Truss Structures - Natural Frequency Manipulation via SDP

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Optimization of Truss Vibration

Motivation

- The design and analysis of trusses are found in a wide variety of scientific applications including engineering mechanics, structural engineering, MEMS, and biomedical engineering.
- As finite approximations to solid structures, a truss is the fundamental concept of Finite Element Analysis.
- The truss problem also arises quite obviously and naturally in the design of scaffolding-based structures such as bridges, the Eiffel tower, and the skeletons for tall buildings.

 73

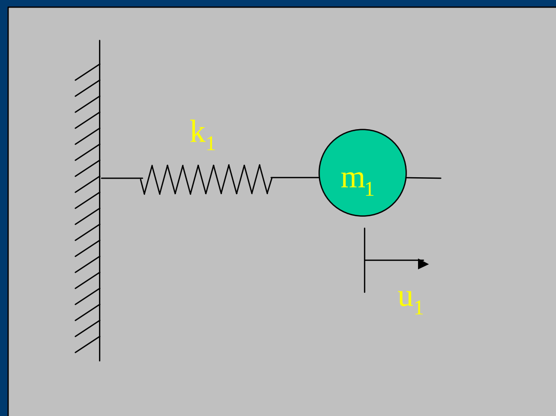
Optimization of Truss Vibration

Motivation

- Using semidefinite programming (SDP) and the interior-point software SDPT3, we will explore an elegant and powerful technique for optimizing truss vibration dynamics.
- The problem we consider here is designing a truss such that the lowest frequency Ω at which it vibrates is above a given lower bound $\bar{\Omega}$.
- November 7, 1940, Tacoma Narrows Bridge in Tacoma, Washington

Background : Mechanical Systems

Simple Mechanical System

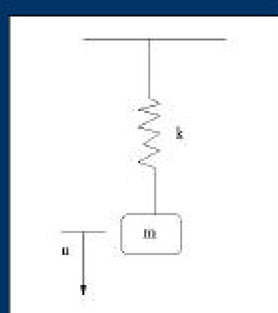


Optimization of Truss Vibration

The Dynamics Model

Newton's Second Law of Motion:

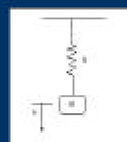
$$F = m \times a \,.$$



 75

Optimization of Truss Vibration

The Dynamics Model



If the mass is pulled down, the displacement u produces a force in the spring tending to move the mass back to its equilibrium point (where $u = 0$).

The displacement u causes an upward force $k \times u$, where k is the spring constant.

We obtain from $F = m \times a$ that:

$$-ku(t) = m\ddot{u}(t)$$

Optimization of Truss Vibration

The Dynamics Model

Law of Motion:

$$-ku(t) = m\ddot{u}(t)$$

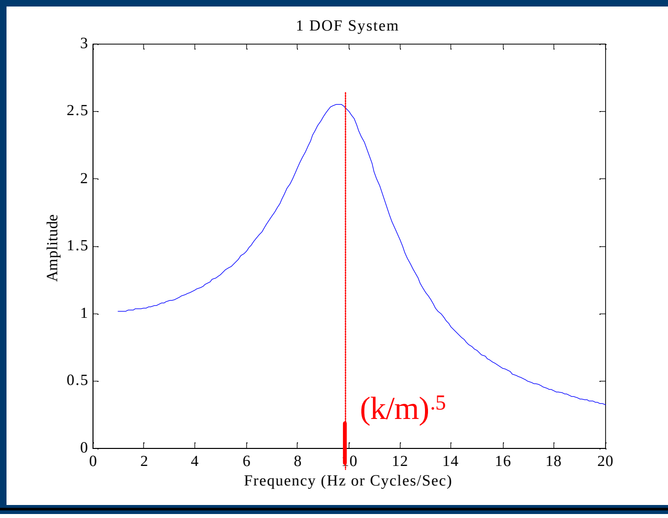
Solution:

$$u(t) = \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

Frequency of vibration:

$$\omega = \sqrt{\frac{k}{m}}.$$

Natural Frequencies – How a Mechanical System wants to vibrate when forced.



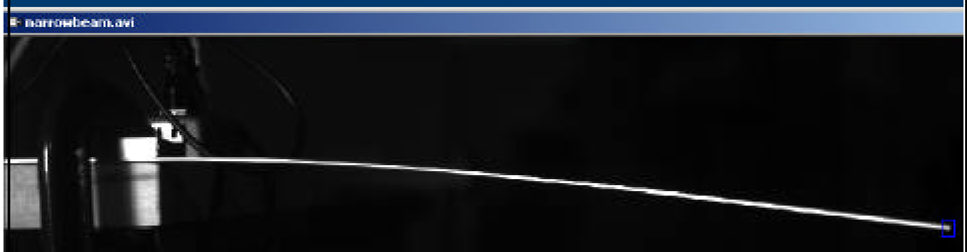
Examples

- Ball on a string
- Beam(s)

- For a simple mechanical systems it is relatively easy to systematically effect a change in the natural frequency.

Beam Vibration

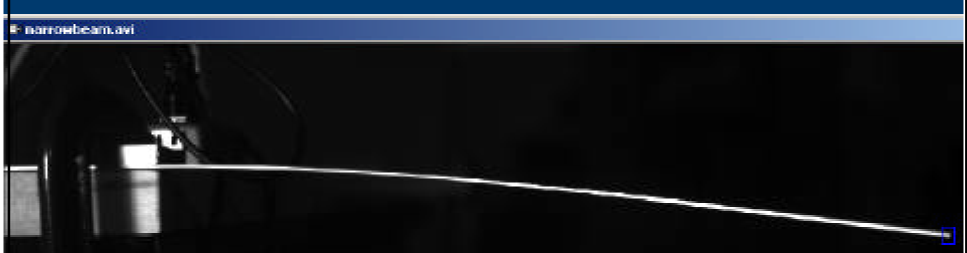
Narrow Beam



How would we change the frequency of vibration?

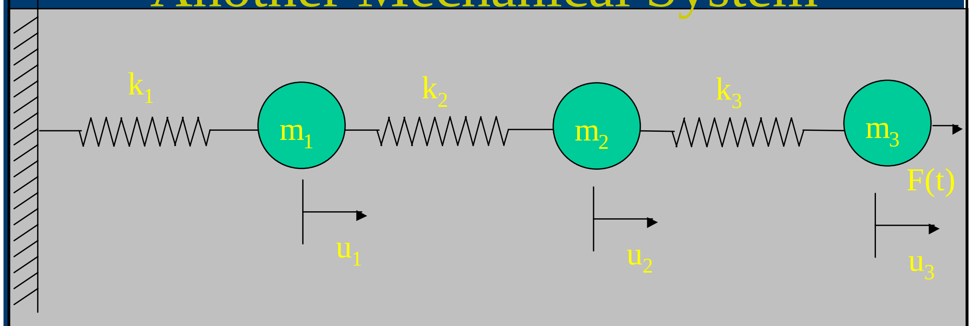
Beam Vibration

Narrow Beam



Narrow Beam	-	Natural Frequency = 5.6 Hz
Narrow and Short Beam	-	Natural Frequency = 11.7 Hz
Wide Beam	-	Natural Frequency = 7.8 Hz

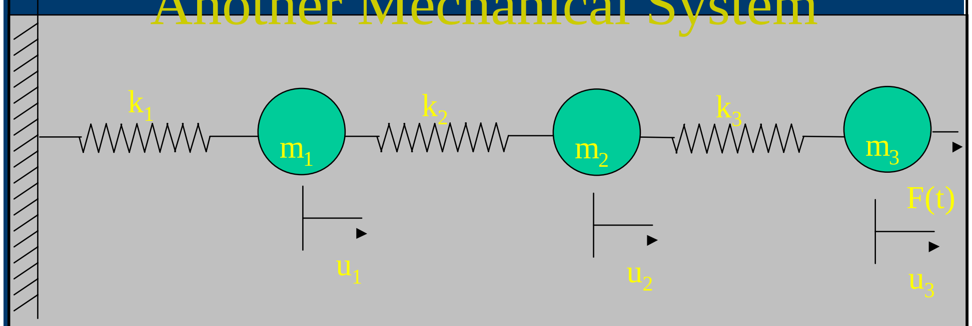
Another Mechanical System



Write Newton's Law for Each Mass

F = ma

Another Mechanical System



The Dynamics Model (The Equations of motion).

$$m_1(d^2u_1/dt^2) + k_1u_1 + k_2(u_1-u_2) = 0$$
$$m_2(d^2u_2/dt^2) + k_2(u_2-u_1) + k_3(u_2-u_3) = 0$$
$$m_3(d^2u_3/dt^2) + k_3(u_3-u_2) = F(t)$$

In Matrix Form

$$\mathbf{M}(d^2\mathbf{U}/dt^2) + \mathbf{KU} = \mathbf{F}(t)$$

Equations of Motion

And Eigenvalue Analysis

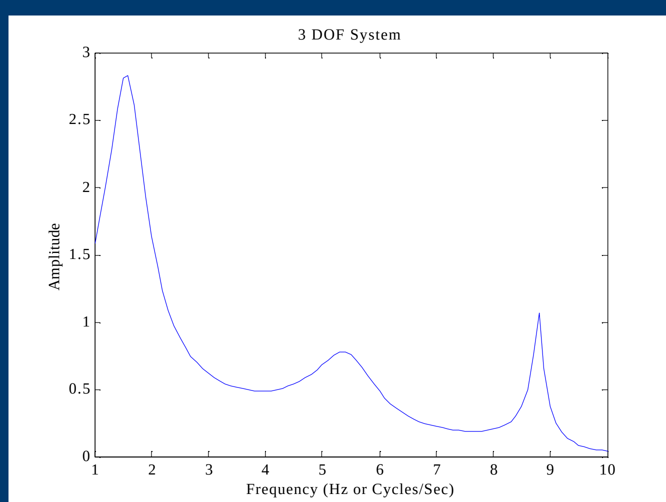
The Equations of motion

$M(d^2U/dt^2) + KU = F(t)$

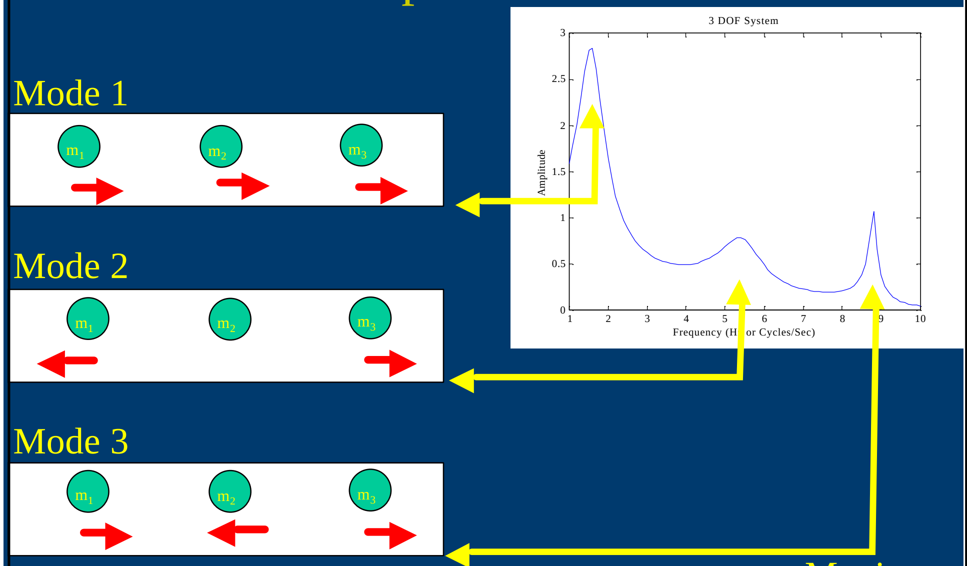
$$M\frac{d^2U}{dt^2} + KU = F(t)$$

- The Eigenvalues of $M^{-1}K$ are the natural frequencies of vibration (squared).
- The Eigenvectors of $M^{-1}K$ are the mode shapes (the relative displacement of each degree of freedom)

Natural Frequencies – The Rate a Mechanical System wants to vibrate

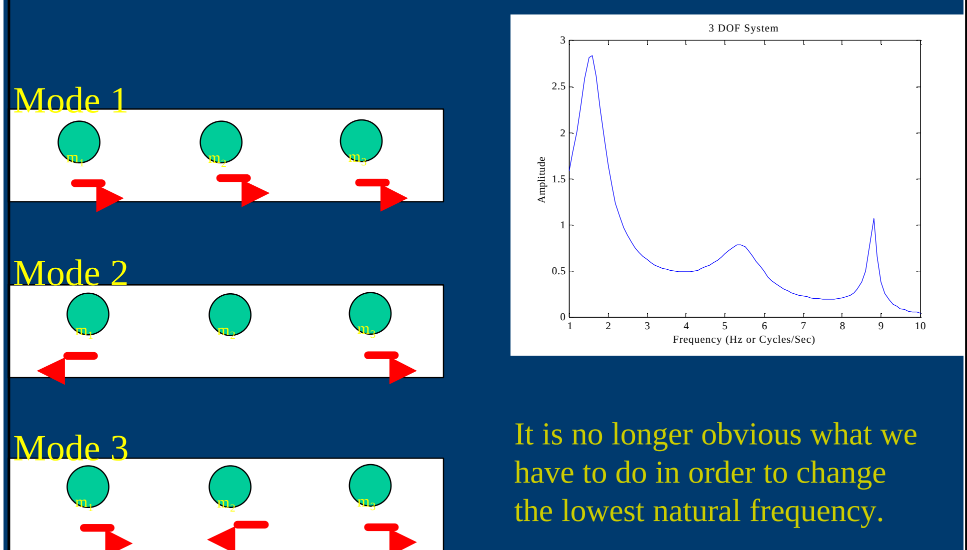


Natural Modes – (The Eigen-Modes) Are the shape of the Vibration



Movies…

Modification?

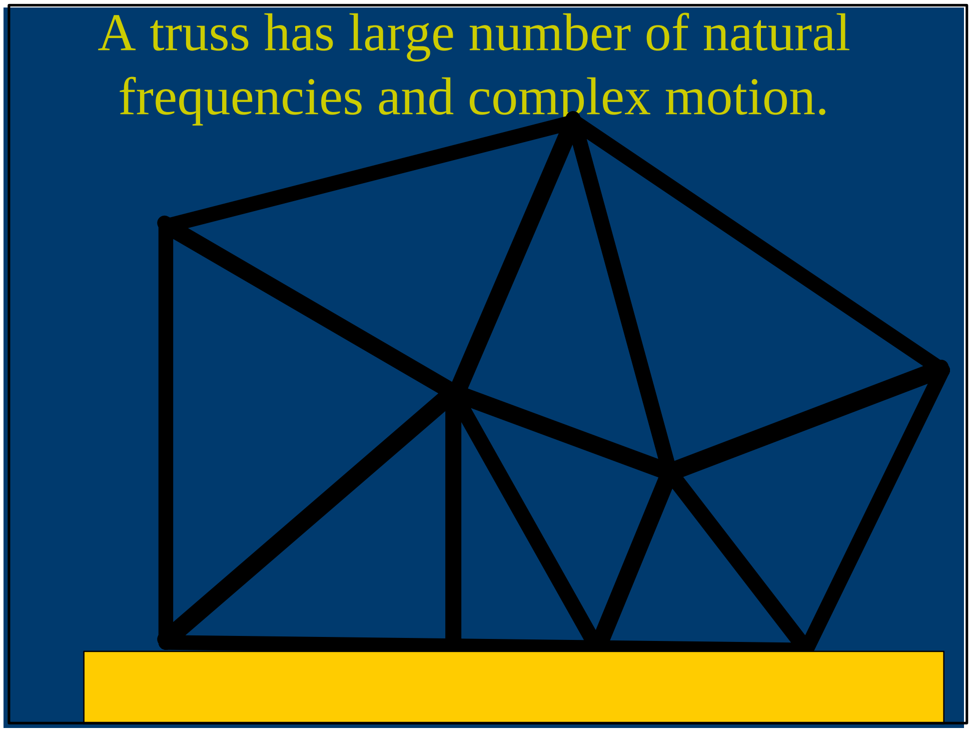


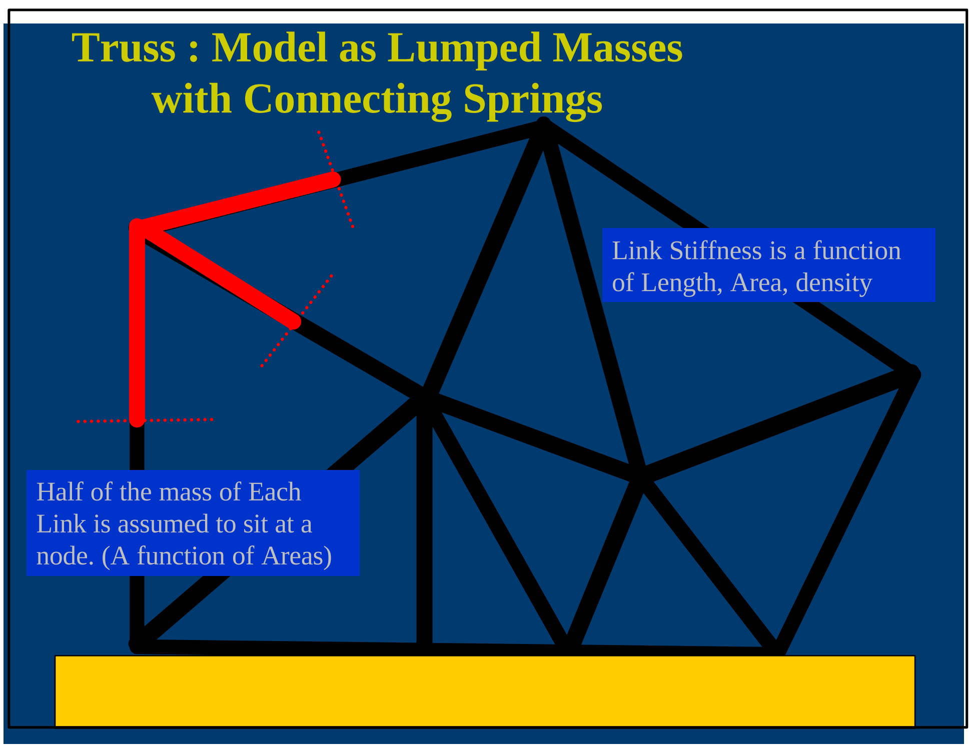
It is no longer obvious what we have to do in order to change the lowest natural frequency.

Truss Structure

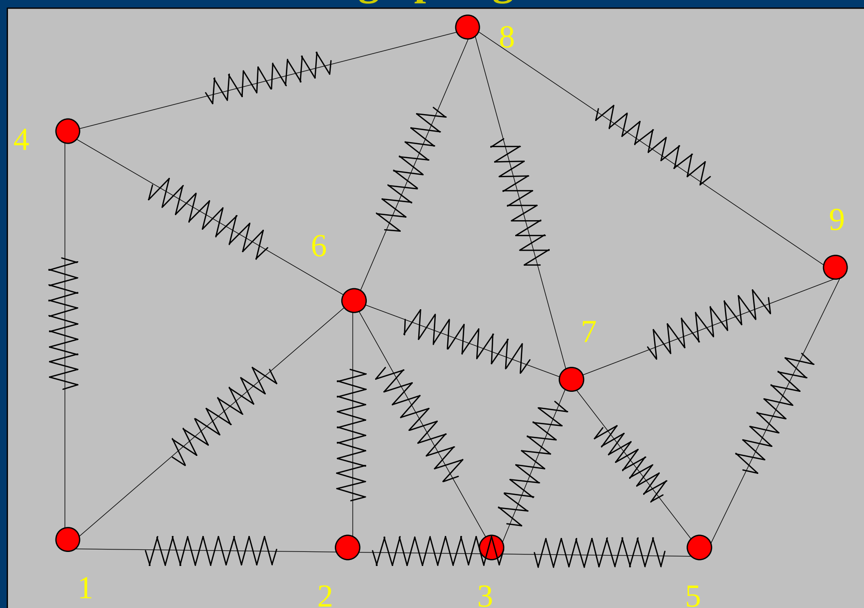
Truss Structures

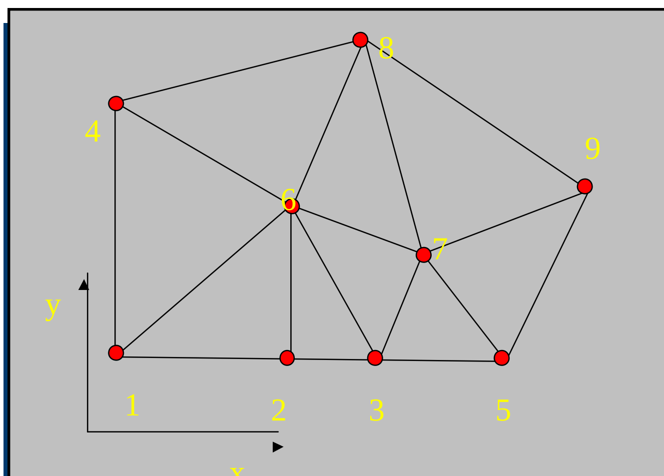
- Rigid beams
  - Axial forces only
- Pin-connected
  - Concentric joints
  - Welded or bolted
- Bridges, towers, exoskeletons


A truss has large number of natural frequencies and complex motion.


Truss : Model as Lumped Masses with Connecting Springs
Link Stiffness is a function of Length, Area, density
Half of the mass of Each Link is assumed to sit at a node. (A function of Areas)

Truss : Modeled as Lumped Masses with Connecting Springs





Step 6. Determine System of Equations by applying Newton's Law at each node (on each ball of mass)

$$M(d^2U/dt^2) + KU = F(t)$$

Optimization of Truss Vibration

The Dynamics Model

Apply to Truss Structure...

Law of Motion:

$$-ku(t) = m\ddot{u}(t)$$

Solution:

$$u(t) = \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$\omega = \sqrt{\frac{k}{m}}$$

For truss structure, we need multidimensional analogs for $k, u(t)$, and m.

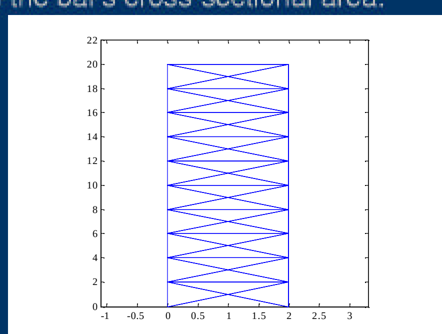
Optimization of Truss Vibration

The Dynamics Model

...Apply to Truss Structure

A simple truss.

Each bar has both stiffness and mass that depend on material properties and the bar's cross-sectional area.



Optimization of Truss Vibration

The Dynamics Model

Analog of k

The spring constant k extends to the stiffness matrix of a truss. We used G to denote the stiffness matrix. Here we will use K.

$$K = G = AB^{-1}A^{T}$$

Each column of A, denoted as a_i, is the projection of bar i onto the degrees of freedom of the nodes that bar i meets.

$$B = \begin{pmatrix} \frac{L_1^2}{E_1 t_1} & & 0 \\ & \ddots & \\ 0 & & \frac{L_m^2}{E_m t_m} \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} \frac{E_1 t_1}{L_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{E_m t_m}{L_m^2} \end{pmatrix}.$$

 80

Optimization of Truss Vibration

The Dynamics Model

Analog of m

Instead of a single displacement scalar $u(t)$, we have N degrees of freedom, and the vector

$$u(t) = (u_1(t), \ldots, u_N(t))$$

is the vector of displacements.

The mass m extends to a mass matrix M

Each Node (Mass) has, in general, 3 Degrees of Freedom

 81

Optimization of Truss Vibration

The Dynamics Model

Laws of Motion...

$$-ku(t) = m\ddot{u}(t)$$

becomes:

$$-Ku(t) = M\ddot{u}(t)$$

Both K and M are SPD matrices, and are easily computed once the truss geometry and the nodal constraints are specified.

 82

Equations of Motion

And Eigenvalue Analysis

The Equations of motion

$M(d^2U/dt^2) + KU = F(t)$

- The Eigenvalues of $M^{-1}K$ are the natural frequencies of vibration (squared).
- The Eigenvectors of $M^{-1}K$ are the mode shapes (the relative displacement of each degree of freedom)

Optimization of Truss Vibration

The Dynamics Model

...Laws of Motion...

$$-Ku(t) = M\ddot{u}(t)$$

The truss structure vibration involves sine functions with frequencies

$$\omega_i = \sqrt{\lambda_i}$$

where

$$\lambda_1, \ldots, \lambda_N$$

are the eigenvalues of

$$M^{-1}K$$

The *threshold frequency* Ω of the truss is the lowest frequency $\omega_i, i = 1, \ldots, N$, or equivalently, the square root of the smallest eigenvalue of $M^{-1}K$.

Optimization of Truss Vibration

The Dynamics Model

...Laws of Motion

$$-Ku(t) = M\ddot{u}(t)$$

The *threshold frequency* Ω of the truss is the square root of the smallest eigenvalue of $M^{-1}K$.

Lower bound constraint on the threshold frequency

$$\Omega \geq \bar{\Omega}$$

Property:

$$\Omega \geq \bar{\Omega} \iff K - \bar{\Omega}^2 M \succeq 0 .$$

Optimization of Truss Vibration

Truss Vibration Design

We wrote the stiffness matrix as a linear function of the volumes t_i of the bars i:

$$K = \sum_{i=1}^{m} t_i \frac{E_i}{L_i^2}(a_i)(a_i)^T \,,$$

L_i is the length of bar i

E_i is the Young's modulus of bar i

t_i is the volume of bar i.

Optimization of Truss Vibration

Truss Vibration Design

Here we use y_i to represent the area of bar i ($y_i = \frac{t_i}{L_i}$)

$$K = K(y) = \sum_{i=1}^{m} \left[\frac{E_i}{L_i}(a_i)(a_i)^T\right] y_i = \sum_{i=1}^{m} K_i y_i$$

where

$$K_i = \left[\frac{E_i}{L_i}(a_i)(a_i)^T\right] \,,\ i = 1, \ldots, m$$

Optimization of Truss Vibration

Truss Vibration Design

There are matrices $M_1, \ldots, M_m$ for which we can write the mass matrix as a linear function of the areas $y_1, \ldots, y_m$:

$$M = M(y) = \sum_{i=1}^{m} M_i y_i$$

Optimization of Truss Vibration

Truss Vibration Design

In truss vibration design, we seek to design a truss of minimum weight whose threshold frequency Ω is at least a pre-specified value $\bar{\Omega}$.

$$TSDP: \quad \text{minimize} \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} (K_i - \bar{\Omega}^2 M_i) y_i \succeq 0$$

$$l_i \le y_i \le u_i \, , \; i = 1, \ldots, m \, .$$

Optimization of Truss Vibration

Truss Vibration Design

The decision variables are $y_1, \ldots, y_m$

l_i, u_i are bounds on the area y_i of bar i (perhaps from the output of the static truss design model)

b_i is the length of bar i times the material density of bar i

Optimization of Truss Vibration

Truss Vibration Design

$$
\begin{array}{lll}
TSDP: & \text{minimize}_y & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & & \sum_{i=1}^{m} (K_i - \bar{\Omega}^2 M_i) y_i \succeq 0 \\
& & l_i \le y_i \le u_i \,,\ i = 1, \ldots, m \,.
\end{array}
$$

Optimization of Truss Vibration

Computational Example

SDPT3...

SDPT3 is the semidefinite programming software developed by "T3":

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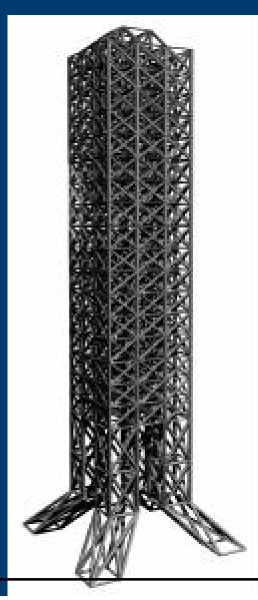
- Michael Todd of Cornell University

 93

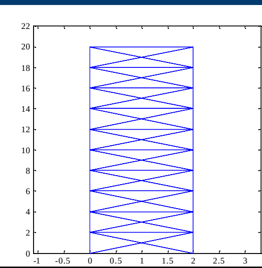
A proposed design of a Cell Phone Tower



Planar Approximation:

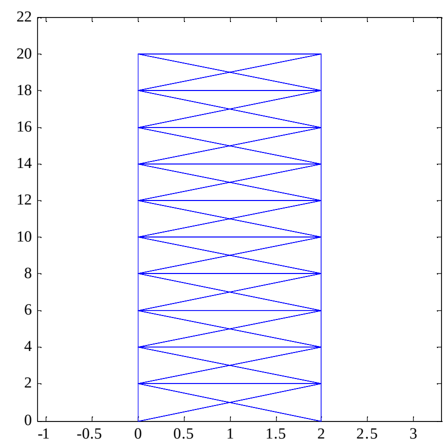
All beams have an
Area of 1 square centimeter

50 Bars
40 Degrees of Freedom

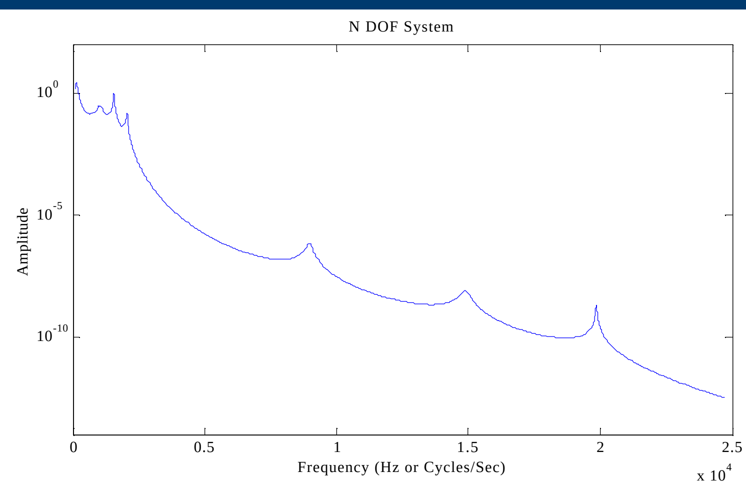


A proposed design of a Cell Phone Tower

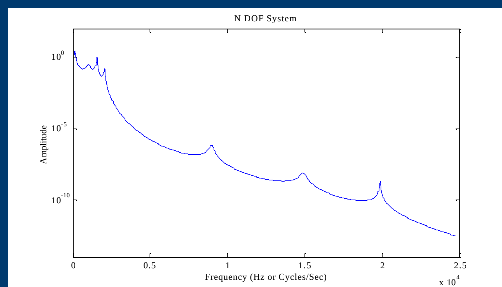
Natural Frequency = 159 Hz
(smallest eigenvalue of $M^{-1}K$)



Natural Frequencies – The Rate a Mechanical System wants to vibrate

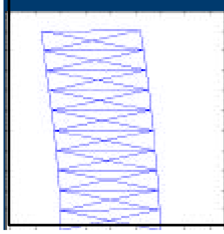


Natural Modes (Eigenvectors or Eigen-modes) - The Shape of the Vibration

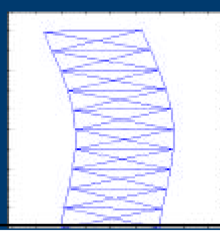


Movies…

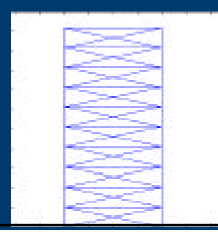
159.7156 Hz



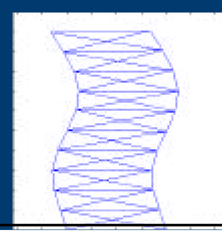
920.3246 Hz



1618.3 Hz



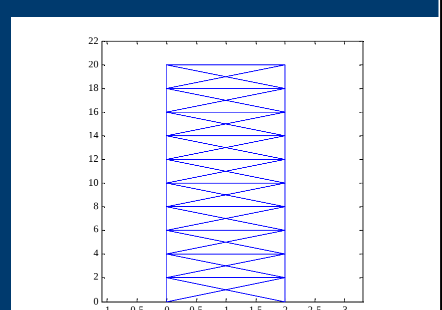
2333.4 Hz



Cell Phone Tower

- The initial design of the cell tower has a natural frequency of 159 Hz.
- We expect ground vibration induced by a nearby railroad near 100 Hz and 159 Hz.

Movies…



Optimized Tower

Natural Frequency = 250 Hz

New Areas:

1.0353

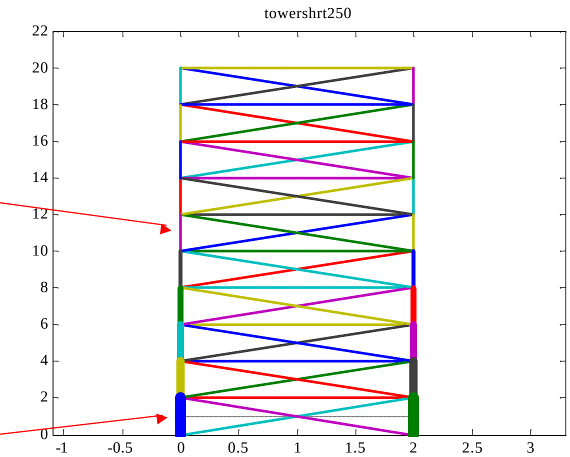
1.5332

2.0785

2.6420

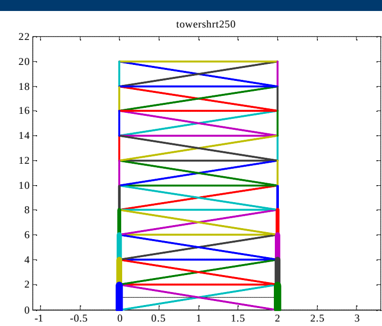
3.2065

3.7667



Cell Phone Tower

- The optimized design of the cell tower has a natural frequency of 250 Hz.
- This vibration of 100 Hz and 159 Hz will now have minimal effect.



Movies…

Optimized Tower

Natural Frequency = 500 Hz

New Areas:

2.0561

4.0845

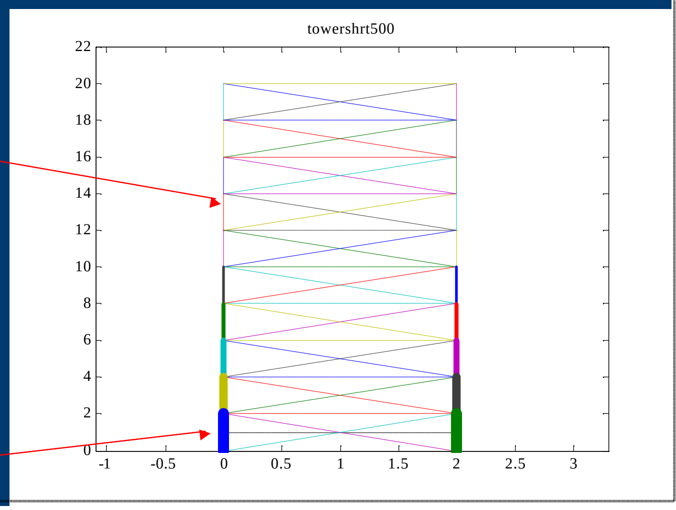
7.3261

11.8377

17.2199

22.8216

28.2383



Optimization of Truss Vibration

Motivation

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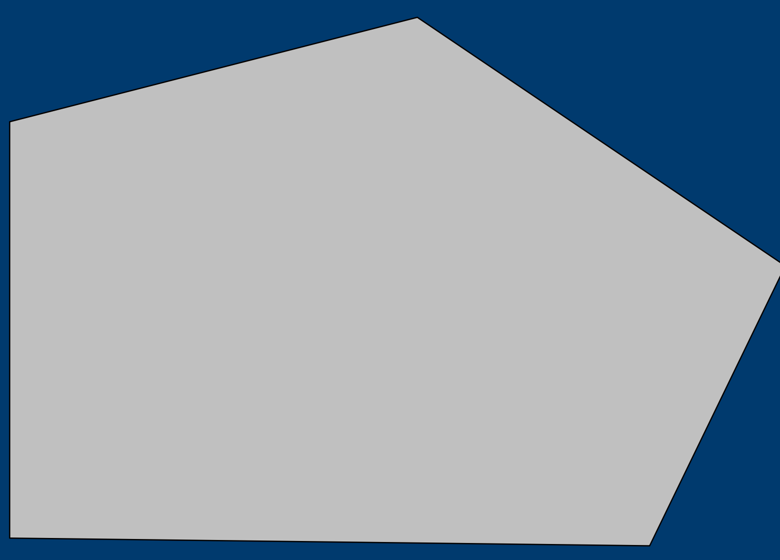
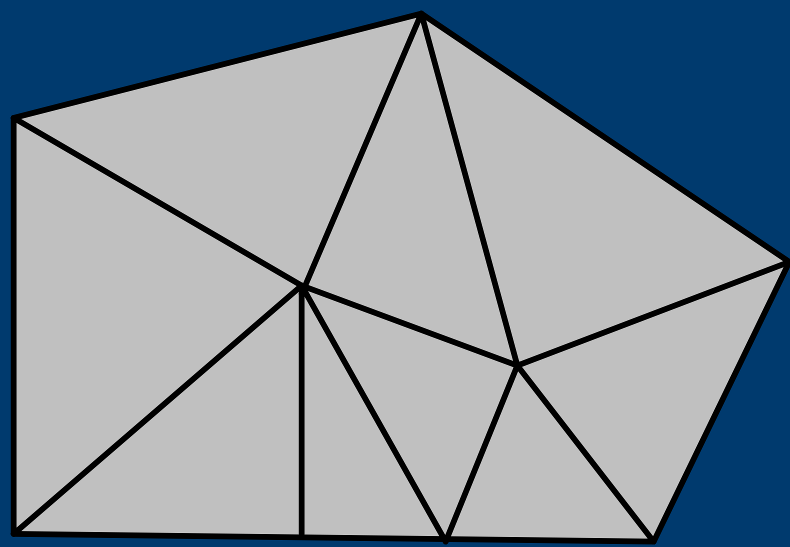
- As finite approximations to solid structures, a truss is the fundamental concept of Finite Element Analysis.
- The truss problem also arises quite obviously and naturally in the design of scaffolding-based structures such as bridges, the Eiffel tower, and the skeletons for tall buildings.

73

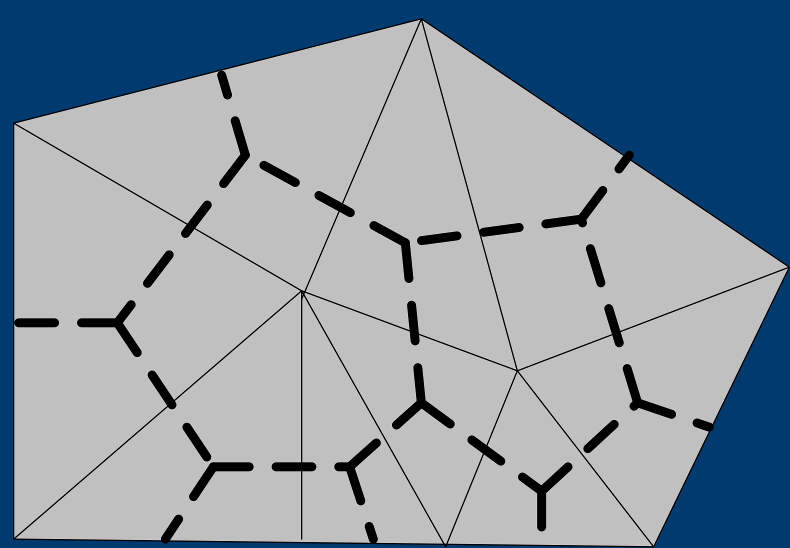
Continuous Mechanical System – Truss Approximation



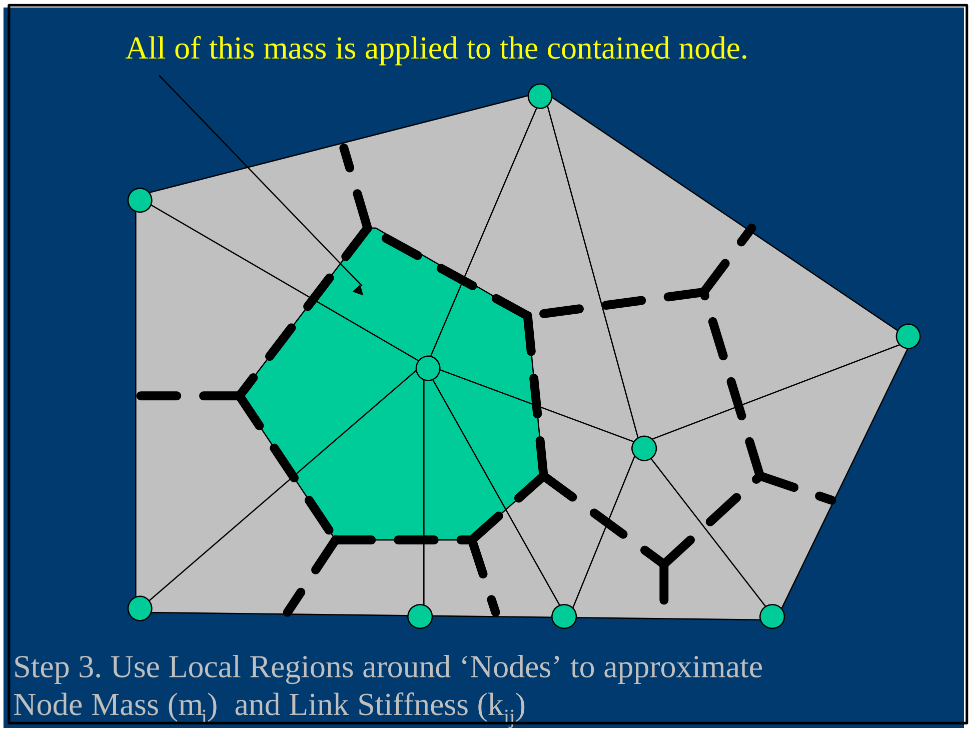
Step 0. Complex (Continuous) Mechanical Structure

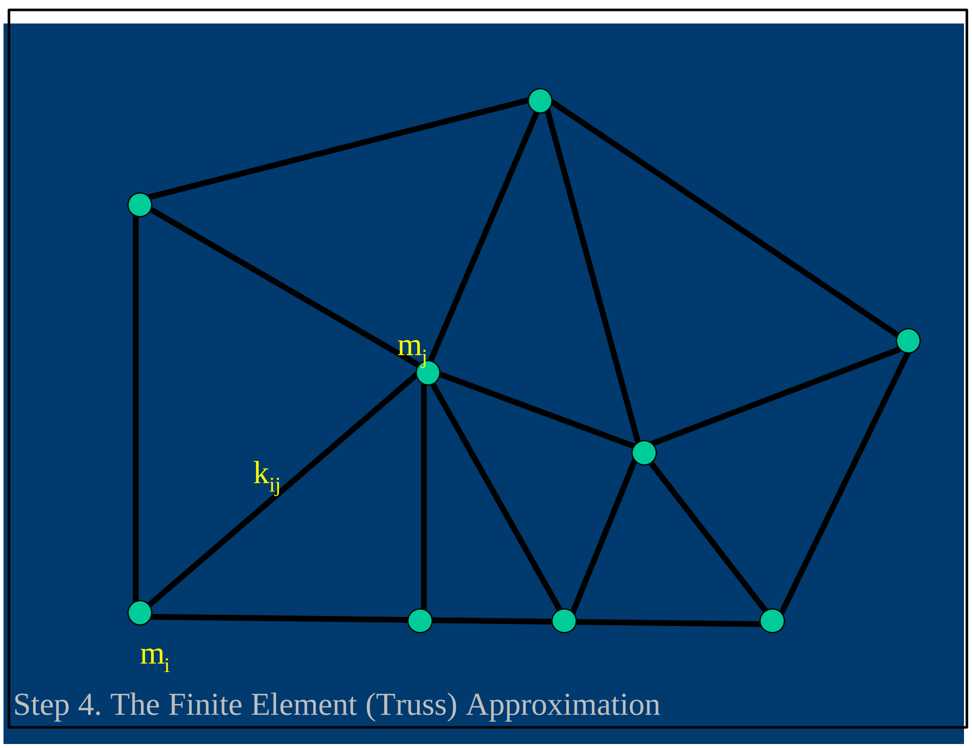


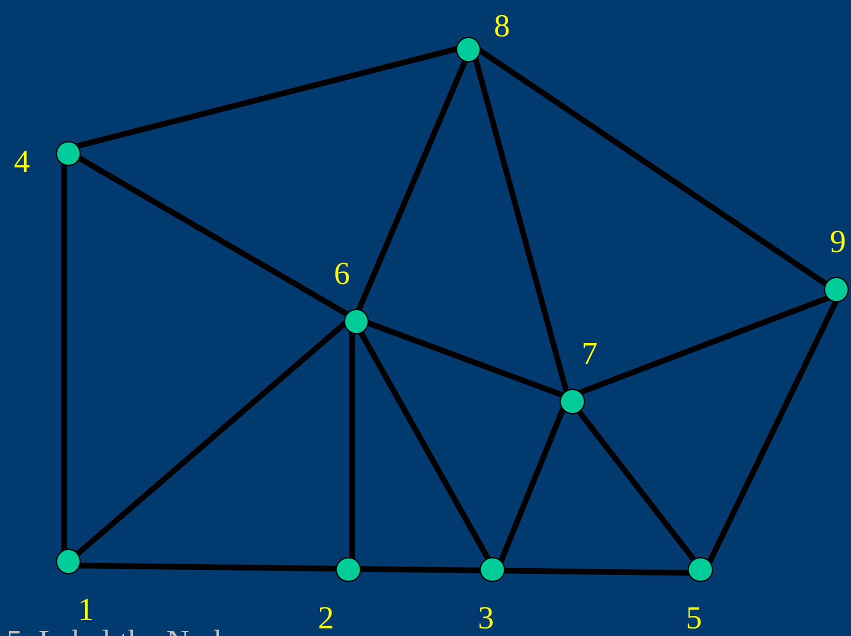
Step 1. Apply a Mesh (Truss) to the Mechanical Structure



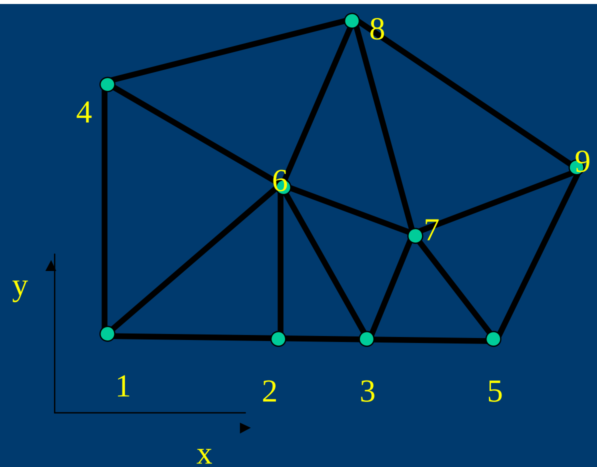
Step 2. Form Local Regions around 'Nodes' of the Mesh


All of this mass is applied to the contained node.
Step 3. Use Local Regions around 'Nodes' to approximate
Node Mass (m_i) and Link Stiffness (k_{ij})


m_j
k_{ij}
m_i
Step 4. The Finite Element (Truss) Approximation



Step 5. Label the Nodes



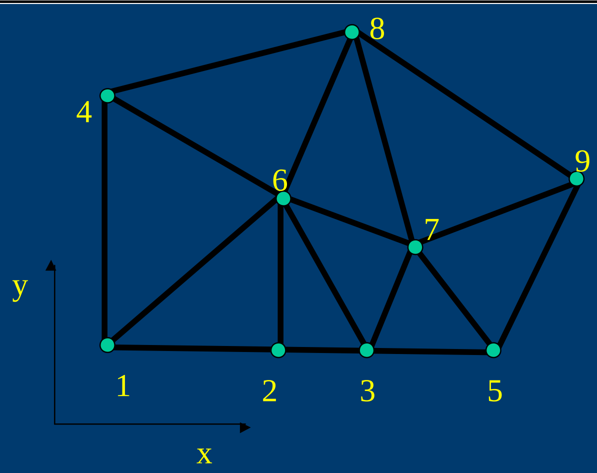
Step 6. Determine System of Equations by applying Newton's Law

…

$$m_6(d^2x_6 /dt^2) + k_{61x}(x_6-x_1) + k_{62x}(x_6-x_2) + \ldots = F_{6x}(t)$$

$$m_6(d^2y_6 /dt^2) + k_{61y}(y_6-y_1) + k_{62y}(y_6-y_2) + \ldots = F_{6y}(t)$$

…



Step 6. Determine System of Equations by applying Newton’s Law

$$M(d^2U/dt^2) + KU = F(t)$$

If Time Remains

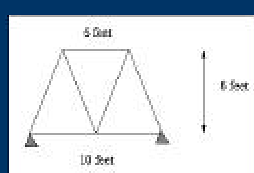
Optimization of Truss Vibration

Computational Example

$$TSDP: \quad \text{minimize}_y \sum_{i=1}^{m} b_i y_i$$

s.t. $\quad {}^2M_i)y_i \succeq 0$

$$l_i \leq y_i \leq u_i \,, \; i = 1, \ldots, m \,.$$



91

Optimization of Truss Vibration

Computational Example

$$
\begin{array}{lll}
TSDP: & \text{minimize}_y & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & & \sum_{i=1}^{m} (K_i - \bar{\Omega}^2 M_i) y_i \succeq 0 \\
 & & l_i \le y_i \le u_i \ , \ i = 1, \ldots, m \ .
\end{array}
$$

- $l_i = 5.0$ square inches for all bars i
- $u_i = 8.0$ square inches for all bars i
- mass density for steel, which is $\rho =$0.736e−03
- Young's modulus for steel, which is 3.0e+07 pounds per square inch
- $\bar{\Omega} = 220$Hz

Optimization of Truss Vibration

Computational Example

SDPT3...

SDPT3 is the semidefinite programming software developed by "T3":

- Kim Chuan Toh of National University of Singapore
- Reha Tütüncu of Carnegie Mellon University
- Michael Todd of Cornell University

Optimization of Truss Vibration

Computational Example

...SDPT3...

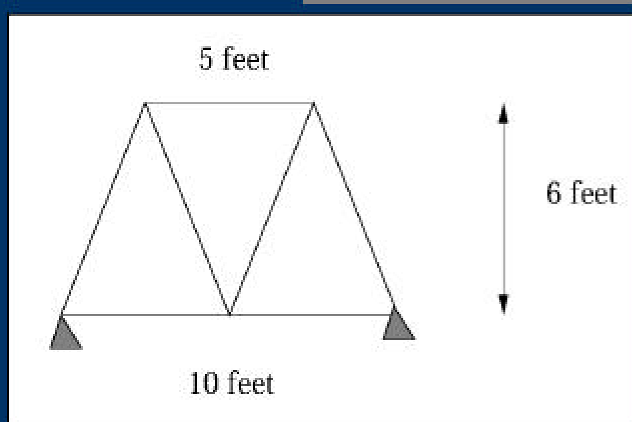
Statistics for TSDP problem run using SDPT3

Linear Inequalities	14
Semidefinite block size	6 × 6
CPU time (seconds):	0.8
IPM Iterations:	15
Optimal Solution	
Bar 1 area (square inches)	8.0000
Bar 2 area (square inches)	8.0000
Bar 3 area (square inches)	7.1797
Bar 4 area (square inches)	6.9411
Bar 5 area (square inches)	5.0000
Bar 6 area (square inches)	6.9411
Bar 7 area (square inches)	7.1797

Optimization of Truss Vibration

Computational Example

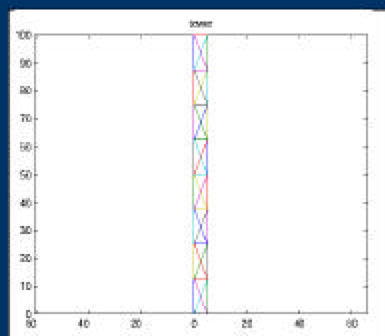
...SDPT3



Optimization of Truss Vibration

More Computation

A truss tower used for computational experiments. This version of the tower has **40** bars and **32** degrees of freedom.



Optimization of Truss Vibration

More Computation

Computational results using SDPT3 for truss frequency optimization.

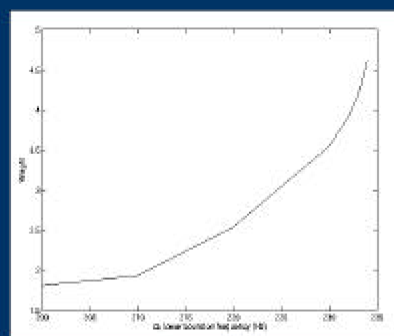
Semidefinite Block	Linear Inequalities	Scalar Variables	IPM Iterations	CPU time (sec)
12 × 12	30	15	17	1.17
20 × 20	50	25	20	1.49
32 × 32	80	40	21	1.88
48 × 48	120	60	20	2.73
60 × 60	150	75	20	3.76
80 × 80	200	100	23	5.34
120 × 120	300	150	23	9.46

Optimization of Truss Vibration

More Computation

Frontier Solutions

Lower bound on Threshold Frequency Ω versus Weight of Structure



 98