} & y_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix} + S = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix} \\ & & S \succeq 0 \end{array}$$

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$$\begin{array}{lll} SDD: & \text{maximize} & 11y_1 + 19y_2 \\ & \text{s.t.} & y_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix} + S = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix} \\ & & S \succeq 0 \end{array}$$

is the same as:

$$\begin{array}{lll} SDD: & \text{maximize} & 11y_1 + 19y_2 \\ & \text{s.t.} & \begin{pmatrix} 1 - 1y_1 - 0y_2 & 2 - 0y_1 - 2y_2 & 3 - 1y_1 - 8y_2 \\ 2 - 0y_1 - 2y_2 & 9 - 3y_1 - 6y_2 & 0 - 7y_1 - 0y_2 \\ 3 - 1y_1 - 8y_2 & 0 - 7y_1 - 0y_2 & 7 - 5y_1 - 4y_2 \end{pmatrix} \succeq 0. \end{array}$$

10.2 Weak Duality

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Weak Duality Theorem: Given a feasible solution X of SDP and a feasible solution (y, S) of SDD, the duality gap is

$$C \bullet X - \sum_{i=1}^{m} y_i b_i = S \bullet X \geq 0 \ .$$

If

$$C \bullet X - \sum_{i=1}^{m} y_i b_i = 0 \ ,$$

then X and (y, S) are each optimal solutions to SDP and SDD, respectively, and furthermore, $SX = 0$.

10.3 Strong Duality

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Strong Duality Theorem: Let z_P^* and z_D^* denote the optimal objective function values of SDP and SDD, respectively. Suppose that there exists a feasible solution $\hat{X}$ of SDP such that $\hat{X} \succ 0$, and that there exists a feasible solution $(\hat{y}, \hat{S})$ of SDD such that $\hat{S} \succ 0$. Then both SDP and SDD attain their optimal values, and

$$z_P^* = z_D^* \ .$$

11 Some Important Weaknesses of SDP

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- There may be a finite or infinite duality gap.
- The primal and/or dual may or may not attain their optima.
- Both programs will attain their common optimal value if both programs have feasible solutions that are SPD.
- There is no finite algorithm for solving SDP.
- There is a simplex algorithm, but it is not a finite algorithm. There is no direct analog of a "basic feasible solution" for SDP.

12 SDP in Combinatorial Optimization

12.0.1 The MAX CUT Problem

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G is an undirected graph with nodes $N = \{1, \ldots, n\}$ and edge set E.
Let $w_{ij} = w_{ji}$ be the weight on edge (i, j), for $(i, j) \in E$.
We assume that $w_{ij} \geq 0$ for all $(i, j) \in E$.
The MAX CUT problem is to determine a subset S of the nodes N for which the sum of the weights of the edges that cross from S to its complement $\bar{S}$ is maximized ($\bar{S} := N \setminus S$).

12.0.2 Formulations

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The MAX CUT problem is to determine a subset S of the nodes N for which the sum of the weights w_{ij} of the edges that cross from S to its complement $\bar{S}$ is maximized ($\bar{S} := N \setminus S$).
Let $x_j = 1$ for $j \in S$ and $x_j = -1$ for $j \in \bar{S}$.

$$\begin{array}{lll} MAXCUT: & \text{maximize}_x & \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(1 - x_i x_j) \\ & & \\ & \text{s.t.} & x_j \in \{-1, 1\}, \quad j = 1, \ldots, n. \end{array}$$

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$$\begin{array}{lll} MAXCUT: & \text{maximize}_x & \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(1 - x_i x_j) \\ & & \\ & \text{s.t.} & x_j \in \{-1, 1\}, \quad j = 1, \ldots, n. \end{array}$$

Let

$$Y = xx^T \ .$$

Then

$$Y_{ij} = x_i x_j \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

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Also let W be the matrix whose $(i,j)^{\text{th}}$ element is w_{ij} for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Then

$$\begin{array}{llll} MAXCUT: & \text{maximize}_{Y,x} & \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y \\ & \text{s.t.} & x_j \in \{-1, 1\}, \quad j = 1, \ldots, n \\ & & Y = xx^T. \end{array}$$

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$$\begin{array}{llll} MAXCUT: & \text{maximize}_{Y,x} & \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y \\ & \text{s.t.} & x_j \in \{-1, 1\}, \quad j = 1, \ldots, n \\ & & Y = xx^T. \end{array}$$

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The first set of constraints are equivalent to $Y_{jj} = 1, j = 1, \ldots, n$.

$$\begin{array}{llll} MAXCUT: & \text{maximize}_{Y,x} & \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y \\ & \text{s.t.} & Y_{jj} = 1, \quad j = 1, \ldots, n \\ & & Y = xx^T. \end{array}$$

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$$\begin{array}{llll} MAXCUT: & \text{maximize}_{Y,x} & \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y \\ & \text{s.t.} & Y_{jj} = 1, \quad j = 1, \ldots, n \\ & & Y = xx^T. \end{array}$$

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Notice that the matrix $Y = xx^T$ is a rank-1 SPSD matrix.
We *relax* this condition by removing the rank-1 restriction:

$$\begin{array}{llll} RELAX: & \text{maximize}_{Y} & \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y \\ & \text{s.t.} & Y_{jj} = 1, \quad j = 1, \ldots, n \\ & & Y \succeq 0. \end{array}$$

It is therefore easy to see that RELAX provides an upper bound on MAXCUT, i.e.,

$$MAXCUT \le RELAX.$$

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$$RELAX: \quad \text{maximize}_Y \quad \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - W \bullet Y$$

$$\text{s.t.} \quad Y_{jj} = 1, \quad j = 1, \ldots, n$$

$$Y \succeq 0.$$

As it turns out, one can also prove without too much effort that:

$$0.87856 \; RELAX \le MAXCUT \le RELAX.$$

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This is an impressive result, in that it states that the value of the semidefinite relaxation is guaranteed to be no more than 12.2% higher than the value of NP-hard problem MAX CUT.

13 SDP for Convex QCQP

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A convex quadratically constrained quadratic program (QCQP) is a problem of the form:

$$\begin{array}{lll} QCQP: & \underset{x}{\text{minimize}} & x^T Q_0 x + q_0^T x + c_0 \\ & \text{s.t.} & x^T Q_i x + q_i^T x + c_i \le 0 \quad, i = 1, \ldots, m, \end{array}$$

where the $Q_0 \succeq 0$ and $Q_i \succeq 0$, $\; i = 1, \ldots, m$. This is the same as:

$$\begin{array}{lll} QCQP: & \underset{x,\theta}{\text{minimize}} & \theta \\ & \text{s.t.} & x^T Q_0 x + q_0^T x + c_0 - \theta \le 0 \\ & & x^T Q_i x + q_i^T x + c_i \le 0 \quad, i = 1, \ldots, m. \end{array}$$

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$$\begin{array}{lll} QCQP: & \underset{x,\theta}{\text{minimize}} & \theta \\ & \text{s.t.} & x^T Q_0 x + q_0^T x + c_0 - \theta \le 0 \\ & & x^T Q_i x + q_i^T x + c_i \le 0 \quad, i = 1, \ldots, m. \end{array}$$

Factor each Q_i into

$$Q_i = M_i^T M_i$$

and note the equivalence:

$$\begin{pmatrix} I & M_i x \\ x^T M_i^T & -c_i - q_i^T x \end{pmatrix} \succeq 0 \quad \Longleftrightarrow \quad x^T Q_i x + q_i^T x + c_i \le 0.$$

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$$QCQP: \quad \underset{x,\theta}{\text{minimize}} \quad \theta$$
$$\text{s.t.} \quad x^T Q_0 x + q_0^T x + c_0 - \theta \leq 0$$
$$x^T Q_i x + q_i^T x + c_i \leq 0 \quad , i = 1, \ldots, m.$$

Re-write $QCQP$ as:

$$QCQP: \quad \underset{x,\theta}{\text{minimize}} \quad \theta$$
$$\text{s.t.} \quad \begin{pmatrix} I & M_0 x \\ x^T M_0^T & -c_0 - q_0^T x + \theta \end{pmatrix} \succeq 0$$
$$\begin{pmatrix} I & M_i x \\ x^T M_i^T & -c_i - q_i^T x \end{pmatrix} \succeq 0 \quad , i = 1, \ldots, m.$$

14 SDP for SOCP

14.1 Second-Order Cone Optimization

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Second-order cone optimization:

$$\text{SOCP}: \quad \min_x \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$\|Q_i x + d_i\| \leq \left(g_i^T x + h_i\right) \quad , \quad i = 1, \ldots, k \ .$$

Recall $\|v\| := \sqrt{v^T v}$

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$$\text{SOCP}: \quad \min_x \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$\|Q_i x + d_i\| \leq \left(g_i^T x + h_i\right) \quad , \quad i = 1, \ldots, k \ .$$

Property:

$$\|Qx + d\| \leq \left(g^T x + h\right) \iff \begin{pmatrix} (g^T x + h)I & (Qx + d) \\ (Qx + d)^T & g^T x + h \end{pmatrix} \succeq 0 \ .$$

This property is a direct consequence of the fact that

$$M = \begin{pmatrix} P & v \\ v^T & d \end{pmatrix} \succeq 0 \iff d - v^T P^{-1} v \geq 0 \ .$$

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$$\text{SOCP}: \quad \min_x \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$\|Q_i x + d_i\| \leq \left(g_i^T x + h_i\right) \quad , \quad i = 1, \ldots, k \ .$$

Re-write as:

$$\text{SDPSOCP}: \quad \min_x \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$\begin{pmatrix} (g_i^T x + h_i)I & (Q_i x + d_i) \\ (Q_i x + d_i)^T & g_i^T x + h_i \end{pmatrix} \succeq 0 \ , \quad i = 1, \ldots, k \ .$$

15 Eigenvalue Optimization

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We are given symmetric matrices B and $A_i, i = 1, \ldots, k$
Choose weights $w_1, \ldots, w_k$ to create a new matrix S:

$$S := B - \sum_{i=1}^{k} w_i A_i \ .$$

There might be restrictions on the weights $Gw \leq d$.
The typical goal is for S is to have some nice property such as:

- $\lambda_{\min}(S)$ is maximized
- $\lambda_{\max}(S)$ is minimized
- $\lambda_{\max}(S) - \lambda_{\min}(S)$ is minimized

15.1 Some Useful Relationships

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Property: $M \succeq tI$ if and only if $\lambda_{\min}(M) \geq t$.

Proof: $M = QDQ^T$. Define

$$R = M - tI = QDQ^T - tI = Q(D - tI)Q^T \ .$$

$$M \succeq tI \iff R \succeq 0 \iff D - tI \succeq 0 \iff \lambda_{\min}(M) \geq t \ .$$

q.e.d.

Property: $M \preceq tI$ if and only if $\lambda_{\max}(M) \leq t$.

15.2 Design Problem

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Consider the design problem:

$$\begin{array}{lll} EOP: & \underset{w,S}{\text{minimize}} & \lambda_{\max}(S) - \lambda_{\min}(S) \\ & \text{s.t.} & S = B - \sum_{i=1}^{k} w_i A_i \\ & & Gw \leq d \ . \end{array}$$

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$$\begin{array}{llll} EOP: & \underset{w,S}{\text{minimize}} & \lambda_{\max}(S) - \lambda_{\min}(S) \\ & \text{s.t.} & S = B - \sum_{i=1}^{k} w_i A_i \\ & & Gw \le d\ . \end{array}$$

This is equivalent to:

$$\begin{array}{llll} EOP: & \underset{w,S,\mu,\lambda}{\text{minimize}} & \mu - \lambda \\ & \text{s.t.} & S = B - \sum_{i=1}^{k} w_i A_i \\ & & Gw \le d \\ & \lambda I \preceq S \preceq \mu I. & \end{array}$$

16 The Logarithmic Barrier Function for SPD Matrices

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Let $X \succeq 0$, equivalently $X \in S^n_+$.
X will have n nonnegative eigenvalues, say $\lambda_1(X), \ldots, \lambda_n(X) \ge 0$ (possibly counting multiplicities).

$$\begin{aligned} \partial S^n_+ = \{X \in S^n \mid \lambda_j(X) \ge 0, j = 1, \ldots, n, \\ \text{and } \lambda_j(X) = 0 \text{ for some } j \in \{1, \ldots, n\}\}. \end{aligned}$$

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$$\begin{aligned} \partial S^n_+ = \{X \in S^n \mid \lambda_j(X) \ge 0, j = 1, \ldots, n, \\ \text{and } \lambda_j(X) = 0 \text{ for some } j \in \{1, \ldots, n\}\}. \end{aligned}$$

A natural barrier function is:

$$B(X) := -\sum_{j=1}^{n} \ln(\lambda_i(X)) = -\ln\left(\prod_{j=1}^{n} \lambda_i(X)\right) = -\ln(\det(X)).$$

This function is called the log-determinant function or the logarithmic barrier function for the semidefinite cone.

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$$B(X) := -\sum_{j=1}^{n} \ln(\lambda_i(X)) = -\ln\left(\prod_{j=1}^{n} \lambda_i(X)\right) = -\ln(\det(X)).$$

Quadratic Taylor expansion at $X = \bar{X}$:

$$B(\bar{X} + \alpha D) \approx B(\bar{X}) + \alpha \bar{X}^{-1} \bullet D + \frac{1}{2}\alpha^2 \left(\bar{X}^{-\frac{1}{2}} D \bar{X}^{-\frac{1}{2}}\right) \bullet \left(\bar{X}^{-\frac{1}{2}} D \bar{X}^{-\frac{1}{2}}\right)\ .$$

$B(X)$ has the same remarkable properties in the context of interior-point methods for SDP as the barrier function $-\sum_{j=1}^{n} \ln(x_j)$ does in the context of linear optimization.

17 The SDP Analytic Center Problem

Slide 52

Given a system:

$$\sum_{i=1}^{m} y_i A_i \preceq C \ ,$$

the *analytic center* is the solution $(\hat{y}, \hat{S})$ of:

$$\begin{array}{lll} \text{(ACP:)} & \text{maximize}_{y,S} & \prod_{i=1}^{n} \lambda_i(S) \\ & \text{s.t.} & \sum_{i=1}^{m} y_i A_i + S = C \\ & & S \succeq 0 \ . \end{array}$$

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$$\begin{array}{lll} \text{(ACP:)} & \text{maximize}_{y,S} & \prod_{i=1}^{n} \lambda_i(S) \\ & \text{s.t.} & \sum_{i=1}^{m} y_i A_i + S = C \\ & & S \succeq 0 \ . \end{array}$$

This is the same as:

$$\begin{array}{lll} \text{(ACP:)} & \text{minimize}_{y,S} & -\ln\det(S) \\ & \text{s.t.} & \sum_{i=1}^{m} y_i A_i + S = C \\ & & S \succ 0 \ . \end{array}$$

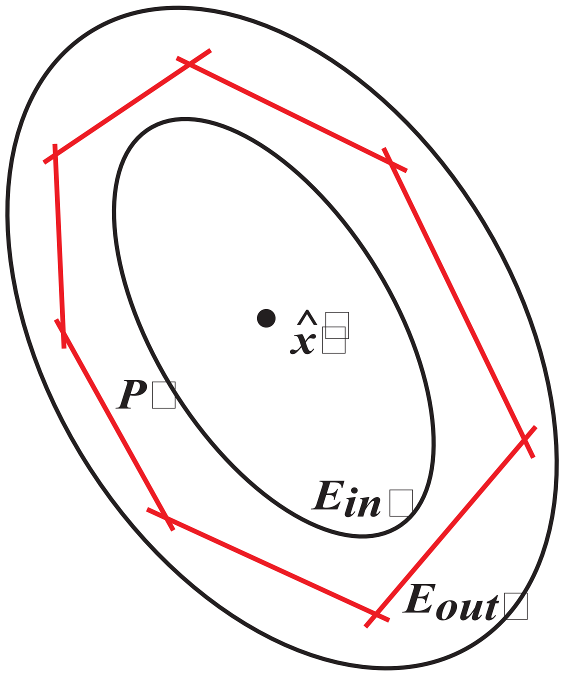
Slide 54

$$\begin{array}{lll} \text{(ACP:)} & \text{minimize}_{y,S} & -\ln\det(S) \\ & \text{s.t.} & \sum_{i=1}^{m} y_i A_i + S = C \\ & & S \succ 0 \ . \end{array}$$

Let $(\hat{y}, \hat{S})$ be the analytic center.
There are easy-to-construct ellipsoids E_{IN} and E_{OUT}, both centered at $\hat{y}$ and where E_{OUT} is a scaled version of E_{IN} with scale factor n, with the property that:

$$E_{\text{IN}} \subset P \subset E_{\text{OUT}}$$

Slide 55



18 Minimum Volume Circumscription

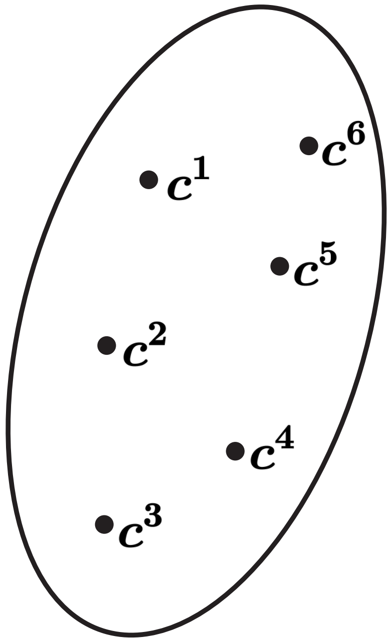
SLIDE 56

$R \succ 0$ and $z \in \Re^n$ define an ellipsoid in $\Re^n$:

$$E_{R,z} := \{y \mid (y-z)^T R(y-z) \leq 1\}.$$

The volume of $E_{R,z}$ is proportional to $\sqrt{\det(R^{-1})}$. SLIDE 57

Given k points $c_1, \ldots, c_k$, we would like to find an ellipsoid circumscribing $c_1, \ldots, c_k$ that has minimum volume:



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$$MCP: \quad \underset{R,z}{\text{minimize}} \quad \text{vol}\ (E_{R,z})$$
$$\text{s.t.} \quad c_i \in E_{R,z}, \quad i = 1, \ldots, k$$

which is equivalent to:

$$MCP: \quad \underset{R,z}{\text{minimize}} \quad -\ln(\det(R))$$
$$\text{s.t.} \quad (c_i - z)^T R (c_i - z) \le 1, \quad i = 1, \ldots, k$$
$$R \succ 0$$

Slide 59

$$MCP: \quad \underset{R,z}{\text{minimize}} \quad -\ln(\det(R))$$
$$\text{s.t.} \quad (c_i - z)^T R (c_i - z) \le 1, \quad i = 1, \ldots, k$$
$$R \succ 0$$

Factor $R = M^2$ where $M \succ 0$ (that is, M is a square root of R):

$$MCP: \quad \underset{M,z}{\text{minimize}} \quad -\ln(\det(M^2))$$
$$\text{s.t.} \quad (c_i - z)^T M^T M (c_i - z) \le 1, \quad i = 1, \ldots, k,$$
$$M \succ 0$$

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$$\begin{array}{llll} MCP: & \underset{M,z}{\text{minimize}} & -\ln(\det(M^2)) & \\ & \text{s.t.} & (c_i - z)^T M^T M(c_i - z) \leq 1, & i = 1, \dots, k, \\ & & M \succ 0. & \end{array}$$

Notice the equivalence:

$$\begin{pmatrix} I & Mc_i - Mz \\ (Mc_i - Mz)^T & 1 \end{pmatrix} \succeq 0 \quad \Longleftrightarrow \quad (c_i - z)^T M^T M(c_i - z) \leq 1$$

Re-write MCP:

$$\begin{array}{lll} MCP: & \underset{M,z}{\text{minimize}} & -2\ln(\det(M)) \\ & \text{s.t.} & \begin{pmatrix} I & Mc_i - Mz \\ (Mc_i - Mz)^T & 1 \end{pmatrix} \succeq 0, \quad i = 1, \dots, k, \\ & & M \succ 0. \end{array}$$

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$$\begin{array}{lll} MCP: & \underset{M,z}{\text{minimize}} & -2\ln(\det(M)) \\ & \text{s.t.} & \begin{pmatrix} I & Mc_i - Mz \\ (Mc_i - Mz)^T & 1 \end{pmatrix} \succeq 0, \quad i = 1, \dots, k, \\ & & M \succ 0. \end{array}$$

Substitute $y = Mz$:

$$\begin{array}{lll} MCP: & \underset{M,y}{\text{minimize}} & -2\ln(\det(M)) \\ & \text{s.t.} & \begin{pmatrix} I & Mc_i - y \\ (Mc_i - y)^T & 1 \end{pmatrix} \succeq 0, \quad i = 1, \dots, k, \\ & & M \succ 0. \end{array}$$

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$$\begin{array}{lll} MCP: & \underset{M,y}{\text{minimize}} & -2\ln(\det(M)) \\ & \text{s.t.} & \begin{pmatrix} I & Mc_i - y \\ (Mc_i - y)^T & 1 \end{pmatrix} \succeq 0, \quad i = 1, \dots, k, \\ & & M \succ 0. \end{array}$$

This problem is very easy to solve.
Recover the original solution R, z by computing:

$$R = M^2 \quad \text{and} \quad z = M^{-1}y.$$

19 SDP in Control Theory

Slide 63

A variety of control and system problems can be cast and solved as instances of SDP. This topic is beyond the scope of this lecturer's expertise.

20 Interior-point Methods for SDP

20.1 Primal and Dual SDP

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$$\begin{array}{lll} SDP: & \text{minimize} & C \bullet X \\ & \text{s.t.} & A_i \bullet X = b_i \quad, i = 1, \ldots, m, \\ & & X \succeq 0 \end{array}$$

and

$$\begin{array}{lll} SDD: & \text{maximize} & \sum_{i=1}^m y_i b_i \\ & \text{s.t.} & \sum_{i=1}^m y_i A_i + S = C \\ & & S \succeq 0 \ . \end{array}$$

If X and (y, S) are feasible for the primal and the dual, the duality gap is:

$$C \bullet X - \sum_{i=1}^m y_i b_i = S \bullet X \geq 0 \ .$$

Also,

$$S \bullet X = 0 \iff SX = 0 \ .$$

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$$B(X) = -\sum_{j=1}^n \ln(\lambda_i(X)) = -\ln\left(\prod_{j=1}^n \lambda_i(X)\right) = -\ln(\det(X)) \ .$$

Consider:

$$\begin{array}{lll} BSDP(\mu): & \text{minimize} & C \bullet X - \mu \ln(\det(X)) \\ & \text{s.t.} & A_i \bullet X = b_i \quad, i = 1, \ldots, m, \\ & & X \succ 0. \end{array}$$

Let $f_\mu(X)$ denote the objective function of $BSDP(\mu)$. Then:

$$-\nabla f_\mu(X) = C - \mu X^{-1}$$

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$$\begin{array}{lll} BSDP(\mu): & \text{minimize} & C \bullet X - \mu \ln(\det(X)) \\ & \text{s.t.} & A_i \bullet X = b_i \quad, i = 1, \ldots, m, \\ & & X \succ 0. \end{array}$$

$\nabla f_\mu(X) = C - \mu X^{-1}$
Karush-Kuhn-Tucker conditions for $BSDP(\mu)$ are:

$$\begin{cases} A_i \bullet X = b_i \quad, i = 1, \ldots, m, \\ X \succ 0, \\ C - \mu X^{-1} = \sum_{i=1}^m y_i A_i. \end{cases}$$

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$$\begin{cases} A_i \bullet X = b_i \quad , i = 1, \ldots, m, \\ X \succ 0, \\ C - \mu X^{-1} = \sum_{i=1}^{m} y_i A_i. \end{cases}$$

Define

$$S = \mu X^{-1} \ ,$$

which implies

$$XS = \mu I \ ,$$

and rewrite KKT conditions as: SLIDE 68

$$\begin{cases} A_i \bullet X = b_i \quad , i = 1, \ldots, m, \quad X \succ 0 \\ \sum_{i=1}^{m} y_i A_i + S = C \\ XS = \mu I. \end{cases}$$

SLIDE 69

$$\begin{cases} A_i \bullet X = b_i \quad , i = 1, \ldots, m, \quad X \succ 0 \\ \sum_{i=1}^{m} y_i A_i + S = C \\ XS = \mu I. \end{cases}$$

If (X, y, S) is a solution of this system, then X is feasible for SDP, (y, S) is feasible for SDD, and the resulting duality gap is

$$S \bullet X = \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij} X_{ij} = \sum_{j=1}^{n} (SX)_{jj} = \sum_{j=1}^{n} (\mu I)_{jj} = n\mu.$$

SLIDE 70

$$\begin{cases} A_i \bullet X = b_i \quad , i = 1, \ldots, m, \quad X \succ 0 \\ \sum_{i=1}^{m} y_i A_i + S = C \\ XS = \mu I. \end{cases}$$

If (X, y, S) is a solution of this system, then X is feasible for SDP, (y, S) is feasible for SDD, the duality gap is

$$S \bullet X = n\mu.$$

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This suggests that we try solving $BSDP(\mu)$ for a variety of values of μ as $\mu \to 0$. Interior-point methods for SDP are very similar to those for linear optimization, in that they use Newton's method to solve the KKT system as $\mu \to 0$.

21 Website for SDP

SLIDE 72

A good website for semidefinite programming is:

`http://www-user.tu-chemnitz.de/ helmberg/semidef.html`.

22 Optimization of Truss Vibration

22.1 Motivation

SLIDE 73

- The design and analysis of trusses are found in a wide variety of scientific applications including engineering mechanics, structural engineering, MEMS, and biomedical engineering.
- As finite approximations to solid structures, a truss is the fundamental concept of Finite Element Analysis.
- The truss problem also arises quite obviously and naturally in the design of scaffolding-based structures such as bridges, the Eiffel tower, and the skeletons for tall buildings.

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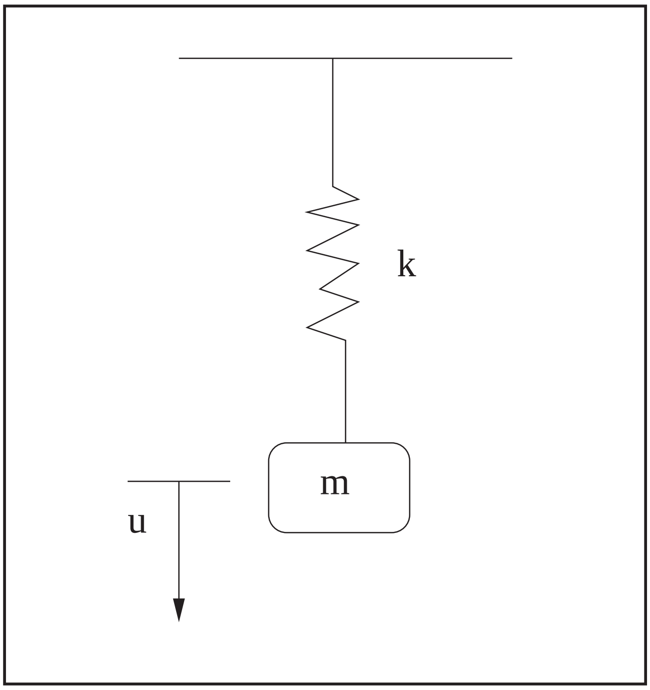
- Using semidefinite programming (SDP) and the interior-point software SDPT3, we will explore an elegant and powerful technique for optimizing truss vibration dynamics.
- The problem we consider here is designing a truss such that the lowest frequency Ω at which it vibrates is above a given lower bound $\bar{\Omega}$.
- November 7, 1940, Tacoma Narrows Bridge in Tacoma, Washington

22.2 The Dynamics Model

SLIDE 75

Newton's Second Law of Motion:

$$F = m \times a \ .$$



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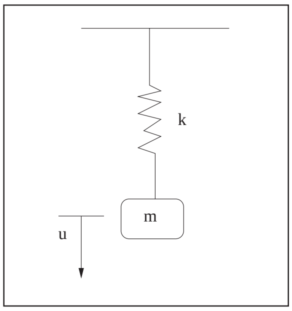

If the mass is pulled down, the displacement u produces a force in the spring tending to move the mass back to its equilibrium point (where $u = 0$).
The displacement u causes an upward force $k \times u$, where k is the spring constant.
We obtain from $F = m \times a$ that:

$$-ku(t) = m\ddot{u}(t)$$

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Law of Motion:

$$-ku(t) = m\ddot{u}(t)$$

Solution:

$$u(t) = \sin\left(\sqrt{\frac{k}{m}}\; t\right)$$

Frequency of vibration:

$$\omega = \sqrt{\frac{k}{m}}\;.$$

22.2.1 Apply to Truss Structure

SLIDE 78

Law of Motion:

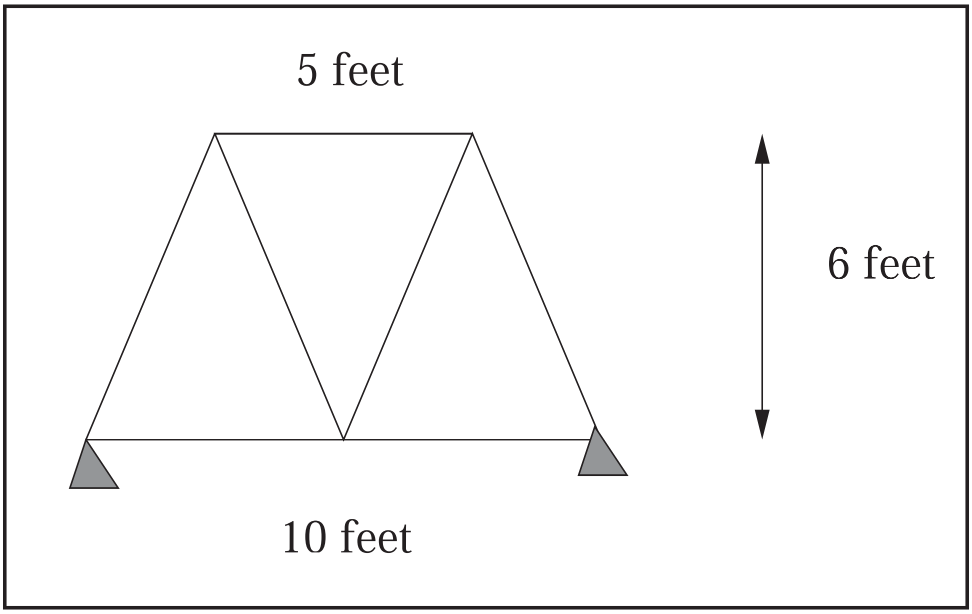
$$-ku(t) = m\ddot{u}(t)$$

Solution:

$$u(t) = \sin\left(\sqrt{\frac{k}{m}}\; t\right)$$

$$\omega = \sqrt{\frac{k}{m}}$$

For truss structure, we need multidimensional analogs for $k, u(t)$, and m. SLIDE 79
A simple truss.
Each bar has both stiffness and mass that depend on material properties and the bar's cross-sectional area.



22.2.2 Analog of k

SLIDE 80

The spring constant k extends to the stiffness matrix of a truss.
We used G to denote the stiffness matrix.
Here we will use K.

$$K = G = AB^{-1}A^T$$

Each column of A, denoted as a_i, is the projection of bar i onto the degrees of freedom of the nodes that bar i meets.

$$B = \begin{pmatrix} \frac{L_1^2}{E_1 t_1} & & 0 \\ & \ddots & \\ 0 & & \frac{L_m^2}{E_m t_m} \end{pmatrix} \quad , \quad B^{-1} = \begin{pmatrix} \frac{E_1 t_1}{L_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{E_m t_m}{L_m^2} \end{pmatrix} .$$

22.2.3 Analog of m

SLIDE 81

Instead of a single displacement scalar $u(t)$, we have N degrees of freedom, and the vector

$$u(t) = (u_1(t), \ldots, u_N(t))$$

is the vector of displacements.
The mass m extends to a mass matrix M

22.2.4 Laws of Motion

SLIDE 82

$$-ku(t) = m\ddot{u}(t)$$

becomes:

$$-Ku(t) = M\ddot{u}(t)$$

Both K and M are SPD matrices, and are easily computed once the truss geometry and the nodal constraints are specified. SLIDE 83

$$-Ku(t) = M\ddot{u}(t)$$

The truss structure vibration involves sine functions with frequencies

$$\omega_i = \sqrt{\lambda_i}$$

where

$$\lambda_1, \dots, \lambda_N$$

are the eigenvalues of

$$M^{-1}K$$

The *threshold frequency* Ω of the truss is the lowest frequency $\omega_i, i = 1, \dots, N$, or equivalently, the square root of the smallest eigenvalue of $M^{-1}K$. SLIDE 84

$$-Ku(t) = M\ddot{u}(t)$$

The *threshold frequency* Ω of the truss is the square root of the smallest eigenvalue of $M^{-1}K$.
Lower bound constraint on the threshold frequency

$$\Omega \geq \bar{\Omega}$$

Property:

$$\Omega \geq \bar{\Omega} \iff K - \bar{\Omega}^2 M \succeq 0 \ .$$

22.3 Truss Vibration Design

SLIDE 85

We wrote the stiffness matrix as a linear function of the volumes t_i of the bars i:

$$K = \sum_{i=1}^{m} t_i \frac{E_i}{L_i^2} (a_i)(a_i)^T \ ,$$

L_i is the length of bar i
E_i is the Young's modulus of bar i
t_i is the volume of bar i.

22.4 Truss Vibration Design

SLIDE 86

Here we use y_i to represent the area of bar i ($y_i = \frac{t_i}{L_i}$)

$$K = K(y) = \sum_{i=1}^{m} \left[\frac{E_i}{L_i} (a_i)(a_i)^T \right] y_i = \sum_{i=1}^{m} K_i y_i$$

where

$$K_i = \left[\frac{E_i}{L_i} (a_i)(a_i)^T \right] \ , \ i = 1, \dots, m$$

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There are matrices $M_1, \ldots, M_m$ for which we can write the mass matrix as a linear function of the areas $y_1, \ldots, y_m$:

$$M = M(y) = \sum_{i=1}^{m} M_i y_i$$

In truss vibration design, we seek to design a truss of minimum weight whose threshold frequency Ω is at least a pre-specified value $\bar{\Omega}$. SLIDE 88

$$\begin{array}{lll} TSDP: & \text{minimize} & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} & & \sum_{i=1}^{m} (K_i - \bar{\Omega}^2 M_i) y_i \succeq 0 \\ & & l_i \leq y_i \leq u_i \ , \ i = 1, \ldots, m \ . \end{array}$$

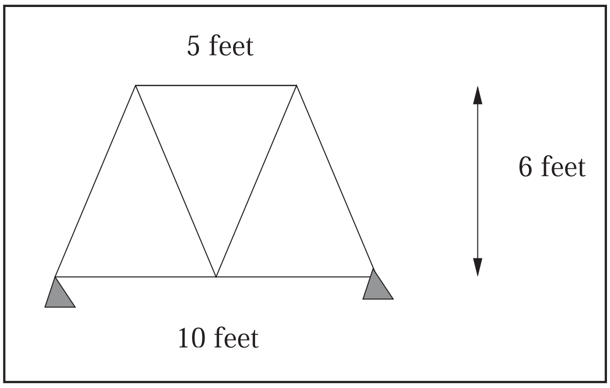
The decision variables are $y_1, \ldots, y_m$ SLIDE 89
l_i, u_i are bounds on the area y_i of bar i (perhaps from the output of the static truss design model)
b_i is the length of bar i times the material density of bar i SLIDE 90

$$\begin{array}{lll} TSDP: & \text{minimize}_{\text{y}} & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} & & \sum_{i=1}^{m} (K_i - \bar{\Omega}^2 M_i) y_i \succeq 0 \\ & & l_i \leq y_i \leq u_i \ , \ i = 1, \ldots, m \ . \end{array}$$

22.5 Computational Example

SLIDE 91

$$\begin{array}{lll} TSDP: & \text{minimize}_{\text{y}} & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} & & \sum_{i=1}^{m} (K_i - \bar{\Omega}^2 M_i) y_i \succeq 0 \\ & & l_i \leq y_i \leq u_i \ , \ i = 1, \ldots, m \ . \end{array}$$



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$$\begin{array}{lll} TSDP: & \text{minimize}_{\text{y}} & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} & & \sum_{i=1}^{m} (K_i - \bar{\Omega}^2 M_i) y_i \succeq 0 \\ & & l_i \leq y_i \leq u_i \ , \ i = 1, \ldots, m \ . \end{array}$$

- $l_i = 5.0$ square inches for all bars i
- $u_i = 8.0$ square inches for all bars i
- mass density for steel, which is ρ =0.736e−03
- Young's modulus for steel, which is 3.0e+07 pounds per square inch
- $\bar{\Omega} = 220$Hz

22.5.1 SDPT3

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SDPT3 is the semidefinite programming software developed by "T3":

- Kim Chuan Toh of National University of Singapore
- Reha Tütünçu of Carnegie Mellon University
- Michael Todd of Cornell University

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Statistics for TSDP problem run using SDPT3

Linear Inequalities	14
Semidefinite block size	6×6
CPU time (seconds):	0.8
IPM Iterations:	15
Optimal Solution	
Bar 1 area (square inches)	8.0000
Bar 2 area (square inches)	8.0000
Bar 3 area (square inches)	7.1797
Bar 4 area (square inches)	6.9411
Bar 5 area (square inches)	5.0000
Bar 6 area (square inches)	6.9411
Bar 7 area (square inches)	7.1797

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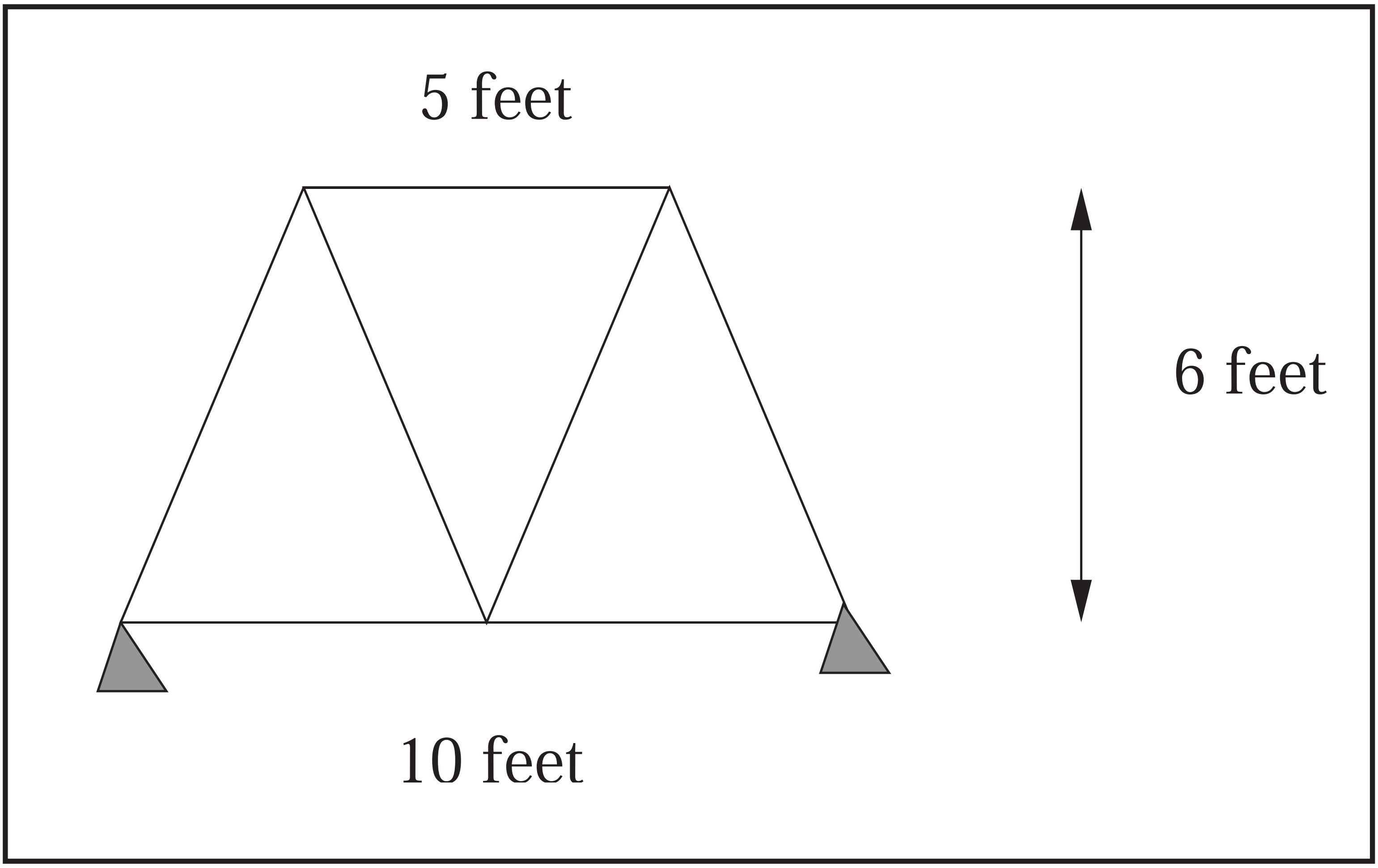
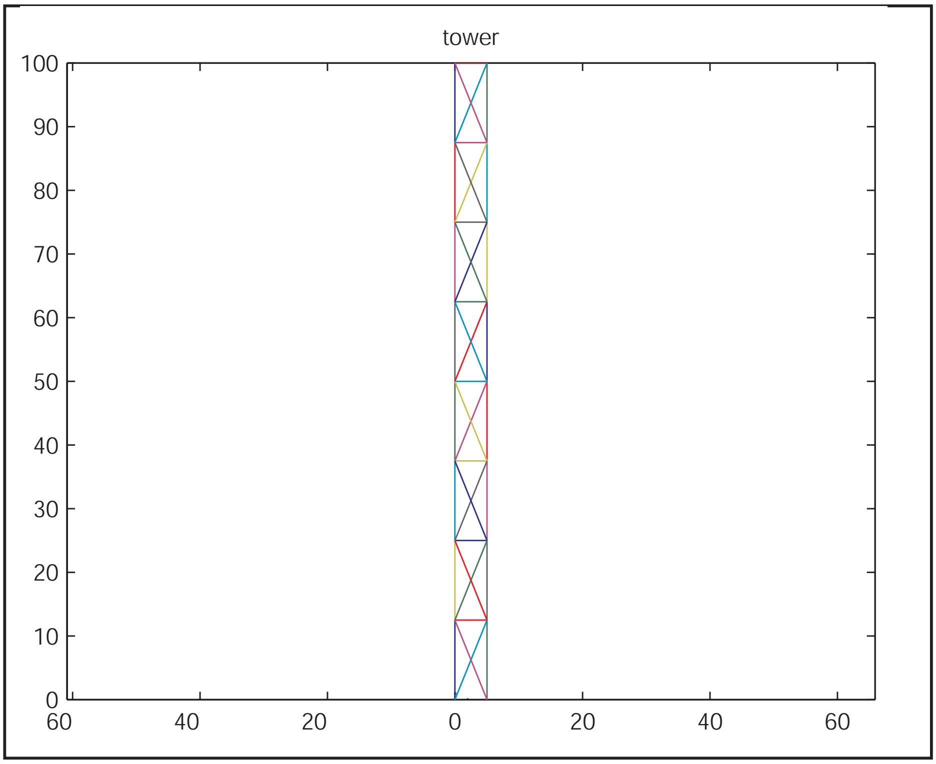


22.6 More Computation

Slide 96

A truss tower used for computational experiments. This version of the tower has 40 bars and 32 degrees of freedom.



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Computational results using SDPT3 for truss frequency optimization.

Semidefinite Block	Linear Inequalities	Scalar Variables	IPM Iterations	CPU time (sec)
12 × 12	30	15	17	1.17
20 × 20	50	25	20	1.49
32 × 32	80	40	21	1.88
48 × 48	120	60	20	2.73
60 × 60	150	75	20	3.76
80 × 80	200	100	23	5.34
120 × 120	300	150	23	9.46

22.6.1 Frontier Solutions

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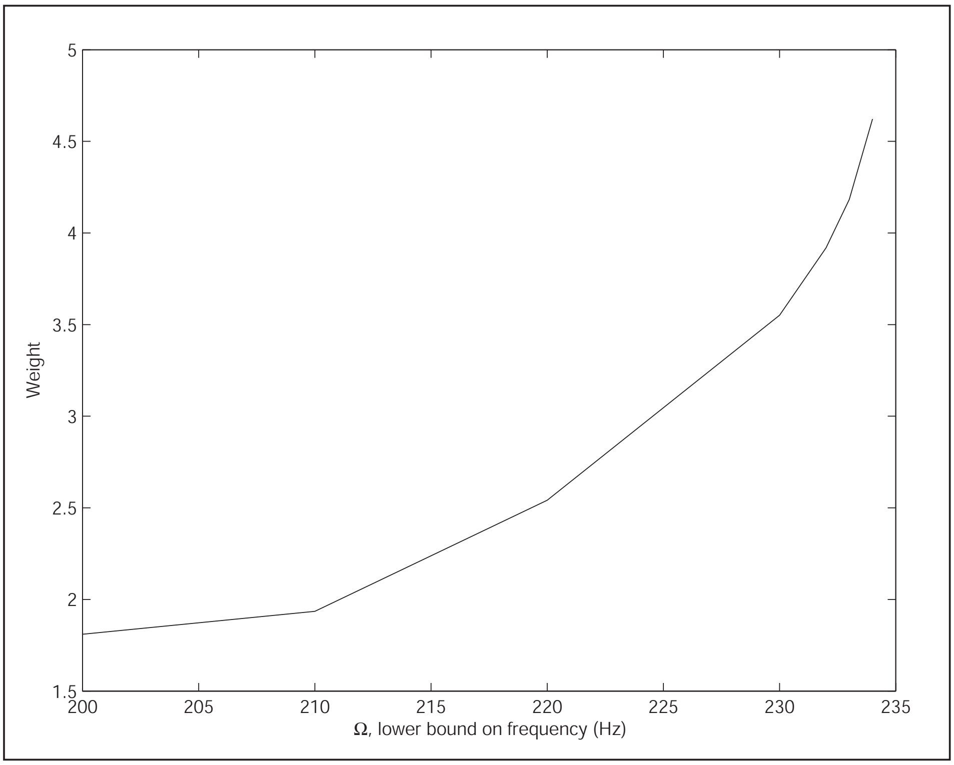
Lower bound on Threshold Frequency Ω versus Weight of Structure



Weight
5
4.5
4
3.5
3
2.5
2
1.5
200
205
210
215
220
225
230
235
Ω, lower bound on frequency (Hz)