

Optimization Methods
MIT 6.255/15.093 – Fall 2008
Midterm exam

Date Given: October 16th, 2008.

Date Due: You have **two hours** (academic time, 110 minutes) to complete the exam.

P1. [30 pts] Classify the following statements as true or false. All answers must be fully justified. Unless stated otherwise, all LP problems are in standard form.

- (a) The feasible set of an LP in standard form is always bounded.
- (b) If a basic feasible solution has nonnegative reduced costs, then it is primal optimal.
- (c) If the shadow price of a constraint is zero, then the constraint is necessarily active.
- (d) The dual variables \mathbf{p} quantify the sensitivity of the optimal cost with respect to variations in the entries of the matrix \mathbf{A} .
- (e) If the dual problem is infeasible, then so is the primal problem.
- (f) The reduced cost of a nonbasic variable is always strictly positive.
- (g) Let $\theta \in \mathbb{R}$. The function $F(\theta) := \{\max \mathbf{c}^T \mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}, x_i \leq \theta \text{ for all } i\}$ is concave.
- (h) If \mathbf{x} and \mathbf{p} are primal and dual optimal, respectively, then by complementary slackness we have that the product of primal and dual variables is always zero (i.e., $p_i x_i = 0$ for all i).
- (i) If the primal LP problem has multiple optimal solutions, then there is at least one degenerate basic feasible primal solution.
- (j) Column generation methods are useful for problems with a very large number of constraints.

Solution: (a) False. A simple counterexample is $\{x \in \mathbb{R}^2 : x_1 - x_2 = 0, x_1 \geq 0, x_2 \geq 0\}$.

- (b) True. This is the optimality criterion of a BFS.
- (c) False. Any non-active constraint will have a shadow price equal to zero.
- (d) False. The dual variables \mathbf{p} quantify the sensitivity with respect to the right-hand side \mathbf{b} .
- (e) False. If the dual is infeasible, the primal can either be unbounded or infeasible.
- (f) False. If the current BFS is not optimal, the reduced cost of a nonbasic variable can have arbitrary sign. Even if the current BFS is optimal, some reduced costs can be zero (e.g., if there are multiple solutions).
- (g) True. Dualizing, we have $F(\theta) = \{\min \mathbf{b}^T \mathbf{p} + \theta \sum_i q_i : \mathbf{A}^T \mathbf{p} + \mathbf{q} = \mathbf{c}, \mathbf{p} \geq 0, \mathbf{q} \geq 0\}$. Since this is equivalent to a minimization over the extreme points of the dual feasible set, we have that the cost function is the minimum of a finite set of affine functions, and therefore it is concave.
- (h) False. The product of primal and dual variables is meaningless! In general there is no correspondence between them, even their number (n vs. m) is different.
- (i) False. As a counterexample, consider

$$\max x_1 + x_2 \quad \text{subject to} \quad \begin{cases} x_1 + x_2 = 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases},$$

that has multiple solutions, but no degenerate BFSs.

- (j) False. Column generation methods are used for problems with a large number of *variables*. For problems with many constraints, cutting plane techniques should be applied instead.

P2. [20 pts] You are organizing a workshop at MIT. The total number of students attending will be either 100 or 200 (with equal probability), but you will not find out the exact number until the day of the workshop. However, you need to commit today to firm dinner arrangements for all attendants.

The cost of a restaurant dinner is the following: \$30 per person for the first 150 guests, and \$35 per person for every additional guest (over the first 150). Additionally, in case you cannot afford a restaurant dinner for everybody, you can order pizzas for the remaining guests at a cost of \$5 per person.

The total initial budget is \$3000.

- (a) Formulate an optimization problem, where you maximize the expected value of the number of guests that will enjoy a restaurant dinner at the workshop.
- (b) In this situation, what would happen if all 200 people show up? What happens if only 100 people show up? (qualitative answer is OK).
- (c) Assume that you absolutely cannot go over your initial budget. Change your formulation accordingly, and write a (possibly different) optimization problem. Explain the differences between the two formulations.

Solution: Due to a slight imprecision in the given formulation, this problem can be interpreted in a couple of different ways. For grading, we will accept alternative interpretations (not necessarily the one discussed below), as long as they are internally consistent.

- (a) We will model this as a two-stage decision problem, where we must commit at the first stage on the number of restaurant dinners. If we let r_i and p_i be the number of restaurant dinners and pizzas ordered in each one of the scenarios, we can write a simple LP formulation of the form

$$\max \frac{1}{2}(r_1 + r_2) \quad \text{subject to} \quad \begin{cases} p_1 + r_1 = 100 \\ 5p_1 + 30r_1 \leq 3000 \\ 5p_1 + 35 \cdot (r_1 - 150) + 30 \cdot 150 \leq 3000 \\ p_2 + r_2 = 200 \\ 5p_2 + 30r_2 \leq 3000 \\ 5p_2 + 35 \cdot (r_2 - 150) + 30 \cdot 150 \leq 3000 \end{cases}$$

We assume here that we will commit to a number of dinners equal to the value of this optimization problem, i.e., $(r_1 + r_2)/2$. Solving this numerically (not required), we have for instance

$$r_1 = 100, \quad p_1 = 0, \quad r_2 = 80 \quad p_2 = 120,$$

with an expected number of dinners equal to $(r_1 + r_2)/2 = 90$. This is the number of dinners that we can decide to commit to in the first stage.

- (b) Under this interpretation, in both situations (100 or 200 people show up) we end up violating our budget constraint, either because we are underspending, or overspending. For instance, if only 100 people show up we will spend a big total of $90 \times \$30 + 10 \times \$5 = \$2750$. Similarly, if 200 people show up the costs will be $90 \times \$30 + 110 \times \$5 = \$3250$. Notice that, *on average*, the budget constraint is preserved.
- (c) If we cannot go over the initial budget, then we can write the following formulation that ensures

that the budget constraint always remains feasible:

$$\max r \quad \text{subject to} \quad \begin{cases} p_1 + r = 100 \\ 5p_1 + 30r \leq 3000 \\ 5p_1 + 35 \cdot (r - 150) + 30 \cdot 150 \leq 3000 \\ p_2 + r = 200 \\ 5p_2 + 30r \leq 3000 \\ 5p_2 + 35 \cdot (r - 150) + 30 \cdot 150 \leq 3000 \end{cases}$$

This reformulation is essentially equivalent to adding the constraint $r_1 = r_2$ (or in this case, to choose $r = \min(r_1, r_2)$). The optimal allocation in this case is

$$r_1 = 80, \quad p_1 = 20, \quad r_2 = 80 \quad p_2 = 120.$$

In other words, we order dinner for 80 people, and accomodate the rest with pizzas. In other words, we allocate our resources preparing for the worst case (if everybody shows up!).

P3. [20 pts] Consider the following simplex tableau, corresponding to an LP problem in standard form:

3	1	0	γ	?
2	0	0	-1	1
α	2	1	β	0

- What should be the value of the missing entry? Why?
- What is the current solution (x_1, x_2, x_3, x_4) ? What is the current cost?
- Give specific values of α, β, γ (if they exist) for which the dual LP is infeasible.
- Find necessary and sufficient conditions on α, β, γ for this tableau to be optimal and the problem to have multiple optimal solutions.
- Assume that the basis associated with this tableau is optimal. Suppose also that \mathbf{c}_1 (the first coefficient of the cost) in the original problem is replaced by $\mathbf{c}_1 + \epsilon$. Give upper and lower bounds on ϵ so that this basis remains optimal.

Solution: (a) The value of the missing entry should be zero, since it corresponds to the reduced cost of a basic variable. This can be easily seen from the fact that the second and fourth columns of $\mathbf{B}^{-1}\mathbf{A}$ are the columns of an identity matrix (and thus, x_2 and x_4 are basic variables).

- The basic variables are x_2 and x_4 , and the current solution is $\mathbf{x} = (0, \alpha, 0, 2)$. The current cost achieved by this solution is equal to -3 .
- For the dual LP to be infeasible, the primal LP must be unbounded (if the primal is feasible). For this, it is certainly sufficient that $\alpha > 0$ (primal feasibility of the current basis), $\gamma < 0$ (reduced cost of x_3 is negative, so can bring it into the basis), and $\beta < 0$ (can move arbitrarily far, and reduce the cost).
- If the tableau contains a feasible primal solution, then $\alpha \geq 0$. Optimality (nonnegativity of the reduced costs) implies in turn that $\gamma \geq 0$. To determine the possibility of optimal solutions, consider the following: the current BFS is equal to $(0, \alpha, 0, 2)$. Multiple solutions will occur if we can bring x_3 into the basis (i.e., $\gamma = 0$), and keep feasibility. This would produce a solution $(0, \alpha, 0, 2) + \theta \cdot (0, -\beta, 1, 1) = (0, \alpha - \theta\beta, \theta, 2 + \theta)$. We need this to remain feasible, for some nontrivial positive range of θ . Thus, if $\alpha = 0$ then we must have $\beta < 0$, and if $\alpha > 0$, then β can take any value.
- If the cost \mathbf{c} is perturbed, the feasibility of the current BFS is not affected. Only the optimality condition (i.e., the nonnegativity of the reduced costs) can change. Since the reduced costs are given by $c_j - \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{A}_j$, and x_1 is a non-basic variable, it is easy to see that the valid range of ϵ is $[-1, +\infty]$.

P4. [30 pts] A computer maker (“Orange, Inc.”) produces 4000 desktops, 3000 laptops, and 9000 MP3 players per month. All models can be sold in either “standard” or “customized” configurations. The total number of items that can be customized during a normal month is equal to 2500. In addition, up to 1500 extra items can be customized on overtime, at a higher cost (increasing this would require negotiating with the workers’ union). The *net* profits are as follows:

	Standard	Customized on Regular time	Customized on Overtime
Desktops	100	300	200
Laptops	120	400	230
MP3 Players	20	40	30

The objective is to find a production schedule that maximizes the total net profit.

- Formulate this as a linear programming problem (not necessarily in standard form).
- Write the corresponding LP dual.
- For your chosen formulation, explain the sign constraints in the dual variables, based on the sensitivity interpretation. For instance, what can you say about the shadow price of the overtime workers’ union constraint?
- After implementing a version of this optimization problem in an LP solver, we obtained the following output file:

```
Status:      OPTIMAL
Objective:   profit = 1795000 (MAXimum)
```

No.	Variable	Value	Marginal
1	DS	3000	
2	LS	0	-10
3	MS	9000	
4	DCR	0	-70
5	LCR	2500	
6	MCR	0	-250
7	DCO	1000	
8	LCO	500	
9	MCO	0	-90

No.	Constraint	Value	Marginal
1	desktops	4000	100
2	laptops	3000	130
3	mp3s	9000	20
4	custom	2500	270
5	overtime	1500	100

Here the variables DS, LS, etc. indicate the amount produced of each product (e.g., DS = “desktop standard”), and similarly for the constraints.

Interpret the solution obtained above by the solver, in terms of your LP formulation. In particular, what is the optimal production schedule? What is the optimal dual solution? Which constraints are tight? Verify whether complementarity slackness holds.

- (e) If you could increase the production of any item by a certain small fixed amount, which product (desktops, laptops, MP3 players) would you choose? Why? How should we expect the net profits to change, by increasing the production of that item by 100 units?
- (f) How should we expect the net profits to change, if we are forced to produce 10 customized MP3 players (e.g., for a gift)? Should we produce them during regular hours, or during overtime? Explain exactly how the optimal production plan should change, and why the profit change is what you computed.
- (g) Assume that, as a gesture of good will, the worker's union offers us to customize additional items. The option is to either customize 50 items during regular hours, or 150 items during overtime. Which option should we choose? Why?

Solution: (a) We have (among other possibilities) the following formulation:

$$\begin{aligned} \text{maximize profit: } & 100 \text{ DS} + 120 \text{ LS} + 20 \text{ MS} + 300 \text{ DCR} + 400 \text{ LCR} + \\ & + 40 \text{ MCR} + 200 \text{ DCO} + 230 \text{ LCO} + 30 \text{ MCO} \end{aligned}$$

Subject To

$$\begin{aligned} \text{desktops:} & \text{DS} + \text{DCR} + \text{DCO} = 4000 \\ \text{laptops:} & \text{LS} + \text{LCR} + \text{LCO} = 3000 \\ \text{mp3s:} & \text{MS} + \text{MCR} + \text{MCO} = 9000 \\ \text{custom:} & \text{DCR} + \text{LCR} + \text{MCR} \leq 2500 \\ \text{overtime:} & \text{DCO} + \text{LCO} + \text{MCO} \leq 1500 \end{aligned}$$

with all decision variables being nonnegative.

- (b) The corresponding dual formulation is:

$$\text{minimize } 4000 \text{ d} + 3000 \text{ l} + 9000 \text{ m} + 2500 \text{ c} + 1500 \text{ ov}$$

Subject To

$$\begin{aligned} \text{DS:} & \text{d} \geq 100 \\ \text{LS:} & \text{l} \geq 120 \\ \text{MS:} & \text{m} \geq 20 \\ \text{DCR:} & \text{d} + \text{c} \geq 300 \\ \text{LCR:} & \text{l} + \text{c} \geq 400 \\ \text{MCR:} & \text{m} + \text{c} \geq 40 \\ \text{DCO:} & \text{d} + \text{ov} \geq 200 \\ \text{LCO:} & \text{l} + \text{ov} \geq 230 \\ \text{MCO:} & \text{m} + \text{ov} \geq 30 \\ & \text{c} \geq 0, \text{ ov} \geq 0 \end{aligned}$$

- (c) In the dual formulation, the shadow prices associated with the c and ov dual variable (corresponding to the `custom` and `overtime` constraints, respectively) are nonnegative, since a positive perturbation on these can only increase our profit. For instance, if we slightly increase the overtime hours from its nominal value of 1500, the feasible set will get bigger, and thus the profit should increase (provided the constraint is active!).
- (d) The optimal production schedule is:
- Desktops: 3000 standard and 1000 customized in overtime.
 - Laptops: 2500 customized in regular hours, 500 customized in overtime.

- MP3s: 9000 standard.

The corresponding prices of constraints (all are active),

$$d = 100, \quad l = 130, \quad m = 20, \quad c = 270, \quad ov=100.$$

It can be easily verified that complementary slackness holds. For instance, the constraint corresponding to LS is not tight (since $130 > 120$), and thus the corresponding primal variable vanishes.

- (e) It would make sense to increase the production of laptops, since they have the largest reduced cost (equal to 130). The increase in profit (assuming the current basis remains optimal) would be equal to $130 \times 100 = 13000$.
- (f) If we are forced to produce 10 customized MP3 players, then we should make them in overtime hours (since their unit shadow price is only -90 , compared to -250 if we produce them during regular hours). The expected profit change is equal to $-90 \times 10 = -900$. To preserve the current basis, the production must change incrementally in the following way:

$$\begin{array}{lll} \Delta DS = +10 & \Delta DCR = 0 & \Delta DCO = -10 \\ \Delta LS = 0 & \Delta LCR = 0 & \Delta LCO = 0 \\ \Delta MS = -10 & \Delta MCR = 0 & \Delta MCO = +10 \end{array}$$

This is easily obtained by keeping the non-basic variables at zero level (i.e., $\Delta LS = \Delta DCR = \Delta MCR = 0$), and solving for the remaining ones using the active constraints. The new solution clearly remains feasible. We can independently verify from here the change in profit:

$$\begin{aligned} \Delta \text{profit} &= 100 \times (\Delta DS) + 20 \times (\Delta MS) + 200 \times (\Delta DCO) + 30 \times (\Delta MCO) \\ &= 100 \times (+10) + 20 \times (-10) + 200 \times (-10) + 30 \times (+10) \\ &= -900. \end{aligned}$$

- (g) We should follow up on this generous offer by choosing the second option. The profit increase in each one of the alternatives is given by the product of the number of additional hours times the shadow price of the respective constraint, i.e.:

$$\begin{aligned} \Delta \text{profit}_1 &= (50) \times 270 = 13500 \\ \Delta \text{profit}_2 &= (150) \times 100 = 15000 \end{aligned}$$

Since the profit increase is greater in the second case, we should choose to customize 150 additional items during overtime.

MIT OpenCourseWare
<http://ocw.mit.edu>

15.093J / 6.255J Optimization Methods
Fall 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.