Mixed-Integer Programming II

Mixed Integer Inequalities

- Consider $S = \left\{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b\right\}$.
- Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$.
- Let $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$.
- Then $\sum_{f_j \leq f_0} f_j x_j + \sum_{f_j > f_0} (f_j - 1) x_j + \sum_{j=1}^{p} g_j y_j = k + f_0$, where k is some integer.
- Since $k \leq -1$ or $k \geq 0$, any $x \in S$ satisfies

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{j=1}^{p} \frac{g_j}{f_0} y_j \geq 1 \tag{1}$$

OR

$$-\sum_{f_j \leq f_0} \frac{f_j}{1 - f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{j=1}^{p} \frac{g_j}{1 - f_0} y_j \geq 1. \tag{2}$$

- This is of the form $\sum_j a_j^1 x_j \geq 1$ or $\sum_j a_j^2 x_j \geq 1$, which implies $\sum_j \max\{a_j^1, a_j^2\} x_j \geq 1$ for any $x \geq 0$.
- For each variable, what is the max coefficient in (1) and (2)?
- We get

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j \geq 1.$$

- This is the *Gomory mixed integer (GMI) inequality.*
- In the pure integer programming case, the GMI inequality reduces to

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j \geq 1.$$

- Since $\frac{1 - f_j}{1 - f_0} < \frac{f_j}{f_0}$ when $f_j > f_0$, the GMI inequality dominates

$$\sum_{j=1}^{n} f_j x_j \geq f_0,$$

which is known as the *fractional cut.*

- Consider now $S = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Gy \leq b\}$.
- Let $P = \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^p : Ax + Gy \leq b\}$ be the underlying polyhedron.
- Let $\alpha x + \gamma y \leq \beta$ be any valid for P.
- Add a nonnegative slack variable s, use $\alpha x + \gamma y + s = \beta$ to derive a GMI inequality, and eliminate $s = \beta - \alpha x - \gamma y$ from it.
- The result is a valid inequality for S.
- These inequalities are called the GMI inequalities for S.
- We illustrate this on a small example:

$$\begin{array}{lrrcl} \max & x & +2y & & \\ \text{s.t.} & -x & +y & \leq & 2 \\ & x & +y & \leq & 5 \\ & 2x & -y & \leq & 4 \\ & x \in \mathbb{Z}_+ & y \in \mathbb{R}_+ & & \end{array}$$

- Adding slack variables $s_1, s_2, s_3 \geq 0$ leads to the system

$$\begin{array}{crrrrrcl} z & -x & -2y & & & & = & 0 \\ & -x & +y & +s_1 & & & = & 2 \\ & x & +y & & +s_2 & & = & 5 \\ & 2x & -y & & & +s_3 & = & 4 \end{array}$$

- The optimal tableau is

$$\begin{array}{cccrrrcl} z & & & +0.5s_1 & +1.5s_2 & & = & 8.5 \\ & & y & +0.5s_1 & +0.5s_2 & & = & 3.5 \\ & x & & -0.5s_1 & +0.5s_2 & & = & 1.5 \\ & & & 0.5s_1 & -0.5s_2 & +s_3 & = & 4.5 \end{array}$$

  and the corresponding solutions is $\bar{x} = 1.5$ and $\bar{y} = 3.5$.

- Since $\bar{x}$ is not integer, we generate a cut from that row:

$$x - 0.5s_1 + 0.5s_2 = 1.5$$

- Here $f_0 = 0.5$ and we get $s_1 + s_2 \geq 1$.
- Since $s_1 + s_2 = 7 - 2y$, this corresponds to $y \leq 3$ in the (x, y)-space.
- In contrast to lift-and-project cuts, it is in general NP-hard to find a GMI inequality that cuts off a point $(\bar{x}, \bar{y}) \in P \setminus S$, or show that none exists.
- However, one can easily find a GMI inequality that cuts off a basic feasible solution.

- On 41 MIPLIB instances, adding the GMI cuts generated from the optimal simplex tableau reduces the integrality gap by 24% on average [Bonami et al. 2008]
- GMI cuts are widely used in commercial codes today.
- Numerical issues need to be addressed, however.

Split cuts

- Let $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \leq b\}$, and let $S = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$.
- For $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, define

$$\begin{aligned} \Pi_1 &= P \cap \{(x, y) : \pi x \leq \pi_0\} \\ \Pi_2 &= P \cap \{(x, y) : \pi x \geq \pi_0 + 1\} \end{aligned}$$

- Clearly, $S \subseteq \Pi_1 \cup \Pi_2$.
- Therefore, $\text{conv}(S) \subseteq \text{conv}(\Pi_1 \cup \Pi_2)$.
- We call the latter set $P^{(\pi,\pi_0)}$. It is a polyhedron.
- An inequality $cx + hy \leq c_0$ is a *split inequality* if it is valid for some $P^{(\pi,\pi_0)}$.
- A *split* is a disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$ where $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$.
- A split defined by (π, π_0) is a *one-side split* for P if

$$\pi_0 \leq z < \pi_0 + 1, \tag{3}$$

  where $z = \max\{\pi x : (x, y) \in P\}$.
- This is equivalent to $\Pi_1 \subseteq P$ and $\Pi_2 = \emptyset$.
- The inequality $\pi x \leq \pi_0$ is valid for S; in fact, it is a Gomory-Chvátal inequality.
- In particular, $\pi x \leq \pi_0$ satisfies (4) iff $\pi_0 = \lfloor z \rfloor$.

Split cuts and Gomory-Chvátal cuts

- Let P^1 be the split closure of P, and, for $k \geq 2$, let P^k denote the split closure relative to P^{k-1}.
- P^1 is a polyhedron (and so is P^k).
- In contrast to the pure integer case and to the mixed 0/1 case, there is in general no finite r such that $P^r = \text{conv}(S)$.

Split cuts and other cuts

- Lift-and-project inequalities are split inequalities (where the disjunction is $x_j \le 0$ or $x_j \ge 1$).
- Gomory's mixed-integer inequalities are split inequalities (where the disjunction is (1) or (2)).
  - We argued that $k = \lfloor b \rfloor - \sum_{f_j \le f_0} \lfloor a_j \rfloor x_j - \sum_{f_j > f_0} \lceil a_j \rceil x_j$ is an integer, and either $k \le -1$ or $k \ge 0$.

Split cuts and GMI cuts

Lemma 1. *Let $P = \{x : Ax \le b\}$ and let $\Pi = P \cap \{x : \pi x \le \pi_0\}$. If $\Pi \ne \emptyset$ and $\alpha x \le \beta$ is valid for Π, then there exists $\lambda \ge 0$ such that*

$$\alpha x - \lambda(\pi x - \pi_0) \le \beta$$

is valid for P.

Proof:

- By LP duality, there exist $u \ge 0$ and $\lambda \ge 0$ such that

$$\alpha = uA + \lambda\pi \quad \text{and} \quad \beta \ge ub + \lambda\pi_0.$$

- Since $uAx \le ub$ is valid for P, so is $uAx \le \beta - \lambda\pi_0$.
- As $uAx = \alpha x - \lambda\pi x$, the claim follows. □

Theorem 2. *Let $P = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \le b\}$ be a rational polyhedron and let $S = P \cap (\mathbb{Z}^n_+ \times \mathbb{R}^p_+)$. The split closure of P is identical to the Gomory mixed integer closure of P.*

Proof:

- Let $cx + hy \le c_0$ be a split inequality. Let (π, π_0) be the corresponding split.
- We may assume that $\Pi_1 \ne \emptyset$ and $\Pi_2 \ne \emptyset$.
- By the previous lemma, there exist $\alpha, \beta \ge 0$ such that

$$\begin{aligned} cx + hy - \alpha(\pi x - \pi_0) &\le c_0 \text{ and} \qquad (4)\\ cx + hy + \beta(\pi x - (\pi_0 + 1)) &\le c_0 \qquad (5)\end{aligned}$$

  are both valid for P.

- We can assume that $\alpha > 0$ and $\beta > 0$; otherwise $cx + hy \le c_0$ is already valid for P.
- We now apply the Gomory procedure to (4) and (5).
- Introduce slack variables s_1 and s_2 and subtract (4) from (5):

$$(\alpha + \beta)\pi x + s_2 - s_1 = (\alpha + \beta)\pi_0 + \beta$$

- Dividing by $\alpha + \beta$ yields

$$\pi x + \frac{s_2}{\alpha + \beta} - \frac{s_1}{\alpha + \beta} = \pi_0 + \frac{\beta}{\alpha + \beta}.$$

- Note that $f_0 = \frac{\beta}{\alpha+\beta}$ and s_2 has a positive coefficient, while s_1 has a negative coefficient. We get

$$\frac{\frac{1}{\alpha+\beta}}{\frac{\beta}{\alpha+\beta}} s_2 + \frac{\frac{1}{\alpha+\beta}}{1 - \frac{\beta}{\alpha+\beta}} s_1 \geq 1.$$

- This simplifies to

$$\frac{1}{\alpha} s_1 + \frac{1}{\beta} s_2 \geq 1.$$

- Now replace s_1 and s_2 as defined in (4) and (5) to get the GMI inequality in the (x, y)-space. The resulting inequality is

$$cx + hy \leq c_0.$$

□

Additional Literature

- W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A. Schrijver: *Combinatorial Optimization*
- M. Grötschel, L. Lovász, A. Schrijver: *Geometric Algorithms and Combinatorial Optimization*
- B. Korte, J. Vygen: *Combinatorial Optimization – Theory and Algorithms*
- E. Lawler: *Combinatorial Optimization: Networks and Matroids*
- E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys: *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*
- J. Lee: *A First Course in Combinatorial Optimization*
- G. Nemhauser, L.A. Wolsey: *Integer and Combinatorial Optimization*
- C.H. Papadimitriou, K. Steiglitz: *Combinatorial Optimization – Algorithms and Complexity*
- A. Schrijver: *Combinatorial Optimization – Polyhedra and Efficiency*
- A. Schrijver: *Theory of Linear and Integer Programming*
- ...

Final Exam

- Tuesday, December 15, 1:30-4:30PM, E51-376

- You can bring/use the textbook, the lecture notes, the homeworks, and homework solutions.

15.083J / 6.859J Integer Programming and Combinatorial Optimization
Fall 2009