15.083J/6.859J Integer Optimization

Lecture 8: Duality I

1 Outline

Slide 1

- Duality from lift and project
- Lagrangean duality

2 Duality from lift and project

Slide 2

- $$\begin{aligned} Z_{\mathrm{IP}} = \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{Ax} = \boldsymbol{b} \\ & x_i \in \{0,1\}. \end{aligned}$$
- $\{\boldsymbol{x} \in \Re^n \mid \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is bounded for all $\boldsymbol{b}$.
- Without of loss of generality $x_i + x_{i+n} = 1$ are included in $\boldsymbol{Ax} = \boldsymbol{b}$.

2.1 LP1

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$$\begin{aligned} Z_{\mathrm{LP1}} = \max \quad & \sum_{S \subseteq N} \left(\sum_{j \in S} c_j \right) w_S \\ \text{s.t.} \quad & \left(\sum_{j \in S} \boldsymbol{A}_j - \boldsymbol{b} \right) w_S = \boldsymbol{0} \quad \forall S \subseteq N, \\ & \sum_{S \subseteq N} w_S = 1, \\ & w_S \geq 0. \end{aligned}$$

Theorem: $Z_{\mathrm{IP}} = Z_{\mathrm{LP1}}$.

2.2 LP2

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$y_S = \sum_{T: S \subseteq T} w_T$.

$$\begin{aligned} Z_{\mathrm{LP2}} = \max \quad & \sum_{j \in N} c_j y_{\{j\}} \\ \text{s.t.} \quad & \left(\sum_{j \in S} \boldsymbol{A}_j - \boldsymbol{b} \right) y_S + \sum_{j \notin S} \boldsymbol{A}_j \, y_{S \cup \{j\}} = \boldsymbol{0},\ \forall S \subseteq N, \\ & \left(\sum_{j \in N} \boldsymbol{A}_j - \boldsymbol{b} \right) y_N = \boldsymbol{0}, \\ & y_S \geq 0, y_\emptyset = 1. \end{aligned}$$

Theorem: $Z_{\mathrm{LP1}} = Z_{\mathrm{LP2}}$.

2.3 Lift-Project

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- Inequality form: $\sum_{j\in N} \boldsymbol{A}_j x_j \le \boldsymbol{b}$
- Multiply constraints with $\prod_{i\in S} x_i$ for all $S \subseteq N$ to obtain using $x_i^2 = x_i$:

$$\sum_{j\in S} \boldsymbol{A}_j \prod_{i\in S} x_i + \sum_{j\notin S} \boldsymbol{A}_j \prod_{i\in S\cup\{j\}} x_i \le \boldsymbol{b} \prod_{i\in S} x_i.$$

- Define $y_S = \prod_{i\in S} x_i$, noting that $y_S \ge \mathbf{0}$ and setting $y_\emptyset = 1$

$$\left(\sum_{j\in S} \boldsymbol{A}_j - \boldsymbol{b}\right) y_S + \sum_{j\notin S} \boldsymbol{A}_j \, y_{S\cup\{j\}} \le \mathbf{0}.$$

2.4 The dual problem

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$$\begin{aligned} \min \quad & \boldsymbol{u}_\emptyset' \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{u}_{\{j\}}' (\boldsymbol{A}_j - \boldsymbol{b}) + \boldsymbol{u}_\emptyset' \boldsymbol{A}_j \ge c_j \ \forall\, j \in N, \\ & \boldsymbol{u}_S' \left(\sum_{j\in S} \boldsymbol{A}_j - \boldsymbol{b}\right) + \sum_{j\in S} \boldsymbol{u}_{S\setminus\{j\}}' \boldsymbol{A}_j \ge 0 \ \forall\, S \subseteq N, \ |S| \ge 2. \end{aligned}$$

2.5 Strong Duality

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Suppose that the only feasible solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, $\boldsymbol{x} \ge \mathbf{0}$ is the vector $\mathbf{0}$.

- **(Weak duality)** If $\boldsymbol{x}$ is a feasible solution to the primal problem and $\boldsymbol{u}$ is a feasible solution to the dual problem, then

$$\boldsymbol{c}'\boldsymbol{x} \le \boldsymbol{u}_\emptyset' \boldsymbol{b}.$$

- **(Strong duality)** If the primal problem has an optimal solution, so does its dual problem, and the respective optimal costs are equal.

2.6 Complementary slackness

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$\boldsymbol{x}$ and $\boldsymbol{u}$ feasible solutions for primal and dual. Then, $\boldsymbol{x}$ and $\boldsymbol{u}$ are optimal solutions if and only if

$$\begin{aligned} \left(\boldsymbol{u}_{\{j\}}' (\boldsymbol{A}_j - \boldsymbol{b}) + \boldsymbol{u}_\emptyset' \boldsymbol{A}_j - c_j\right) x_j &= 0 \ \forall\, j \in N, \\ \left(\boldsymbol{u}_S' \left(\sum_{j\in S} \boldsymbol{A}_j - \boldsymbol{b}\right) + \sum_{j\in S} \boldsymbol{u}_{S\setminus\{j\}}' \boldsymbol{A}_j\right) \prod_{j\in S} x_j &= 0 \ \forall\, S \subseteq N, \ |S| \ge 2. \end{aligned}$$

2.7 Example

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$$\begin{aligned}\text{maximize} \quad & x_1 + 2x_2 + 3x_3 + 5x_4 \\ \text{subject to} \quad & 3x_1 + 5x_2 + 7x_3 + 9x_4 = 12, \\ & x_i \in \{0.1\}, \qquad i = 1, 2, 3, 4.\end{aligned}$$

Dual

$$\begin{aligned}\text{minimize} \quad & 12u_\emptyset & \\ \text{subject to} \quad & -9u_1 + 3u_\emptyset & \geq 1 \\ & -7u_2 + 5u_\emptyset & \geq 2 \\ & -5u_3 + 7u_\emptyset & \geq 3 \\ & -3u_4 + 9u_\emptyset & \geq 5 \\ & -4u_{1,2} + 5u_1 + 3u_2 & \geq 0 \\ & -2u_{1,3} + 7u_1 + 3u_3 & \geq 0 \\ & 0u_{1,4} + 9u_1 + 3u_4 & \geq 0 \\ & 0u_{2,3} + 7u_2 + 5u_3 & \geq 0 \\ & 2u_{2,4} + 9u_2 + 5u_4 & \geq 0 \\ & 4u_{3,4} + 9u_3 + 7u_4 & \geq 0 \\ & 3u_{1,2,3} + 7u_{1,2} + 5u_{1,3} + 3u_{2,3} & \geq 0 \\ & 5u_{1,2,4} + 9u_{1,2} + 5u_{1,4} + 3u_{2,4} & \geq 0 \\ & 7u_{1,3,4} + 9u_{1,3} + 7u_{1,4} + 3u_{3,4} & \geq 0 \\ & 9u_{2,3,4} + 9u_{2,3} + 7u_{2,4} + 5u_{3,4} & \geq 0 \\ & 12u_{1,2,3,4} + 9u_{1,2,3} + 7u_{1,2,4} + 5u_{1,3,4} + 3u_{2,3,4} & \geq 0.\end{aligned}$$

Optimal solution

$$u_\emptyset = \frac{1}{2},\ u_1 = \frac{1}{18},\ u_4 = -\frac{1}{6},\ u_{2,4} = \frac{5}{12},\ u_{3,4} = \frac{7}{24}$$

Complementary slackness condition: $x_1 = 1$, $x_2 = x_3 = 0$ and $x_4 = 1$, the dual constraints associated with the subsets $S = \{1\},\ \{4\},\ \{1,4\}$

$$\begin{aligned}-9u_1 + 3u_\emptyset & \geq 1 \\ -3u_4 + 9u_\emptyset & \geq 5 \\ 0u_{1,4} + 9u_1 + 3u_4 & \geq 0\end{aligned}$$

are all satisfied with equality.

3 Lagrangean duality

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$$\begin{aligned}Z_{\text{IP}} = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \qquad (*) \\ & \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d} \\ & \boldsymbol{x} \in \mathcal{Z}^n,\end{aligned}$$

$$X = \{\boldsymbol{x} \in \mathcal{Z}^n \mid \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d}\}.$$

Let $\boldsymbol{\lambda} \geq \mathbf{0}$.

$$\begin{aligned} Z(\boldsymbol{\lambda}) = \min \quad & \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{\lambda}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{x} \in X, \end{aligned}$$

3.1 Weak duality

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- If problem (*) has an optimal solution, then $Z(\boldsymbol{\lambda}) \leq Z_{\mathrm{IP}}$ for $\boldsymbol{\lambda} \geq \mathbf{0}$.
- The function $Z(\boldsymbol{\lambda})$ is concave.
- Lagrangean dual

$$\begin{aligned} Z_{\mathrm{D}} = \max \quad & Z(\boldsymbol{\lambda}) \\ \text{s.t.} \quad & \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}$$

- $Z_{\mathrm{D}} \leq Z_{\mathrm{IP}}$.

3.2 Characterization

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$$\begin{aligned} Z_{\mathrm{D}} = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \in \operatorname{conv}(X). \end{aligned}$$

3.3 Proof outline

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- $Z(\boldsymbol{\lambda}) = \min\limits_{\boldsymbol{x} \in X} \big(\boldsymbol{c}'\boldsymbol{x} + \boldsymbol{\lambda}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\big)$.
- $Z(\boldsymbol{\lambda}) = \min_{\boldsymbol{x} \in \operatorname{conv}(X)} \big(\boldsymbol{c}'\boldsymbol{x} + \boldsymbol{\lambda}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\big)$.
- $Z_{\mathrm{D}} = \max\limits_{\boldsymbol{\lambda} \geq \mathbf{0}} \min\limits_{\boldsymbol{x} \in \operatorname{conv}(X)} \big(\boldsymbol{c}'\boldsymbol{x} + \boldsymbol{\lambda}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\big)$.
- Let $\boldsymbol{x}^k,\ k \in K$, and $\boldsymbol{w}^j,\ j \in J$, be the extreme points and a extreme rays of $\operatorname{conv}(X)$

$$Z(\boldsymbol{\lambda}) = \begin{cases} -\infty, & \text{if } (\boldsymbol{c}' - \boldsymbol{\lambda}'\boldsymbol{A})\boldsymbol{w}^j < 0, \\ & \text{for some } j \in J, \\ \min\limits_{k \in K} \big(\boldsymbol{c}'\boldsymbol{x}^k + \boldsymbol{\lambda}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^k)\big), & \text{otherwise.} \end{cases}$$

- $$\begin{aligned} Z_{\mathrm{D}} = \text{maximize} \quad & \min_{k \in K} \big(\boldsymbol{c}'\boldsymbol{x}^k + \boldsymbol{\lambda}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^k)\big) \\ \text{subject to} \quad & (\boldsymbol{c}' - \boldsymbol{\lambda}'\boldsymbol{A})\boldsymbol{w}^j \geq 0, \qquad j \in J, \\ & \boldsymbol{\lambda} \geq \mathbf{0}, \end{aligned}$$
- $$\begin{aligned} \text{maximize} \quad & y \\ \text{subject to} \quad & y + \boldsymbol{\lambda}'(\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}) \leq \boldsymbol{c}'\boldsymbol{x}^k, \qquad k \in K, \\ & \boldsymbol{\lambda}'\boldsymbol{A}\boldsymbol{w}^j \leq \boldsymbol{c}'\boldsymbol{w}^j, \qquad j \in J, \\ & \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}$$

- $$\begin{aligned} \text{Dual} \quad \text{minimize} \quad & \boldsymbol{c}' \left(\sum_{k\in K} \alpha_k \boldsymbol{x}^k + \sum_{j\in J} \beta_j \boldsymbol{w}^j \right) \\ \text{subject to} \quad & \sum_{k\in K} \alpha_k = 1 \\ & \boldsymbol{A} \left(\sum_{k\in K} \alpha_k \boldsymbol{x}^k + \sum_{j\in J} \beta_j \boldsymbol{w}^j \right) \geq \boldsymbol{b} \\ & \alpha_k, \beta_j \geq 0, \qquad k \in K,\ j \in J. \end{aligned}$$
- $\text{conv}(X) = \left\{ \sum_{k\in K} \alpha_k \boldsymbol{x}^k + \sum_{j\in J} \beta_j \boldsymbol{w}^j \;\middle|\; \sum_{k\in K} \alpha_k = 1,\ \alpha_k, \beta_j \geq 0, \quad k \in K,\ j \in J \right\}$.

3.4 Example

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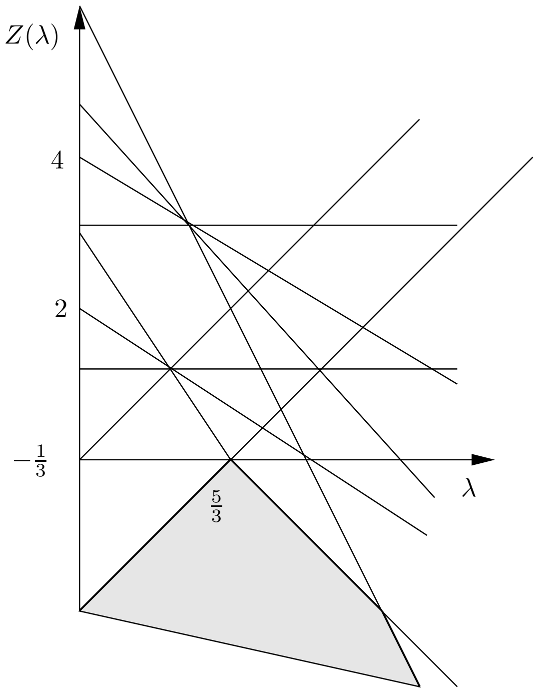
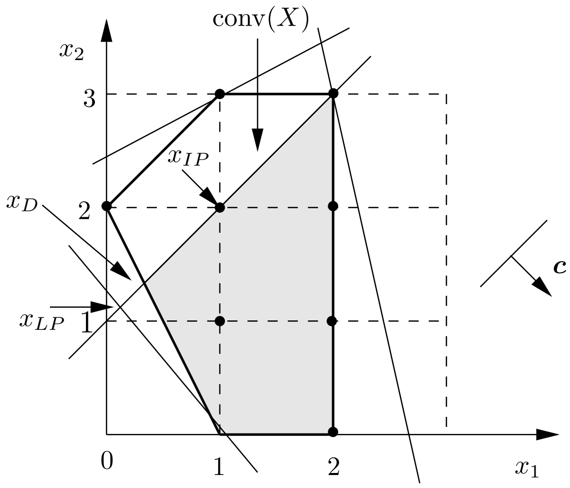
$$\begin{aligned} \text{minimize} \quad & 3x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \geq -1 \\ & -x_1 + 2x_2 \leq 5 \\ & 3x_1 + 2x_2 \geq 3 \\ & 6x_1 + x_2 \leq 15 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathcal{Z}. \end{aligned}$$

- Relax $x_1 - x_2 \geq -1$
- $X = \{(1,0), (2,0), (1,1), (2,1), (0,2), (1,2), (2,2), (1,3), (2,3)\}$.
- $Z(\lambda) = \min_{(x_1,x_2)\in X} \big(3x_1 - x_2 + \lambda(-1 - x_1 + x_2)\big)$,
- $Z(\lambda) = \begin{cases} -2 + \lambda, & 0 \leq \lambda \leq 5/3, \\ 3 - 2\lambda, & 5/3 \leq \lambda \leq 3, \\ 6 - 3\lambda, & \lambda \geq 3. \end{cases}$
- $\lambda^* = 5/3$, and the optimal value is $Z_{\text{D}} = Z(5/3) = -1/3$. For $\lambda = 5/3$, the corresponding elements of X are $(1,0)$ and $(0,2)$.


$Z(\lambda)$
4
2
$-\frac{1}{3}$
$\frac{5}{3}$
λ


conv(X)
x_2
3
x_{IP}
x_D
2
$\boldsymbol{c}$
x_{LP}
1
0
1
2
x_1