15.083J/6.859J Integer Optimization

Lecture 4: Methods to enhance formulations II

1 Outline

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- Independence set systems and Matroids
- Strength of valid inequalities
- Nonlinear formulations

2 Independence set systems

2.1 Definition

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- N finite set, $\mathcal{I}$ collection of subsets of N.
- $(N, \mathcal{I})$ is an *independence system* if:

  (a) $\emptyset \in \mathcal{I}$;

  (b) if $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
- Combinatorial structures that exhibit hereditary properties

2.2 Examples

SLIDE 3

- Node disjoint paths; $G = (V, E)$, $\mathcal{I}_1$ collection of node disjoint paths in G. $(E, \mathcal{I}_1)$ is IS. Why?
- Acyclic subgraphs. $\mathcal{I}_2$ collection of acyclic subgraphs (forests) in $G = (V, E)$. $(E, \mathcal{I}_2)$ is IS. Why?
- Linear independence; $\boldsymbol{A}$ matrix; N index set of columns of $\boldsymbol{A}$; $\mathcal{I}_3$ collection of linearly independent columns of $\boldsymbol{A}$. $(N, \mathcal{I}_3)$ is IS. Why?
- Feasible solutions to packing problems. $S = \{\boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$, $\boldsymbol{A} \geq \boldsymbol{0}$, $N = \{1, 2, \ldots, n\}$. For $\boldsymbol{x} \in S$, $A(\boldsymbol{x}) = \{i \mid x_i = 1\}$. $\mathcal{I}_4 = \cup_{\boldsymbol{x} \in S} A(\boldsymbol{x})$. $(N, \mathcal{I}_4)$ is IS. Why?

2.3 Rank

SLIDE 4

- $(N, \mathcal{I})$ independence system
- An independent set of maximal cardinality contained in $T \subseteq N$ is called *a basis* of T. The maximum cardinality of a basis of T, denoted by $r(T)$, is called the *rank of* T.
- $S \subseteq T$; $|A| = r(T)$. $A \cap S$ and $A \cap (T \setminus S)$ are independent sets contained in S and $T \setminus S$
- $r(S) + r(T \setminus S) \geq |A \cap S| + |A \cap (T \setminus S)| = |A| = r(T)$.

2.4 Matroids

SLIDE 5

- $(N, \mathcal{I})$ is a matroid if: Every maximal independent set contained in F has the same cardinality $r(F)$ for all $F \subset N$.
- $(E, \mathcal{I}_1)$ (node disjoint paths in G). Is $(E, \mathcal{I}_1)$ a matroid?
- $F = \{(1,2), (2,3), (2,4), (4,5), (4,6)\}$. Maximal independent sets in F: $\{(1,2), (2,4), (4,5)\}$ and $\{(1,2), (2,3), (4,5), (4,6)\}$.
- Is $(E, \mathcal{I}_2)$ of forests a matroid?
- $(N, \mathcal{I}_3)$ of linearly independent columns of $\boldsymbol{A}$ is a matroid.
  $T \subset N$ index of columns of $\boldsymbol{A}$, $\boldsymbol{A}_T = [\boldsymbol{A}_j]_{j \in T}$. $r(T) = \text{rank}(\boldsymbol{A}_T)$.
- Is $(N, \mathcal{I}_4)$ of feasible solutions to packing problems a matroid?

2.5 Valid Inequalities

SLIDE 6

- $C \subseteq N$ a circuit in $(N, \mathcal{I})$.

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{subject to} \quad & \sum_{i \in C} x_i \leq |C| - 1 \text{ for all } C \in \mathcal{C} \\ & \boldsymbol{x} \in \{0,1\}^n. \end{aligned}$$

- Rank inequality $\sum_{i \in T} x_i \leq r(T)$
- BW contains conditions for rank inequalities to be facet defining. For matroids, rank inequalities completely characterize convex hull.

3 Strength of valid inequalities

SLIDE 7

- S set of integer feasible vectors.
- $P_i = \{\boldsymbol{x} \in \Re^n_+ \mid \boldsymbol{A}_i\boldsymbol{x} \geq \boldsymbol{b}_i\}$, $i = 1, 2$, $\boldsymbol{A}_i, \boldsymbol{b}_i \geq \boldsymbol{0}$; covering type polyhedra.
- The **strength** of P_1 with respect to P_2 denoted by $t(P_1, P_2)$ is the minimum value of $\alpha > 0$ such that $\alpha P_1 \subset P_2$.
- $P_1 = \{x \in \mathcal{R} \mid x \geq 0\}$, $P_2 = \{x \in \mathcal{R} \mid x \geq 1\}$. Strength ?

3.1 Characterization

3.1.1 Theorem

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- $\alpha P_1 \subset P_2$ if and only if for all $\boldsymbol{c} \geq \boldsymbol{0}$, $Z_2 \leq \alpha Z_1$, where $Z_i = \min\ \boldsymbol{c}'\boldsymbol{x}$: $\boldsymbol{x} \in P_i$.
- Proof If $\alpha P_1 \subset P_2$, then $Z_2 \leq \alpha Z_1$ for all $\boldsymbol{c} \geq \boldsymbol{0}$.
- For converse, assume $Z_2 \leq \alpha Z_1$, for all $\boldsymbol{c} \geq \boldsymbol{0}$, and there exists $\boldsymbol{x}_0 \in \alpha P_1$, but $x_0 \notin P_2$.
- By the separating hyperplane theorem, there exists $\boldsymbol{c}$: $\boldsymbol{c}'\boldsymbol{x}_0 < \boldsymbol{c}'\boldsymbol{x}$ for all $\boldsymbol{x} \in P_2$, i.e., $\boldsymbol{c}'\boldsymbol{x}_0 < Z_2$.
- $\boldsymbol{x}_0 \in \alpha P_1$, $\boldsymbol{x}_0 = \alpha \boldsymbol{y}_0$, $\boldsymbol{y}_0 \in P_1$. $Z_1 \leq \boldsymbol{c}'\boldsymbol{y}_0$, i.e., $\alpha Z_1 \leq \boldsymbol{c}'\boldsymbol{x}_0$, and thus $\alpha Z_1 < Z_2$. Contradiction.
- $t(P_1, P_2) = \sup_{\boldsymbol{c} \geq \boldsymbol{0}} \frac{Z_2}{Z_1}$.

3.1.2 Computation

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$P_i = \{\boldsymbol{x} \in \Re^n_+ \mid \boldsymbol{a}_i'\boldsymbol{x} \geq b_i,\ i = 1, \ldots, m\}$, and $\boldsymbol{a}_i \geq \boldsymbol{0},\ b_i \geq 0$ for all $i = 1, \ldots, m$. Then,

$$t(P_1, P_2) = \max_{i=1,\ldots,m} \frac{b_i}{d_i},$$

where $d_i = \min\ \boldsymbol{a}_i'\boldsymbol{x}$: $\boldsymbol{x} \in P_1$. (If $d_i = 0$, then $t(P_1, P_2)$ is defined to be $+\infty$.

3.2 Strength of an inequality

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- The **strength** of$\boldsymbol{f}'\boldsymbol{x} \geq g$, $\boldsymbol{f} \geq \boldsymbol{0}$, $g > 0$ with respect to $P = \{\boldsymbol{x} \in \Re^n_+ \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$ of covering type is defined as g/d, where $d = \min_{\boldsymbol{x}\in P} \boldsymbol{f}'\boldsymbol{x}$.
- By strong duality,

$$\begin{aligned} d = \max \quad & \boldsymbol{b}'\boldsymbol{p} \\ \text{s.t.} \quad & \boldsymbol{A}'\boldsymbol{p} \leq \boldsymbol{f} \\ & \boldsymbol{p} \geq \boldsymbol{0}. \end{aligned}$$

- $\overline{\boldsymbol{p}}$ feasible dual solution. $\boldsymbol{b}'\overline{\boldsymbol{p}} \leq d$. Then, the strength of inequality $\boldsymbol{f}'\boldsymbol{x} \geq g$ with respect to P is at most $g/(\boldsymbol{b}'\overline{\boldsymbol{p}})$.

4 Nonlinear formulations

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$$\begin{aligned} Z_{IP} = \min \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n \boldsymbol{A}_j x_j = \boldsymbol{b} \\ & x_j \in \{0, 1\}. \end{aligned}$$

4.1 SDP relaxation

SLIDE 12

- Multiply each constraint by x_i: $\sum_{j=1}^n \boldsymbol{A}_j x_j x_i = \boldsymbol{b} x_i$.
- Introduce $z_{ij} = x_i x_j$.

$$\begin{aligned} & z_{ii} = x_i^2 = x_i && \forall\ i = 1, \ldots, n. \\ & x_i x_j \geq 0 \iff z_{ij} \geq 0 && \forall\ i, j; i \neq j. \\ & x_i(1 - x_j) \geq 0 \iff z_{ij} \leq z_{ii} && \forall\ i, j,\ i \neq j. \\ & (1 - x_i)(1 - x_j) \geq 0 \iff z_{ii} + z_{jj} - z_{ij} \leq 1 && \forall\ i, j,\ i \neq j. \end{aligned}$$

- Matrix $\mathbf{Z} = \boldsymbol{x}\boldsymbol{x}'$ is positive semidefinite, $\mathbf{Z} \succeq \mathbf{0}$, i.e., for $\boldsymbol{u} \in \Re^n$,

$$\boldsymbol{u}'\mathbf{Z}\boldsymbol{u} = ||\boldsymbol{u}'\boldsymbol{x}||^2 \geq 0.$$

4.2 SDP relaxation

SLIDE 13

$$
\begin{aligned}
Z_{SD} = \min \quad & \sum_{j=1}^{n} c_j z_{jj} \\
\text{s.t.} \quad & \sum_{j=1}^{n} \boldsymbol{A}_j z_{ij} - \boldsymbol{b} z_{ii} = \boldsymbol{0}, && i = 1, \dots, n, \\
& \sum_{j=1}^{n} \boldsymbol{A}_j z_{jj} = \boldsymbol{b}, \\
& 0 \le z_{ij} \le z_{ii}, && i, j = 1, \dots, n, i \ne j, \\
& 0 \le z_{ij} \le z_{jj}, && i, j = 1, \dots, n, i \ne j, \\
& 0 \le z_{ii} \le 1, && j = 1, \dots, n, \\
& z_{ii} + z_{jj} - z_{ij} \le 1 && i, j = 1, \dots, n, i \ne j, \\
& \mathbf{Z} \succeq \mathbf{0}.
\end{aligned}
$$

$Z_{LP} \le Z_{SD} \le Z_{IP}$. Why?

4.3 Stable set

SLIDE 14

$$
\begin{aligned}
Z_{IP} = \max \quad & \sum_{i=1}^{n} w_i x_i \\
\text{s.t.} \quad & x_i + x_j \le 1, && \forall \{i, j\} \in E, \\
& x_i \in \{0, 1\}, && i \in V.
\end{aligned}
$$

$$
\begin{aligned}
Z_{SD} = \max \quad & \sum_{i=1}^{n} w_i z_{ii} \\
\text{s.t.} \quad & z_{ij} = 0, && \forall \{i, j\} \in E, \\
& z_{ii} + z_{jj} \le 1, && \forall \{i, j\} \in E, \\
& z_{ik} + z_{kj} \le z_{kk}, && \forall \{i, j\} \in E, \\
& z_{ii} + z_{jj} + z_{kk} \le 1 + z_{ik} + z_{jk}, && \forall \{i, j\} \in E, \\
& \mathbf{Z} \succeq \mathbf{0}.
\end{aligned}
$$

4.4 Max-Cut

SLIDE 15

$$
\begin{aligned}
\max \quad & \sum_{\{i,j\} \in E} w_{ij}(x_i + x_j - 2 x_i x_j) \\
\text{s.t.} \quad & x_s = 1, \quad x_t = 0, \\
& x_i \in \{0, 1\}, && \forall\, i \in V.
\end{aligned}
$$

$$
\begin{aligned}
Z_{SD} = \max \quad & \sum_{\{i,j\} \in E} w_{ij}(z_{ii} + z_{jj} - 2 z_{ij}) \\
\text{s.t.} \quad & z_{ss} = 1, \quad z_{tt} = 0, \quad z_{st} = 0 \\
& \mathbf{Z} \succeq \mathbf{0}.
\end{aligned}
$$

Also

$$0 \le z_{ii} \le 1,\ z_{ij} \le z_{ii},\ z_{ij} \le z_{jj},\ z_{ii} + z_{jj} - z_{ij} \le 1.$$

4.5 Scheduling

SLIDE 16

- Jobs $J = \{1, \ldots, n\}$ and m machines.
- p_{ij} processing time of job j on machine i.
- Completion time C_j. Objective: assign jobs to machines, and schedule each machine to minimize $\sum_{j \in J} w_j C_j$.
- If jobs j and k are assigned to machine i, then job j is scheduled before job k on machine i, denoted by $j \prec_i k$ if and only if

$$\frac{w_k}{p_{ik}} > \frac{w_j}{p_{ij}}.$$

4.5.1 Formulation

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- x_{ij} is one, if job j is assigned to machine i, and zero, otherwise.
- $C_j = \sum_{i=1}^m x_{ij} \left(p_{ij} + \sum_{k \prec_i j} x_{ik} p_{ik} \right)$,
-

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in J} w_j \sum_{i=1}^m x_{ij} \left(p_{ij} + \sum_{k \prec_i j} x_{ik} p_{ik} \right) \\ \text{subject to} \quad & \sum_{i=1}^m x_{ij} = 1, \qquad \forall j \in J \\ & x_{ij} \in \{0, 1\}. \end{aligned}$$

- $c_{ij} = w_j p_{ij}$,

$$d_{(ij),(hk)} = \begin{cases} 0, & \text{if } i \ne h \text{ or } j = k, \\ w_j p_{ik} & \text{if } i = h \text{ and } k \prec_i j, \\ w_k p_{ij} & \text{if } i = h \text{ and } j \prec_i k, \end{cases}$$

- $x_{ij}^2 = x_{ij}$:

$$\begin{aligned} Z_{\text{CP}} = \min \quad & \frac{1}{2} \boldsymbol{c}' \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}' \left(\boldsymbol{D} + \text{diag}(\boldsymbol{c}) \right) \boldsymbol{x} \\ \text{s.t.} \quad & \sum_{i=1}^m x_{ij} = 1, \qquad \forall j \in J \\ & 0 \le x_{ij} \le 1. \end{aligned}$$