

**15.094/SMA5223 Systems Optimization: Models and Computation**  
**Assignment 5 (100 points)**  
**Due April 27, 2004**

## 1 Some Convex Analysis (20 points)

(a) Given positive scalars  $L$  and  $E$ , consider the following set in three-dimensional space:

$$S = \left\{ (f, t, s) \mid \frac{L^2}{E} f^2 \leq s \cdot t, s \geq 0, t \geq 0 \right\} .$$

This set arises in the optimization of load bearing truss structures that you have seen in the lectures on truss design. Show that  $S$  is a convex set.

(b) Let  $S^{k \times k}$  denote the set of  $k \times k$  square symmetric matrices. A matrix  $X \in S^{k \times k}$  is symmetric positive semidefinite (SPSD) if

$$v^T X v \geq 0 \text{ for all } v \in \mathfrak{R}^k .$$

Let  $S_+^{k \times k}$  denote the set of  $k \times k$  SPSD matrices. Prove that  $S_+^{k \times k}$  is a convex set.

(c) A matrix  $X \in S^{k \times k}$  is symmetric positive definite (SPD) if

$$v^T X v > 0 \text{ for all } v \in \mathfrak{R}^k, v \neq 0 .$$

Let  $S_{++}^{k \times k}$  denote the set of  $k \times k$  SPD matrices. Prove that  $S_{++}^{k \times k}$  is a convex set.

## 2 Truss Design (20 points)

The purpose of this problem is to give you some experience in solving truss design models using AMPL and LOQO. Figure 1 shows an  $11 \times 8$  grid for a bridge design problem. The left and right bottom points in the grid, which are the ordered pairs  $(0, 0)$  and  $(10, 0)$ , are fixed points in the grid. There is a downward force at each grid point on the bottom row of the grid.

The AMPL model file `bridge18soc.mod` contains a truss design model of this bridge design problem (modeled using the second-order cone formulation discussed in class) that uses the data file `basic18.dat` .

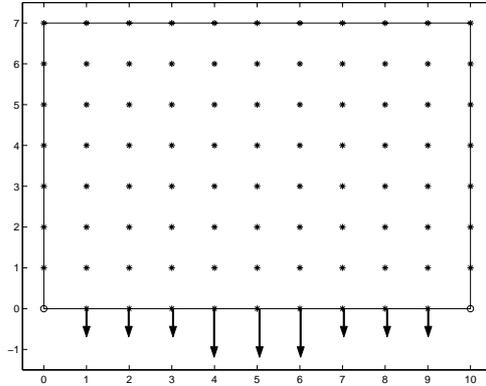


Figure 1: An  $11 \times 8$  bridge design grid.

- (a) Run this model using LOQO. What is the optimal compliance? What is the running time? How many iterations did the algorithm take?
- (b) Create and hand in a picture of the optimal solution using the MATLAB file `respm.m`.
- (c) In reality, structures need to be designed to handle many different external forces. For example, the bridge should be designed to handle forces from winds in various directions that may shift over time as well as vary in intensity. In order to see how sensitive the optimal solution might be to different external forces, we ask you to try several different external force (load) profiles on the bottom of the bridge. As a common basis of comparison use the following external loads:

$$(i) F = \left( \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right)$$

$$(ii) F = \left( \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix} \right)$$

$$(iii) F = \left( \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix} \right)$$

$$(iv) F = \left( \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \end{pmatrix} \right)$$

$$(v) F = \left( \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \right)$$

- (c) What do you observe? How might you take load variations into account when solving this problem in practice?

### 3 Truss Design Formulations (20 points)

In this problem, you are asked to explore possible changes in the the model parameters and in the formulation for the bridge design model shown in Figure 1.

- (a) **Maximum Volume Constraint.** The current total volume constraint upper bound is 1,600.0 units. Change this bound to values between 100.0 units and 100,000.0 units and solve the resulting model. Record the optimized compliance and the number of iterations it takes to solve the model as functions of the volume constraint upper bound. What do you observe? Can you suggest any reasons for this?
- (b) **Tolerance Parameter  $\epsilon$ .** If the value of the Young's modulus is  $E = 1.0$ , then the second-order cone constraint in the truss design problem is of the form:

$$\sqrt{L^2 f^2 + y^2} \leq w .$$

However, you will notice in the AMPL file `bridge6soc.mod` that the constraint in the model is actually:

$$\sqrt{\epsilon^2 + L^2 f^2 + y^2} \leq w ,$$

where  $\epsilon$  is a tolerance parameter given in the input data file `basic6.dat` . The value of  $\epsilon$  given in `basic6.dat` is  $\epsilon = 0.000001$  . If  $\epsilon = 0.0$ , then the AMPL model is the same as the true formulation of the problem. If  $\epsilon > 0$ , then the AMPL model will have a slightly smaller feasible region than the true model of the problem. The purpose of introducing  $\epsilon > 0$  is to speed up solution time by “smoothing” the second-order cone constraint a bit.

Change the value of the tolerance parameter  $\epsilon$  in the range from  $\epsilon = 0.00000001$  up to  $\epsilon = 1.0$  and solve the resulting model. Record the optimal compliance and the number of iterations it takes to solve the model as functions of the value of  $\epsilon$ . What do you observe? Can you suggest any reasons for this?

### 4 High-Percentage Covering Disk Problem (20 points)

Consider the problem of determining the smallest disk that contains the  $m$  given points  $c_1, \dots, c_m$ . The decision variables in the problem are the center  $x$  and the radius  $R$  of the enclosing disk, and the problem yields the simple formulation:

$$\begin{aligned} \text{CDP : } & \text{minimize}_{x,R} && R \\ & \text{s.t.} && \|x - c_i\| \leq R \quad i = 1, \dots, m \\ & && x \in \mathfrak{R}^n, R \in \mathfrak{R} , \end{aligned}$$

where in this case  $n = 2$  is the dimension in which the problem arises.

The Ampl file **CDPql.mod** contains a reformulation of this problem with quadratic objective and linear constraints. **CDPql2.mod** is the same formulation but demonstrates the *problem* command in Ampl. The Ampl file **CDPlq.mod** contains the version of the problem with a linear objective and quadratic constraints. All of these formulations of the problem solve very rapidly, with zero duality gap.

- (a) The data files **cover18.dat** and **cover19.dat** each contain the locations of 100 points in  $n = 2$  dimensions. The first column in the file is the index of the point, and the second and third columns contain the first and second coordinates of the point. Solve the problem using these data sets, and create figures illustrating your solution.

Solve **CDPql2.mod** with the LOQO convex option turned on and off. Is there a difference in how LOQO solves the problem? If so, why?

Solve **CDPlq.mod** with the LOQO convex option turned on and off. Is there a difference in how LOQO solves the problem? If so, why?

- (b) Consider the following variation of the problem, where we would like to find the smallest disk that contains 90% of the points. This problem arises in data mining, for example. We will use the sigmoid function:

$$f_\alpha(s) := \frac{1}{1 + e^{-\alpha s}}$$

to account for points that might lie outside the disk. The sigmoid function with parameter  $\alpha > 0$  has the following attractive properties:

$$\begin{aligned} f_\alpha(s) &\rightarrow 0 \text{ as } s \rightarrow -\infty \\ f_\alpha(s) &\rightarrow 1 \text{ as } s \rightarrow +\infty \\ f_\alpha(s) &= f_\alpha(-s) \\ f_\alpha(0) &= \frac{1}{2} \end{aligned}$$

A graph of this function is shown in Figure 2.

Using the sigmoid function, our problem can be approximated as the following smooth nonlinear problem:

$$\begin{aligned} \text{HPCDP : } & \text{minimize}_{x,R,s_1,\dots,s_m} && R \\ & \text{s.t.} && \|x - c_i\| \leq R + s_i \quad i = 1, \dots, m \\ & && x \in \mathfrak{R}^n, R \in \mathfrak{R}, s_1, \dots, s_m \in \mathfrak{R} \\ & && \sum_{i=1}^m f_\alpha(s_i) \leq 0.1m . \end{aligned}$$

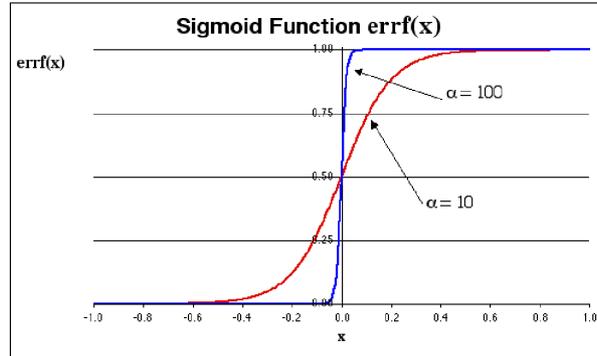


Figure 2: The sigmoid function.

In class we considered a reformulation of this problem where we squared the constraint “ $\|x - c_i\| \leq R + s_i$ ” after adding the relaxation variables  $s_i$ , yielding the new constraint “ $(x - c_i)^T(x - c_i) \leq R^2 + 2Rs_i + s_i^2$ ”. Instead of this approach, create a reformulation where you square the constraint “ $\|x - c_i\| \leq R$ ” and then add the relaxation variables  $s_i$ , namely “ $(x - c_i)^T(x - c_i) \leq R^2 + s_i$ ”. Modify the AMPL model from part (a) and solve HPCDP using the data files `cover18.dat` and `cover19.dat`.

Compute solution results for  $\alpha = 7, 10$ , and  $20$ . What do you observe?

## 5 Disk and Spherical Packing Problem (20 points)

Consider the problem of packing a variety of wires into a cable. The  $m$  wires will have given radii  $r_1, r_2, \dots, r_m$ . We would like to determine the minimum width of a cable that will be used to enclose the wires. We can conceptualize this problem by considering a cross-section of the cable. The decision variables in the problem are the centers  $x_1, \dots, x_m$  of the  $m$  disks of radii  $r_1, \dots, r_m$ , and the radius  $R$  of the disk that is the cross-section of the enclosing cable. This problem has the following formulation:

$$\begin{aligned}
 \text{PP : } & \text{minimize}_{x_1, \dots, x_m, R} && R \\
 & \text{s.t.} && \|x_i - x_j\| \geq r_i + r_j && i = 1, \dots, m - 1, j = i + 1, \dots, m \\
 & && \|x_i\| + r_i \leq R && i = 1, \dots, m \\
 & && x_1, \dots, x_m \in \mathfrak{R}^n, R \in \mathfrak{R},
 \end{aligned}$$

where in this case  $n = 2$  is the dimension in which the problem arises.

The first set of constraints ensures that no two wires overlap the same space, and the second set of constraints ensures that the cable encloses all of the wires.

- (a) One problem with the formulation of PP is that the constraint functions are not differentiable. Convert this formulation into an equivalent form where all functions

are differentiable.

- (b) Construct an iterative AMPL model to solve the problem. Do *not* use the model **PP2.com** that we designed and presented in class; instead we ask you to develop your own iterative AMPL model. The data files `pack18.dat` and `pack18a.dat` contain radius data for  $m = 60$  radii of wires. In each of these files, the first column contains the index (1, 2, etc.), the second column contains the number of points of that index, and the third column contains the radius associated with that index. For example, suppose that the file data is as shown in Table 1. This indicates that there are 50 wires of radius 10.0, also 40 wires of radius 5.0, and also 10 wires of radius 8.7 to be packed.

Index	Number of Wires with given Radius	Radius
1	50	10.0
2	40	5.0
3	10	8.7

Table 1: Example of a packing data file.

Describe your iterative solution method. Use your model to solve the problem for these two data sets, and create figures illustrating your final computed solution.