Optimization Methods
MIT 2.098/6.255/15.093
Final exam

Date Given: December 19th, 2006

P1. [***30 pts***] Classify the following statements as true or false. All answers must be well-justified, either through a short explanation, or a counterexample. Unless stated otherwise, all LP problems are in standard form.

(a) If there is a unique primal optimal solution to a linear programming problem, then the reduced costs of all the nonbasic variables are strictly positive.

(b) For a network flow problem with capacity constraints, there always exists an optimal solution that is tree-structured.

(c) When minimizing a convex function over a convex set, the optimal solution is always on the boundary of the set.

(d) For a convex optimization problem with constraints, if a feasible point satisfies the KKT conditions then it is a global optimum.

(e) For a nonlinear optimization problem, if Newton's method converges, then it converges to a local minimum.

(f) For an LP in standard form, if $\mathbf{c} \geq 0$ then the primal is either bounded or infeasible.

(g) On very degenerate LP problems, the simplex method performs better than interior point methods.

(h) The primal iterates $\mathbf{x}_k$ generated by the affine scaling algorithm are always in the interior of the primal feasible set.

(i) The feasible set of a semidefinite programming problem is always convex.

(j) For a quadratic function $f(x) = x^T Ax + b^T x + c$, the convergence rate of Newton's method depends on the condition number of the matrix A.

Solution: **(a)** FALSE. This is true only if the unique optimal solution is nondegenerate.

(b) FALSE. In the capacitated case, optimal solutions do not necessarily have to be trees, a simple counterexample is a network with nodes $\{A, B, C, D\}$ and edges $A - B, A - C, B - D, C - D$ of unit capacity.

(c) FALSE. Solutions can be in the interior. As an example, consider minimizing x^2 on $[-1, 1]$.

(d) TRUE. This is proven in the book and in the lecture notes.

(e) FALSE. Newton's method can converge to global maxima.

(f) TRUE. If c is nonnegative, then if the problem is feasible we have $c^T x \geq 0$, and thus is it bounded.

(g) FALSE. For degenerate problems, interior point methods are a much better choice.

(h) TRUE. By construction, the primal iterates in the affine scaling method (with $\beta < 1$) are always strictly feasible.

(i) TRUE. The feasible set of an SDP problem is convex, since it is the intersection of two convex sets (an affine subspace and the cone of PSD matrices).

(j) FALSE. For quadratic functions, Newton's method converges exactly in one iteration.

P2. [*25 pts*] You are planning on having access to a car for the next N years, where N is a fixed number. The price of a new car is P dollars. For reliability reasons, you will only own relatively new cars, at most m years old. The yearly cost of repairing and maintaining a car during its kth year is r_k, and it satisfies $r_1 < r_2 < \cdots < r_m$ (i.e., it increases over time). At the end of any given year, you have the option of exchanging your k-year old car for a new one, with the corresponding trade-in value t_k of your old car satisfying $t_1 > t_2 > \cdots > t_m$ (i.e., depreciating over time).

You want to find the most economical sequence of buys and trade-ins, i.e., want to minimize the total cost over the N-year period. This includes all the money spent, either in buying or repairing (notice that you can sell your car at the end of the N years).

(a) Propose a shortest path (or network flow) formulation for this problem.

(b) Propose a dynamic programming formulation for this problem. Express clearly what are the state and decision variables, and the corresponding iteration.

(c) Use your DP formulation to solve the problem for the following data: $N = 5$, $m = 3$, $P = 20000$, the repairing costs

$$r_1 = 1200, \quad r_2 = 1600, \quad r_3 = 2400,$$

and the trade-in values

$$t_1 = 16000, \quad t_2 = 12000, \quad t_3 = 10000.$$

What is the optimal sequence of actions? Is it unique?

Solution: (a) Define a network whose nodes are on a $N \times m$ grid with two additional nodes: a source node s and a sink node t. A node on the gride is indexed by (time index) $t = 1, \ldots, N$ and (car age) $a = 1, \ldots, m$.

From node (t, a), there is a directed edge

- to node $(t+1, a+1)$ with cost r_a if $t < N$ and $a < m$ (car is maintained for one year)
- to node $(t+1, 1)$ with cost $P - t_k + r_1$ if $t < N$ (car is traded in)
- to a sink node t if $t = N$ with cost $-t_a$.

Node s is connected to the grid node $(1, 1)$ with cost $P + r_1$.

Node s a supply of 1 and node t a demand of 1.

A min-cost flow solution will correspond to an optimal purchase/maintenance plan for the car over the time horizon N

(b) Let the time index t run from 1 to N. Define the state a at time t as the age of the car at the end of the current period. Let $V_t(a)$ be the expected cost given that the car is a-year old at the end of the time period t.

At time $t < N$, if $a = m$, the car has to be exchanged and maintained for one year with cost $P - t_m + r_1$, whereas there are two available options in state $a \in \{1, \ldots, m-1\}$:

- maintain the car, with a cost of r_a, leading to $(a+1)$-year old car at the next period
- trade-in the car for a new one and maintain it for a year, with a cost $P - t_a + r_1$, leading to a one-year old car in the next period.

At time $t = N$, the car is sold for its value t_a.

Bellman's equations for this problem are

$$\begin{aligned} V_N(a) &= -t_a, && a = 1, \ldots, m, \\ V_t(m) &= P - t_m + r_1 + V_{t+1}(1), && t < N \\ V_t(a) &= \min\left(r_a + V_{t+1}(a+1), P - t_a + r_1 + V_{t+1}(1)\right), && a = 1, \ldots, m-1,\ t < N. \end{aligned}$$

(c) Solving Bellman's recursion yields the optimal cost of $\$26,500$.

P3. [***20 pts***] Consider the following transshipment problem:

The supply nodes are A, B, the demand nodes are D, E, and the transshipment node is node C. There are four unknowns a, b, c and d. The supply/demand amounts in the different nodes are:

$$A:\ a, \qquad B:\ 400, \qquad D:\ -b, \qquad E:\ -200,$$

where as usual a positive amount indicates supply and a negative amount indicates demand. We are interested in finding an optimal (minimum cost) transshipment plan.

(a) State conditions on a, b, c, d such that the above problem is feasible.

(b) Consider the spanning tree given by the edges $\{(A, D), (B, C), (C, D), (C, E)\}$. State conditions on a, b, c, d for which the spanning tree solution will be feasible.

(c) State conditions on a, b, c, d for which the spanning tree solution will be optimal.

(d) State conditions on a, b, c, d for which there will be multiple solutions, including the spanning tree solution indicated above.

Solution: (a) For the problem to be solvable, supply and demand must balance, giving the necessary condition $a + 400 = 200 + b$, or equivalently, $b - a = 200$.

(b) If the solution has the structure of the indicated spanning tree, then the following relations must hold:

$$a + 200 = b.$$

Here we can calculate all flows on this spanning tree and the condition again is the flow balance condition.

(c) The optimality condition for the spanning tree solution is that all reduced costs of non-basic arcs are non-negative. In order to calculate the reduced costs, we need to calculate node potentials. Without loss of generality, set $p_C = 0$. Knowing the fact that $\bar{c}_{ij} = c_{ij} - (p_i - p_j) = 0$ for all basic arcs (i, j), we obtain $p_B = 5$, $p_D = -3$, $p_A = -1$, and $p_E = -2$. The reduced costs for non-basic arcs then can be calculated as follows:

$$\bar{c}_{BA} = c - 6, \bar{c}_{CA} = d - 1, \bar{c}_{BE} = 2, \bar{c}_{ED} = 10$$

Thus in addition to the feasibility condition $b = a + 200$, we obtain the optimality condition for the indicated spanning tree solution:

$$\begin{cases} c - 6 \geq 0 \\ d - 1 \geq 0 \end{cases}.$$

(d) We need to have reduced costs of some non-basic arcs to be zero in order to have multiple optimal solutions, including this spanning tree solution. It means either $c = 6$ or $d = 1$ (in addition to the feasibility and optimality conditions mentioned previously in (c)). We can see that there is a cost-equivalent path from B to D via A as compared to the path $B \to C \to D$ in both cases. Thus, other optimal solutions can be constructed by rerouting flows from B to D via A if either $c = 6$ or $d = 1$. The final conditions for multiple optimal solutions are: $a + 200 = b$, and $\begin{cases} c - 6 = 0 \\ d - 1 \geq 0 \end{cases}$ or $\begin{cases} c - 6 \geq 0 \\ d - 1 = 0 \end{cases}$.

P4. [***25 pts***] Consider a set of n points $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ in the plane. We want to find a point (x, y) such that the sum of the Euclidean distances from this point to all the other points is minimized.

(a) Give an nonlinear optimization formulation of this problem.

(b) Is the objective function differentiable? Is this a convex optimization problem?

(c) Write the corresponding optimality conditions. Give a geometric interpretation of this condition.

(d) Are the optimality conditions necessary? Sufficient? State clearly your assumptions.

(e) Provide a semidefinite programming formulation of this problem.

All answers and explanations must be *fully* justified.

Solution: (a) A simple formulation as an unconstrained nonlinear optimization problem is the following:

$$\min_{x,y} \sum_{i=1}^{n} \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

(b) The objective function is differentiable everywhere, except at the points where (x, y) is equal to one of the (x_i, y_i). The objective function is convex, since it is a sum of n convex functions.

(c) The optimality conditions are obtained by setting the gradient equal to zero (we assume that the minimum occurs at a differentiable point).

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{n} \frac{x - x_i}{||(x, y) - (x_i, y_i)||} = 0$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{n} \frac{y - y_i}{||(x, y) - (x_i, y_i)||} = 0$$

This condition can be interpreted as requiring the sum of the normalized vectors from the point (x, y) to the (x_i, y_i) to be equal to zero.

For instance, in the case $n = 2$, then any point in the line segment between the two given points will be optimal. Similarly, for $n = 3$, the optimality condition implies that the angles between the vectors from (x, y) to the other points are all equal.

(d) The optimality conditions are sufficient, because the problem is convex. They are necessary if the minimum occurs at a differentiable point.

(e) A semidefinite formulation of the problem can be easily obtained if we introduce slack variables d_i and formulate the problem as:

$$\min \sum_{i=1}^{n} d_i \qquad \text{s.t.} \qquad (x - x_i)^2 + (y - y_i)^2 \leq d_i^2 \quad i = 1, \ldots, n.$$

The constraints are equivalent to:

$$d_i^2 - \begin{bmatrix} x - x_i \\ y - y_i \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_i \\ y - y_i \end{bmatrix} \geq 0$$

Dividing by d_i and using Schur complements, we can rewrite this as:

$$\min \sum_{i=1}^{n} d_i \qquad \text{s.t.} \qquad \begin{bmatrix} d_i & x - x_i & y - y_i \\ x - x_i & d_i & 0 \\ y - y_i & 0 & d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, n.$$

which is an SDP formulation.

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