15.082J and 6.855J and ESD.78J

Lagrangian Relaxation 2

- Applications
- Algorithms
- Theory

Lagrangian Relaxation and Inequality Constraints

$$z^* = \text{Min } cx$$
$$\text{subject to } Ax \le b, \qquad (P^*)$$
$$x \in X.$$

$$L(\mu) = \text{Min } cx + \mu(Ax - b) \qquad (P^*(\mu))$$
$$\text{subject to } x \in X,$$

Lemma. $L(\mu) \le z^*$ for $\mu \ge 0$.

The Lagrange Multiplier Problem: maximize $(L(\mu) : \mu \ge 0)$.

Suppose L^* denotes the optimal objective value, and suppose x is feasible for P^* and $\mu \ge 0$. Then $L(\mu) \le L^* \le z^* \le cx$.

Lagrangian Relaxation and Equality Constraints

$$z^* = \text{Min } cx \quad \text{subject to } Ax = b, \; x \in X. \qquad (P^*)$$

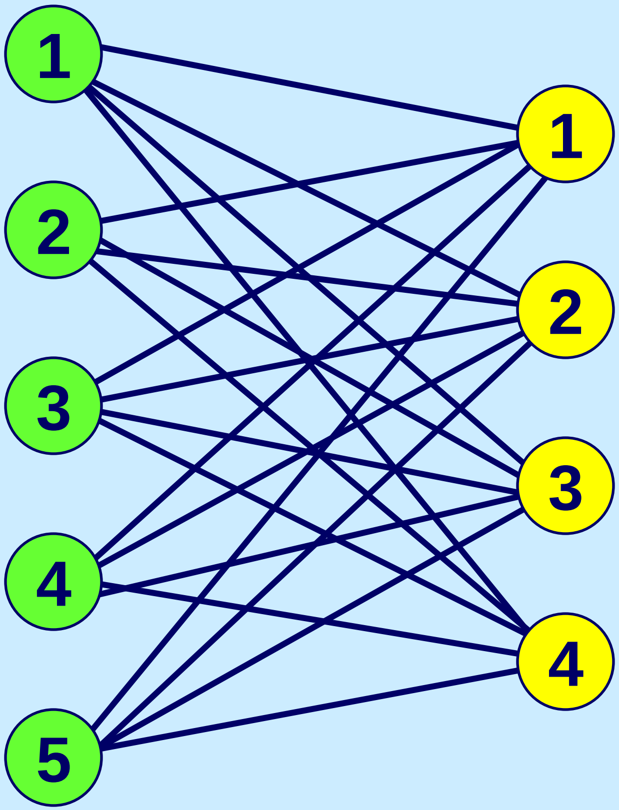
$$L(\mu) = \text{Min } cx + \mu(Ax - b) \quad \text{subject to } x \in X, \qquad (P^*(\mu))$$

Lemma. $L(\mu) \le z^*$ for all $\mu \in R^n$

The Lagrange Multiplier Problem: maximize $(L(\mu) : \mu \in R^n)$.

Suppose L* denotes the optimal objective value, and suppose x is feasible for P^* and $\mu \ge 0$. Then $L(\mu) \le L^* \le z^* \le cx$.

Generalized assignment problem ex. 16.8
Ross and Soland [1975]



Set I of jobs

Set J of machines

a_{ij} = the amount of processing time of job i on machine j

x_{ij} = 1 if job i is processed on machine j
= 0 otherwise

Job i gets processed.

Machine j has at most d_j units of processing

Generalized assignment problem ex. 16.8 Ross and Soland [1975]

Minimize $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$ (16.10a)

$\sum_{j \in J} x_{ij} = 1$ for each $i \in I$ (16.10b)

$\sum_{i \in I} a_{ij} x_{ij} \le d_j$ for each $j \in J$ (16.10c)

$x_{ij} \ge 0$ and integer for all $(i, j) \in A$ (16.10d)

Generalized flow with integer constraints.

Class exercise: write two different Lagrangian relaxations.

Facility Location Problem ex. 16.9
Erlenkotter 1978

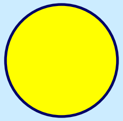
Consider a set J of potential facilities

- **Opening facility $j \in J$ incurs a cost F_j.**
- **The capacity of facility j is K_j.**

Consider a set I of customers that must be served

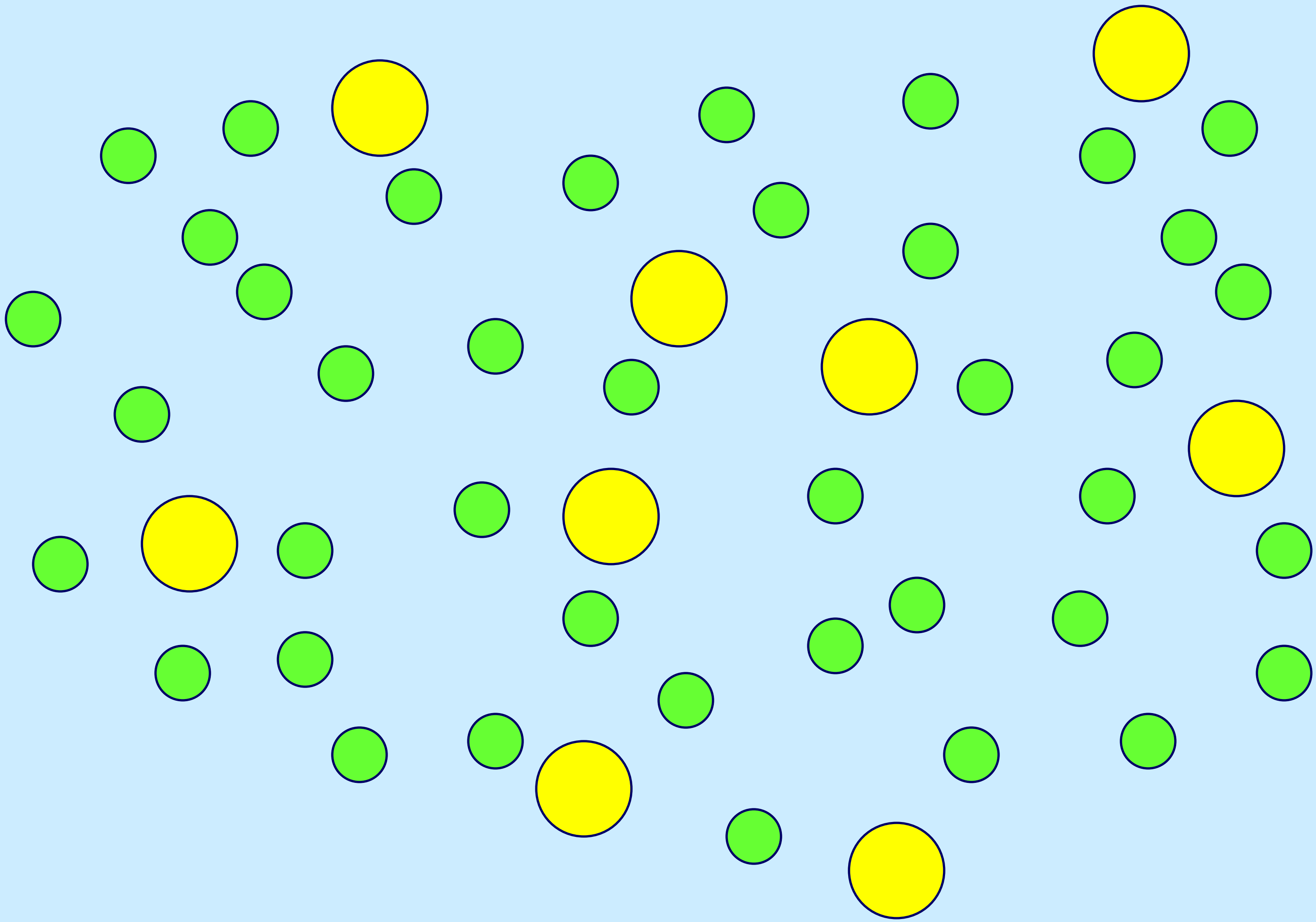
- **The total demand of customer i is d_i.**
- **Serving one unit of customer i's from location j costs c_{ij}.**

customer

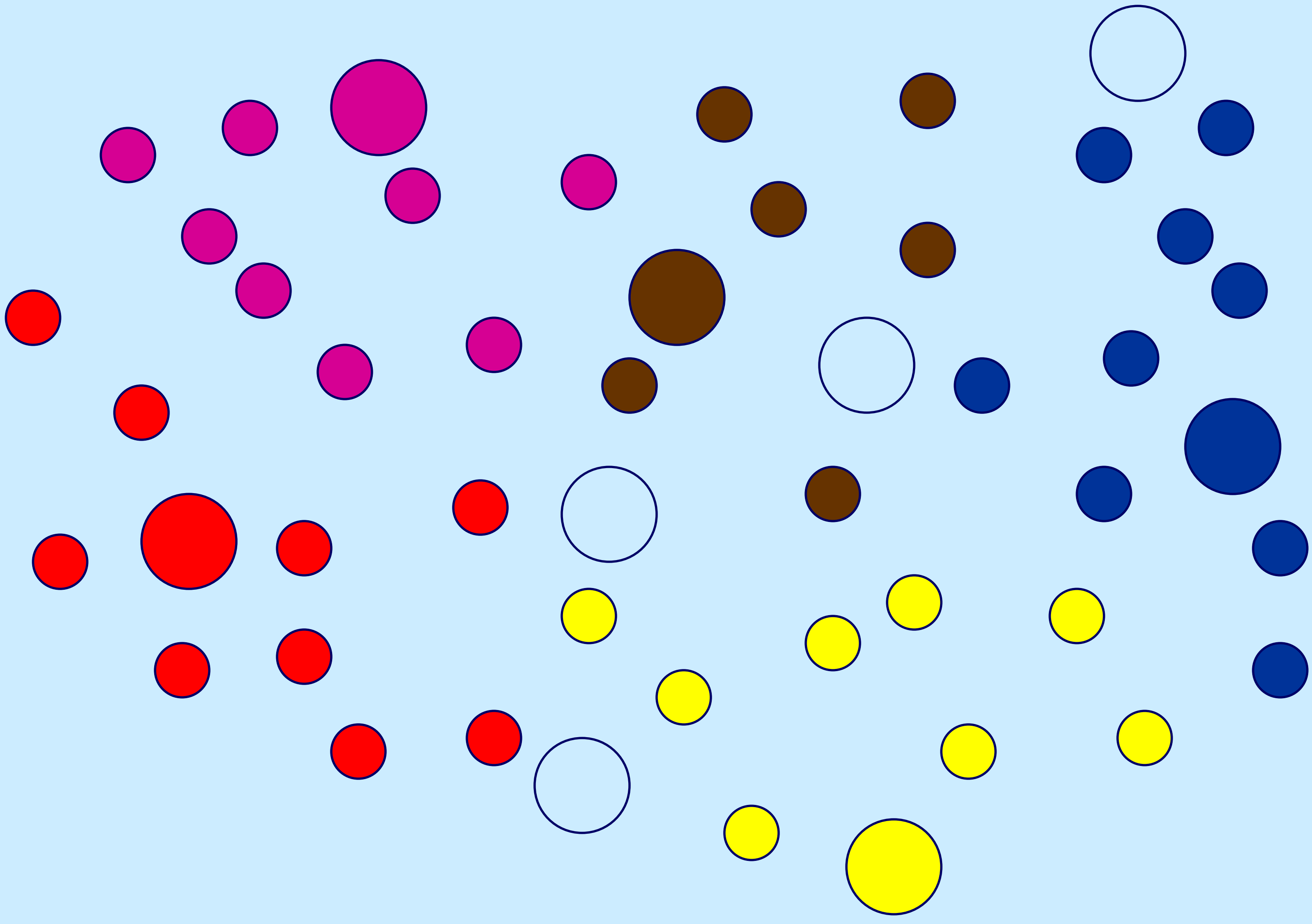


potential facility

A pictorial representation



A possible solution



Class Exercise

Formulate the facility location problem as an integer program. Assume that a customer can be served by more than one facility.

Suggest a way that Lagrangian Relaxation can be used to help solve this problem.

Let x_{ij} be the amount of demand of customer i served by facility j.

Let y_j be 1 if facility j is opened, and 0 otherwise.

The facility location model

Minimize $$\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J} F_j y_j$$

subject to

$$\sum_{j\in J} x_{ij} = 1$$ **for all** $i \in I$

$$\sum_{i\in I} d_i x_{ij} \le K_j y_j$$ **for all** $j \in J$

$$0 \le x_{ij} \le 1$$ **for all** $i \in I$ **and** $j \in J$

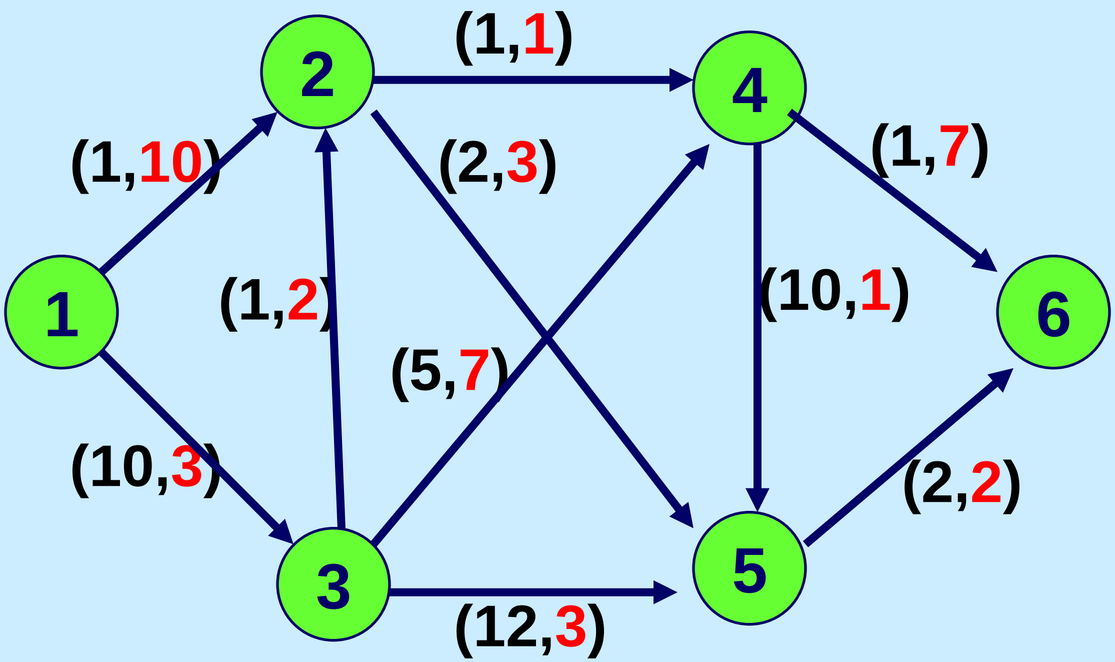
$$y_j = 0 \text{ or } 1$$ **for all** $j \in J$

Solving the Lagrangian Multiplier Problem

Approach 1: represent the LP feasible region as the convex combination of corner points. Then use a constraint generation approach.

Approach 2: subgradient optimization

The Constrained Shortest Path Problem



Find the shortest path from node 1 to node 6 with a transit time at most 14.

Constrained Shortest Paths: Path Formulation

Given: a network $G = (N,A)$

c_{ij}	cost for arc (i,j)
$c(P)$	cost of path P
t_{ij}	traversal time for arc (i,j)
T	upper bound on transit times.
$t(P)$	traversal time for path P
$\mathbf{P}$	set of paths from node 1 to node n

$$\begin{aligned} \text{Min} \quad & c(P) \\ \text{s.t.} \quad & t(P) \le T \\ & P \in \mathbf{P} \end{aligned}$$

Constrained Problem

$$\begin{aligned} L(\mu) = \text{Min} \quad & c(P) + \mu\, t(P) - \mu T \\ \text{s.t.} \quad & P \in \mathbf{P} \end{aligned}$$

Lagrangian

The Lagrangian Multiplier Problem

Step 0. Formulate the Lagrangian Problem.

$$L(\mu) = \text{Min} \quad c(P) + \mu\, t(P) - \mu T$$
$$\text{s.t.} \quad P \in \mathbf{P}$$

Step 1. Rewrite as a maximization problem

$$L(\mu) = \text{Max } w$$
$$\text{s.t} \quad w \leq c(P) + \mu\, t(P) - \mu T \quad \text{for all } P \in \mathbf{P}$$

Step 2. Write the Lagrangian multiplier problem

$$L^* = \max \{L(\mu): \mu \geq 0\} =$$
$$= \quad \text{Max } w$$
$$\text{s.t} \quad w \leq c(P) + \mu\, t(P) - \mu T \quad \text{for all } P \in \mathbf{P}$$
$$\mu \geq 0$$

Max { w: w ≤ c(P) + μ t(P) – μT ∀P ∈ $\mathbf{P}$, and μ ≥ 0}

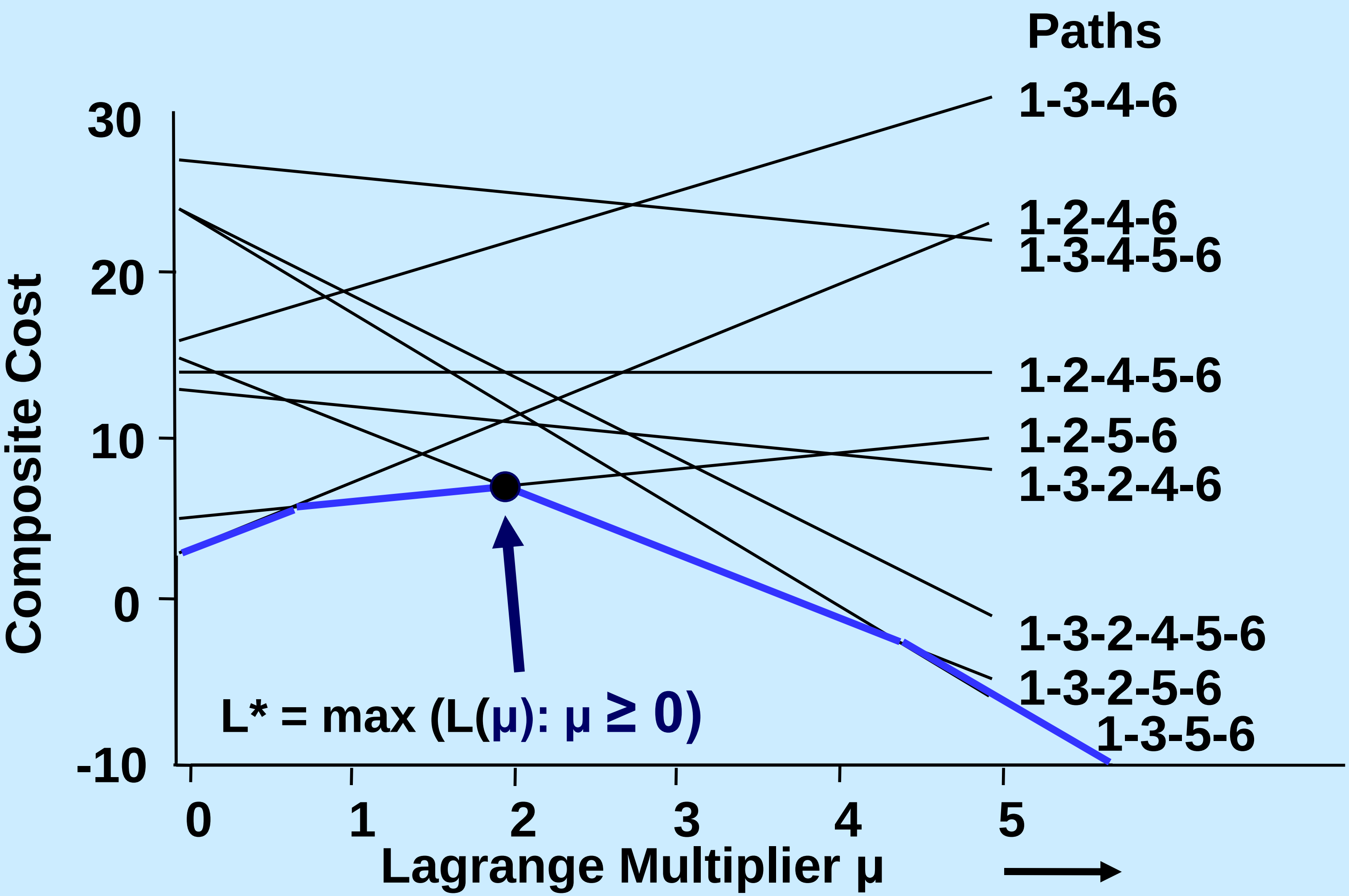


Figure 16.3 The Lagrangian function for T = 14.

The Restricted Lagrangian

$\mathcal{P}$: the set of paths from node 1 to node n

$\mathcal{S} \subseteq \mathcal{P}$: a subset of paths

$L^* = \max w$

s.t $w \leq c(P) + \mu\, t(P) - T$

for all $P \in \mathcal{P}$

$\mu \geq 0$

Lagrangian Multiplier Problem

$L^*_S = \max w$

s.t $w \leq c(P) + \mu\, t(P) - \mu T$

for all $P \in \mathcal{S}$

$\mu \geq 0$

Restricted Lagrangian Multiplier Problem

$L(\mu) \leq L^* \leq L^*_S$ ⇒ If $L(\mu) = L^*_S$ then $L(\mu) = L^*$.

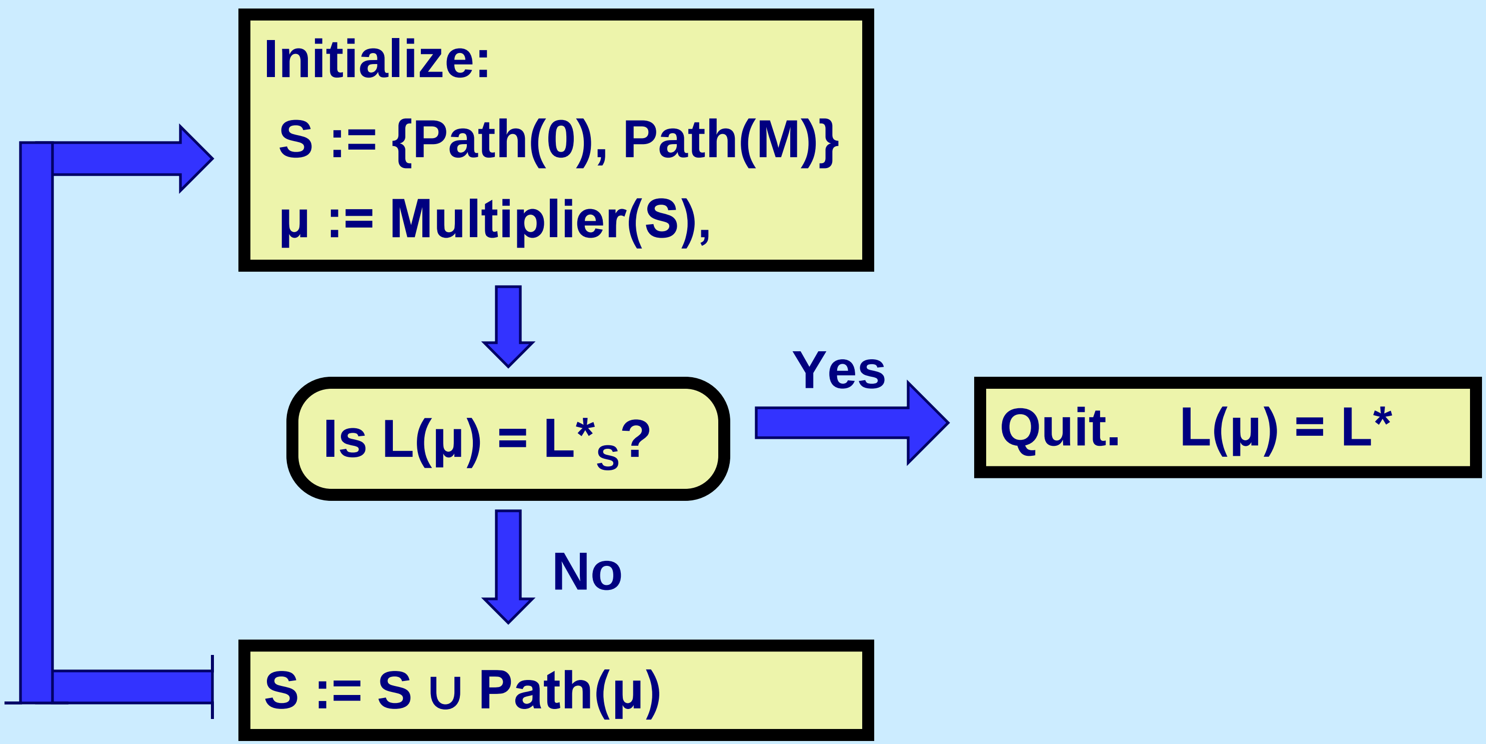
Optimality Conditions

Constraint Generation for Finding L*

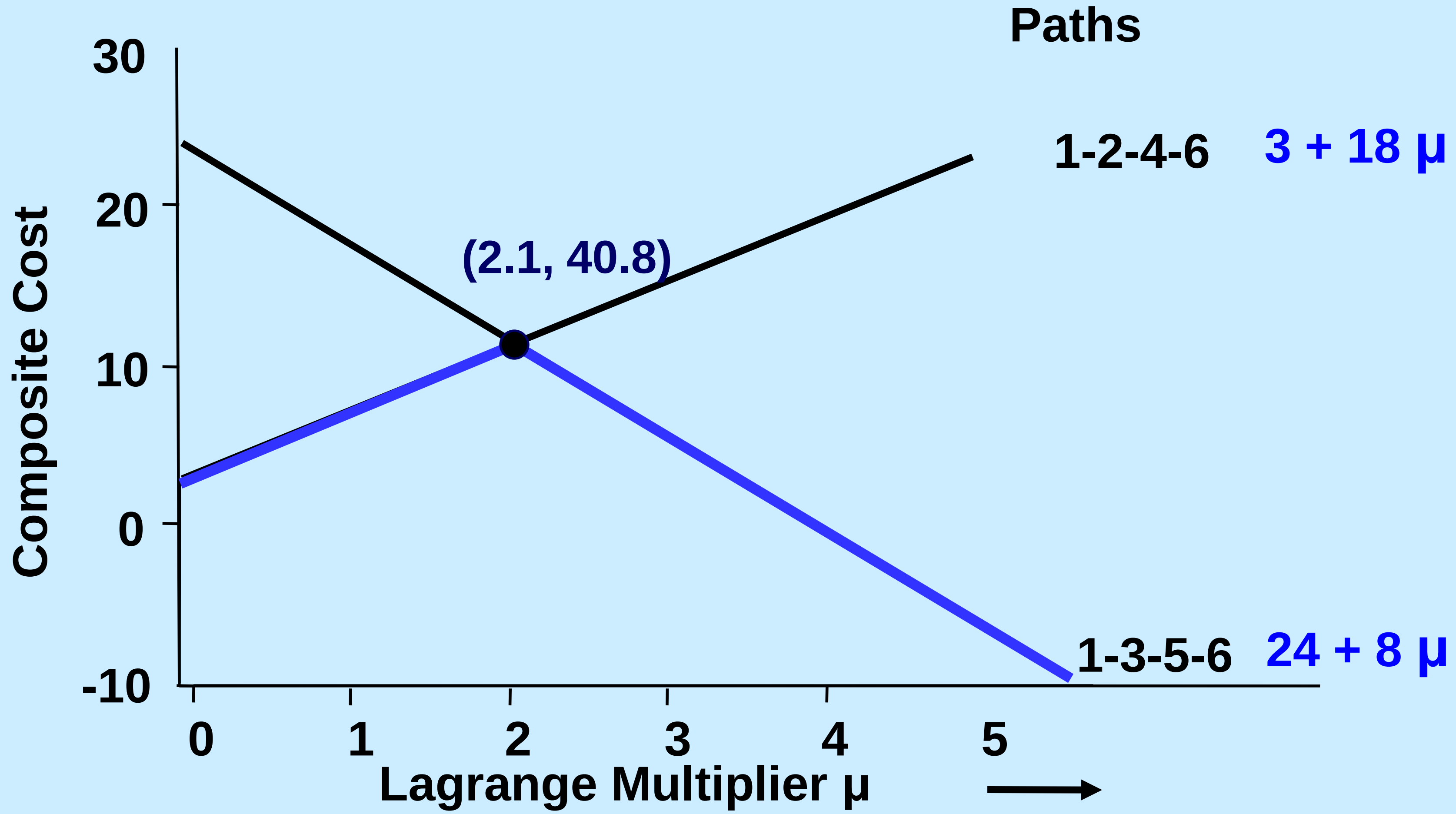
Let Path(μ) be the path that optimizes the Lagrangian.

Let Multiplier(S) be the value of μ that optimizes $L_S(\mu)$.

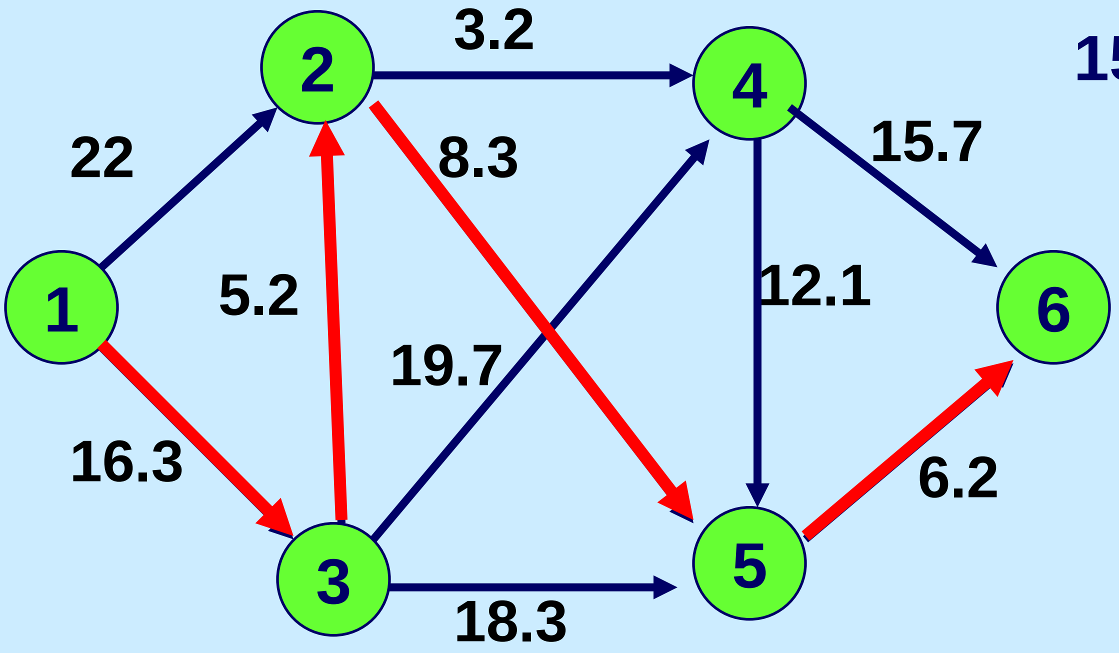
M is some large number



We start with the paths 1-2-4-6, and 1-3-5-6 which are optimal for L(0) and L(∞).



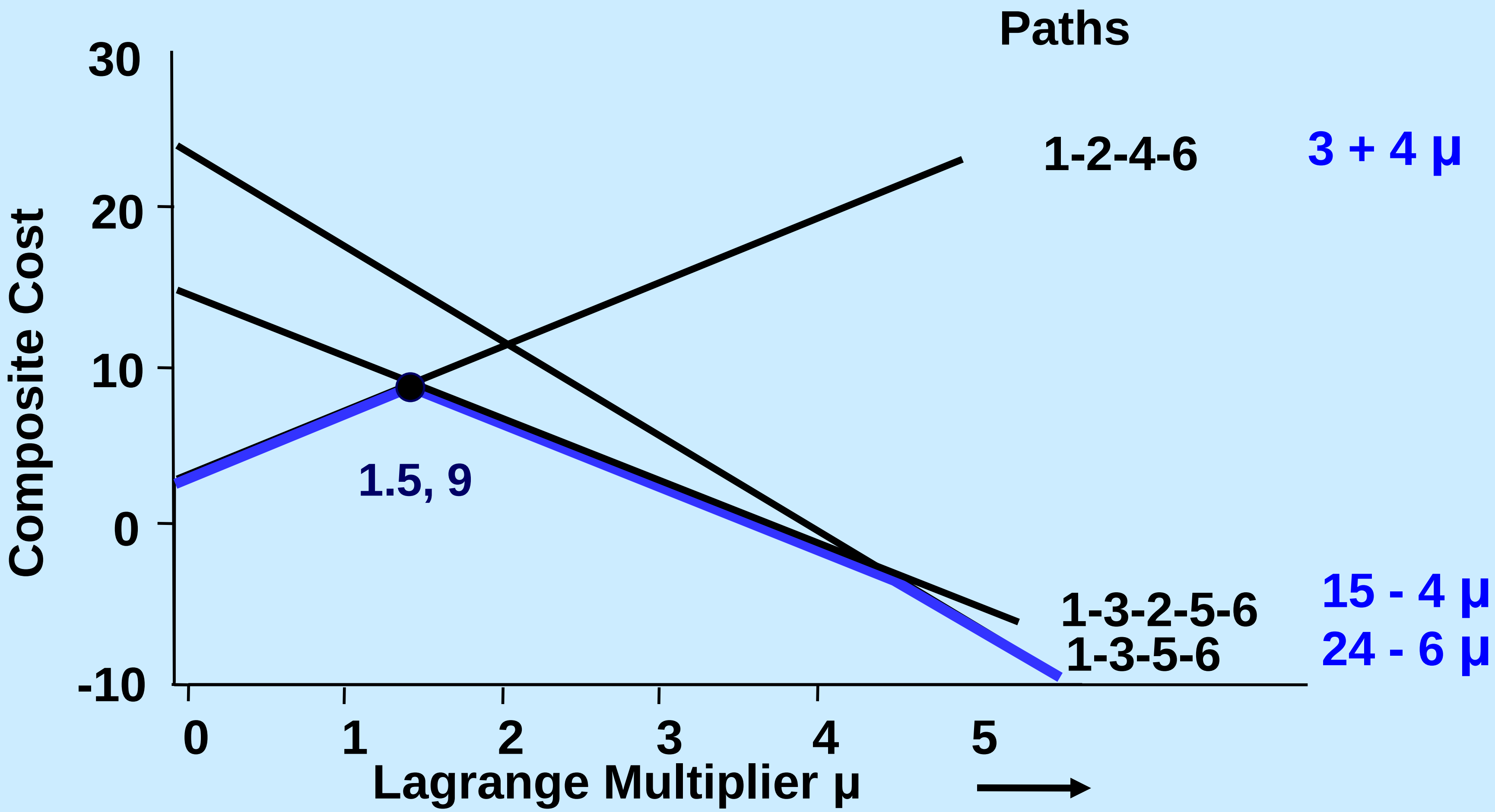
Set μ = 2.1 and solve the constrained shortest path problem



The optimum path is 1-3-2-5-6

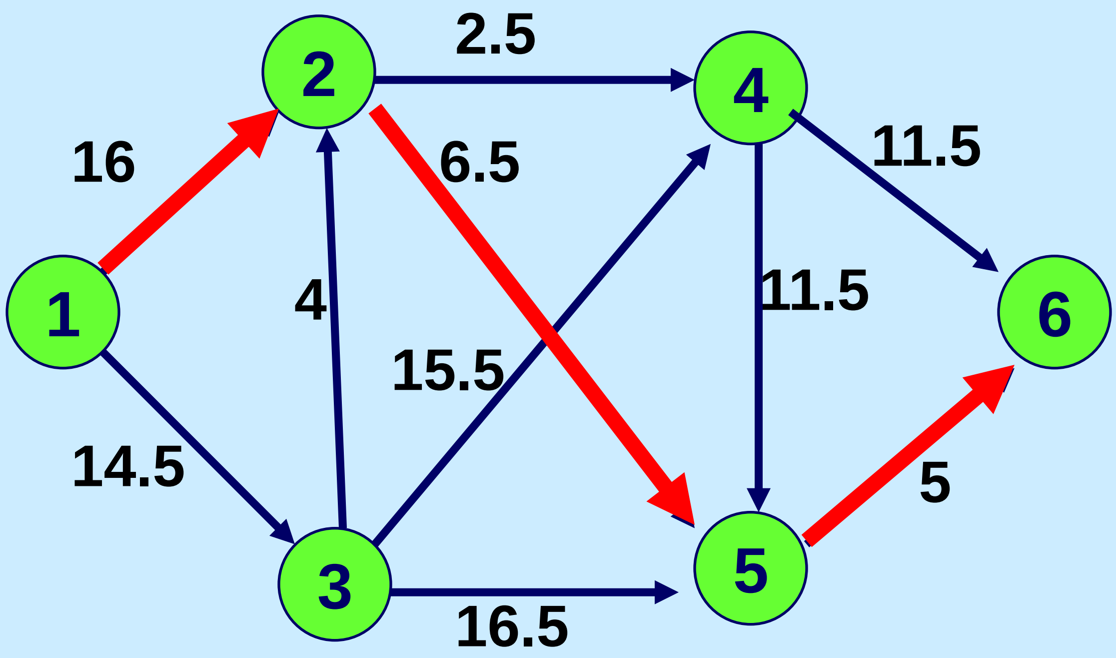
15 + 10 μ

Path(2.1) = 1-3-2-5-6. Add it to S and reoptimize.

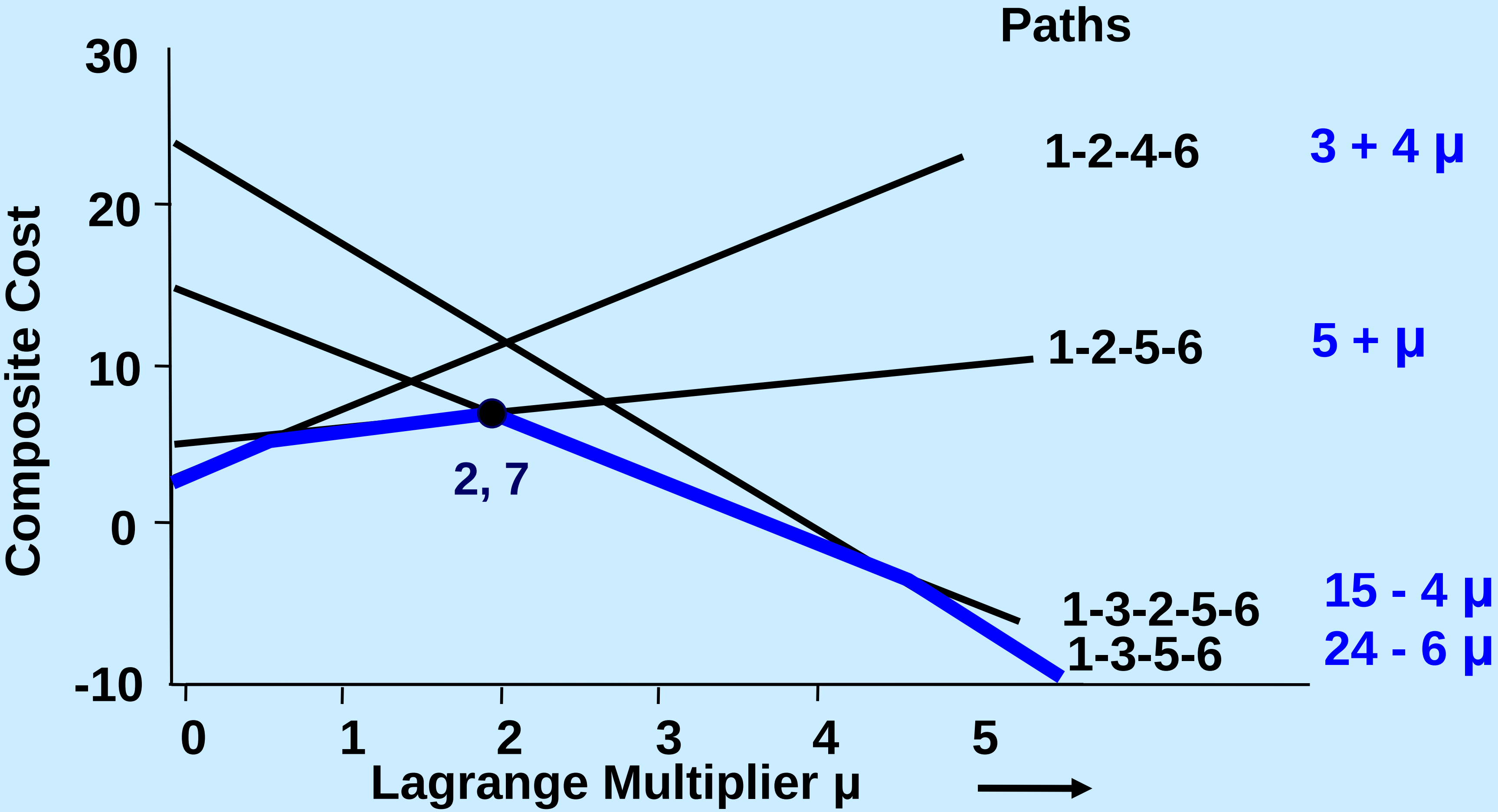


Set μ = 1.5 and solve the constrained shortest path problem

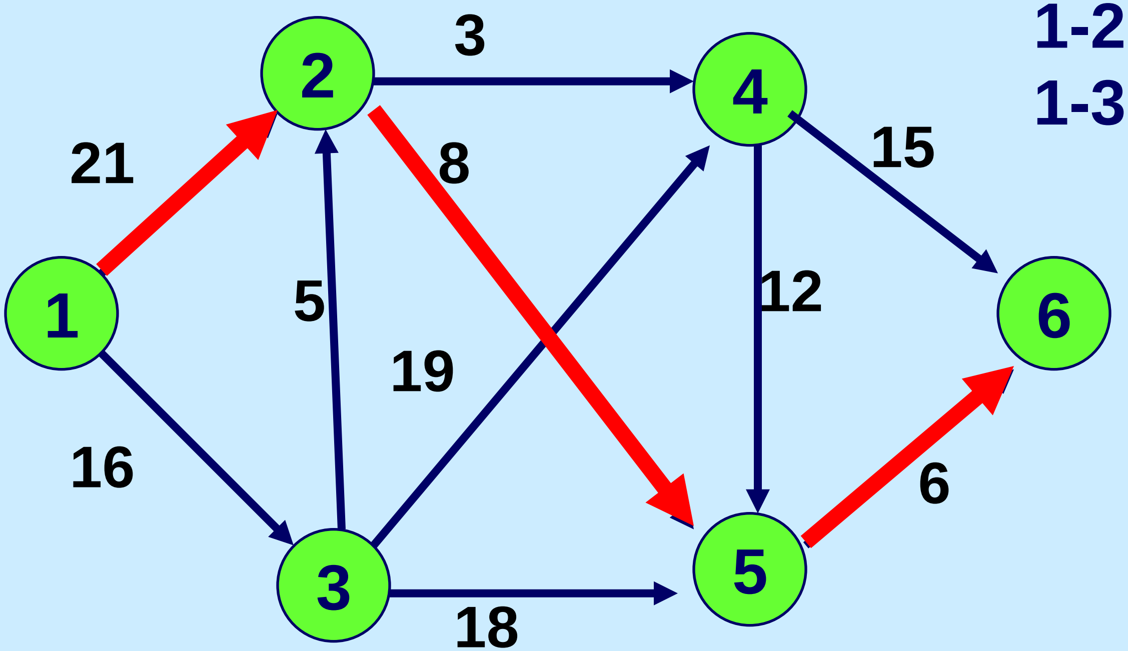
The optimum path is 1-2-5-6.



Add Path 1-2-5-6 and reoptimize

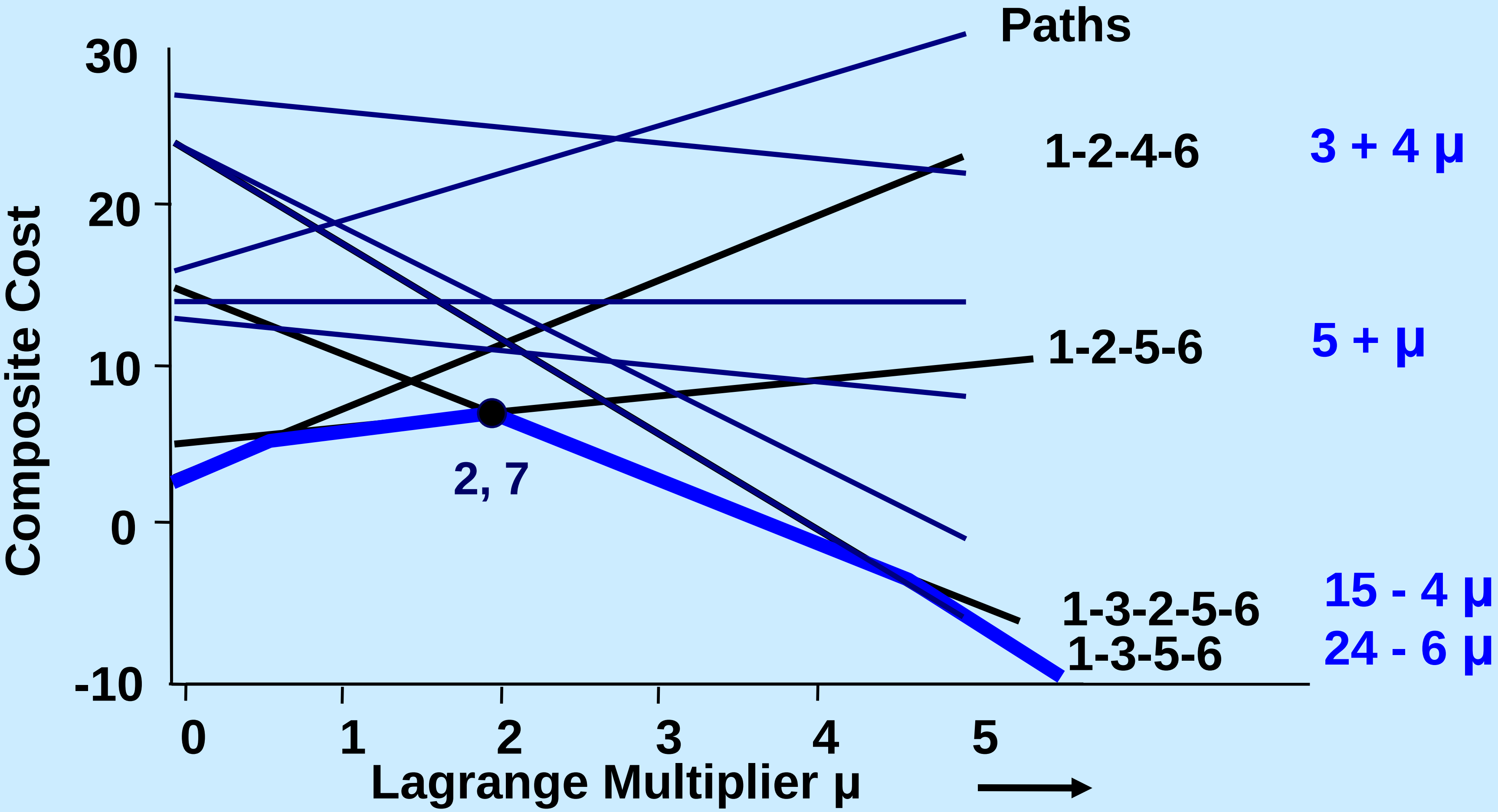


Set μ = 2 and solve the constrained shortest path problem



The optimum paths are 1-2-5-6 and 1-3-2-5-6

There are no new paths to add. μ* is optimal for the multiplier problem



Mental Break

Where did the name Gatorade come from?

The drink was developed in 1965 for the Florida Gators football team. The team credits in 1967 Orange Bowl victory to Gatorade.

What percentage of people in the world have never made or received a phone call?

50%

What causes the odor of natural gas?

Natural gas has no odor.   They add the smell artificially to make leaks easy to detect.

Mental Break

What is the most abundant metal in the earth's crust?

Aluminum

How fast are the fastest shooting stars?

Around 150,000 miles per hour.

The Malaysian government banned car commercials starring Brad Pit. What was their reason for doing so?

Brad Pitt was not sufficiently Asian looking. Using a Caucasian such as Brad was considered "an insult to Asians."

Towards a general theory

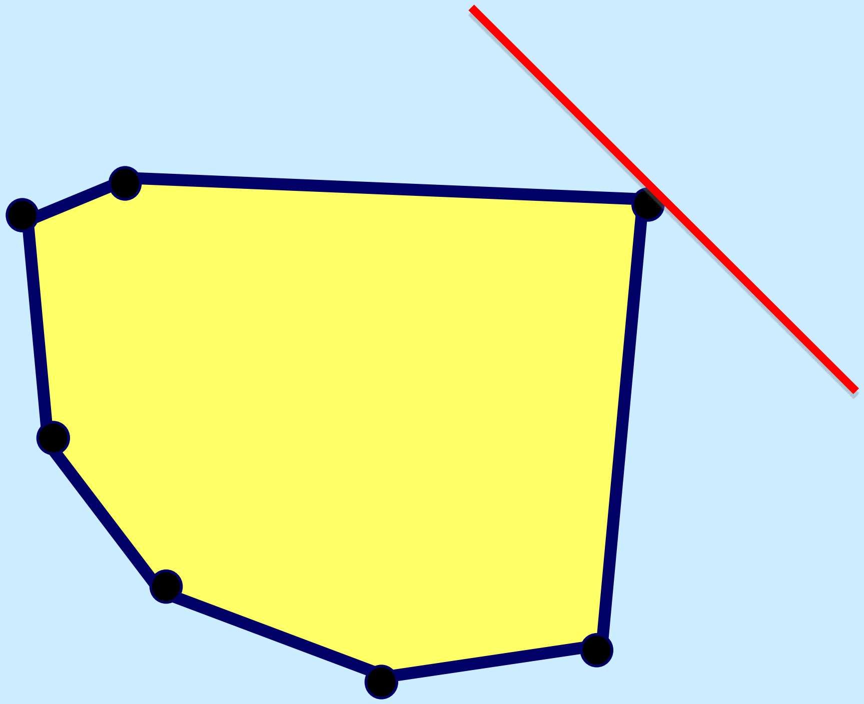
Next: a way of generalizing Lagrangian Relaxation for the time constrained shortest path problem to LPs in general.

Key fact: bounded LPs are optimized at extreme points.

Extreme Points and Optimization

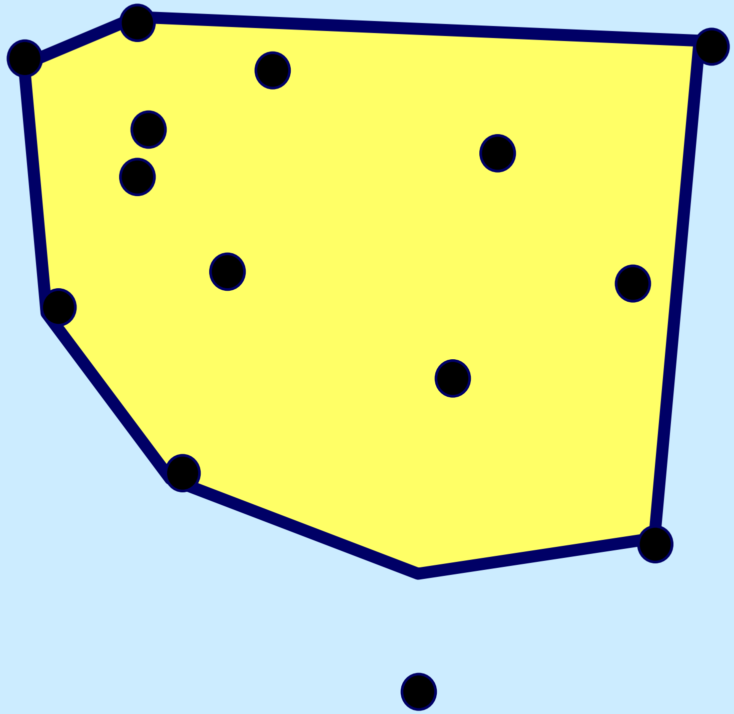
Paths ⇔ Extreme Points

Optimizing over paths ⇔ Optimizing over extreme points



If an LP region is bounded, then there is a minimum cost solution that occurs at an extreme (corner) point)

Convex Hulls and Optimization:



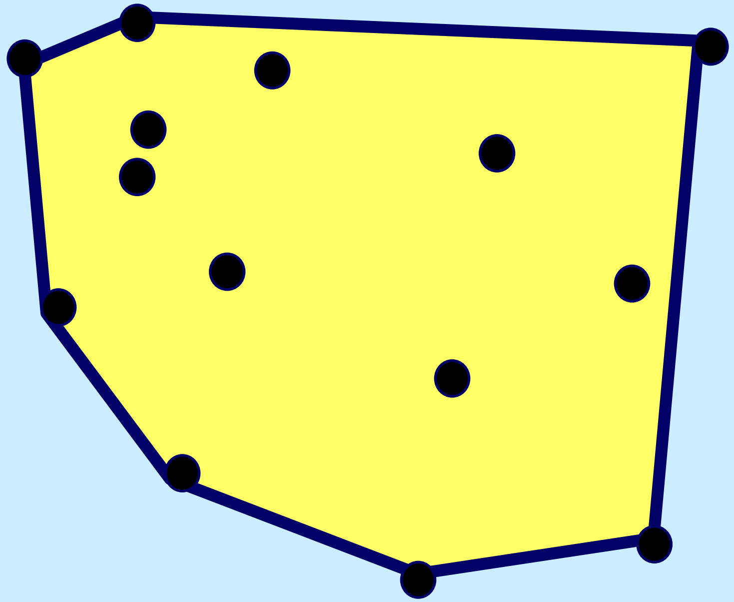
The convex hull of a set X of points is the smallest LP feasible region that contains all points of X. The convex hull will be denoted as H(X) .

Min	**cx**
s.t	**x ∈ X**

$\Longleftrightarrow$

Min	**cx**
s.t.	**x ∈ H(X)**

Convex Hulls and Optimization:



Let S = {x : Ax = b, x ≥ 0}.

Suppose that S has a bounded feasible region.

Let Extreme(S) be the set of extreme points of S.

Min	**cx**
s.t	**Ax = b**
	x ≥ 0

⟺

Min	**cx**
s.t.	**x ∈ Extreme(S)**

Convex Hulls

Suppose that X = {x^1, x^2, …, x^K} is a finite set.

Vector y is a *convex combination* of X = {x^1, x^2, …, x^K} if there is a feasible solution to

$$y = \sum_{k=1}^{K} \lambda_k x^k$$

$$\sum_{k=1}^{K} \lambda_k = 1$$

$$\lambda_k \geq 0 \text{ for } k = 1 \text{ to K}$$

The *convex hull* of X is H(X) = { x : x can be expressed as a convex combination of points in X.}

Lagrangian Relaxation and Inequality Constraints

$$z^* = \min \; cx$$
$$\text{subject to} \quad Ax \leq b, \qquad \text{(P)}$$
$$x \in X.$$

$$L(\mu) = \min \; cx + \mu(Ax - b) \qquad \text{(P}(\mu)\text{)}$$
$$\text{subject to} \quad x \in X.$$

$L^* = \max\,(L(\mu) : \mu \geq 0)$.

So we want to maximize over μ, while we are minimizing over x.

An alternative representation

Suppose that $X = \{x^1, x^2, x^3, \ldots, x^K\}$. Possibly K is exponentially large; e.g., X is the set of paths from node s to node t.

$$L(\mu) \quad = \quad \min \quad cx + \mu(Ax - b) = (c + \mu A)x - \mu b$$

$$\text{subject to} \quad x \in X.$$

$$L(\mu) \quad = \quad \min \ \{(c + \mu A)x^k - \mu b : k = 1 \text{ to } K\}$$

$$L(\mu) \quad = \quad \max \ w$$

$$\text{s.t.} \quad w \le (c + \mu A)x^k - \mu b \ \text{ for all } k$$

The Lagrange Multiplier Problem

$$L^* = \max \; w$$
$$\text{s.t.} \quad w \le (c + A)x^k - \mu b \text{ for all } k \in [1,K]$$
$$\mu \in R^n$$

Suppose that $S \subseteq [1,K]$

For a fixed value μ

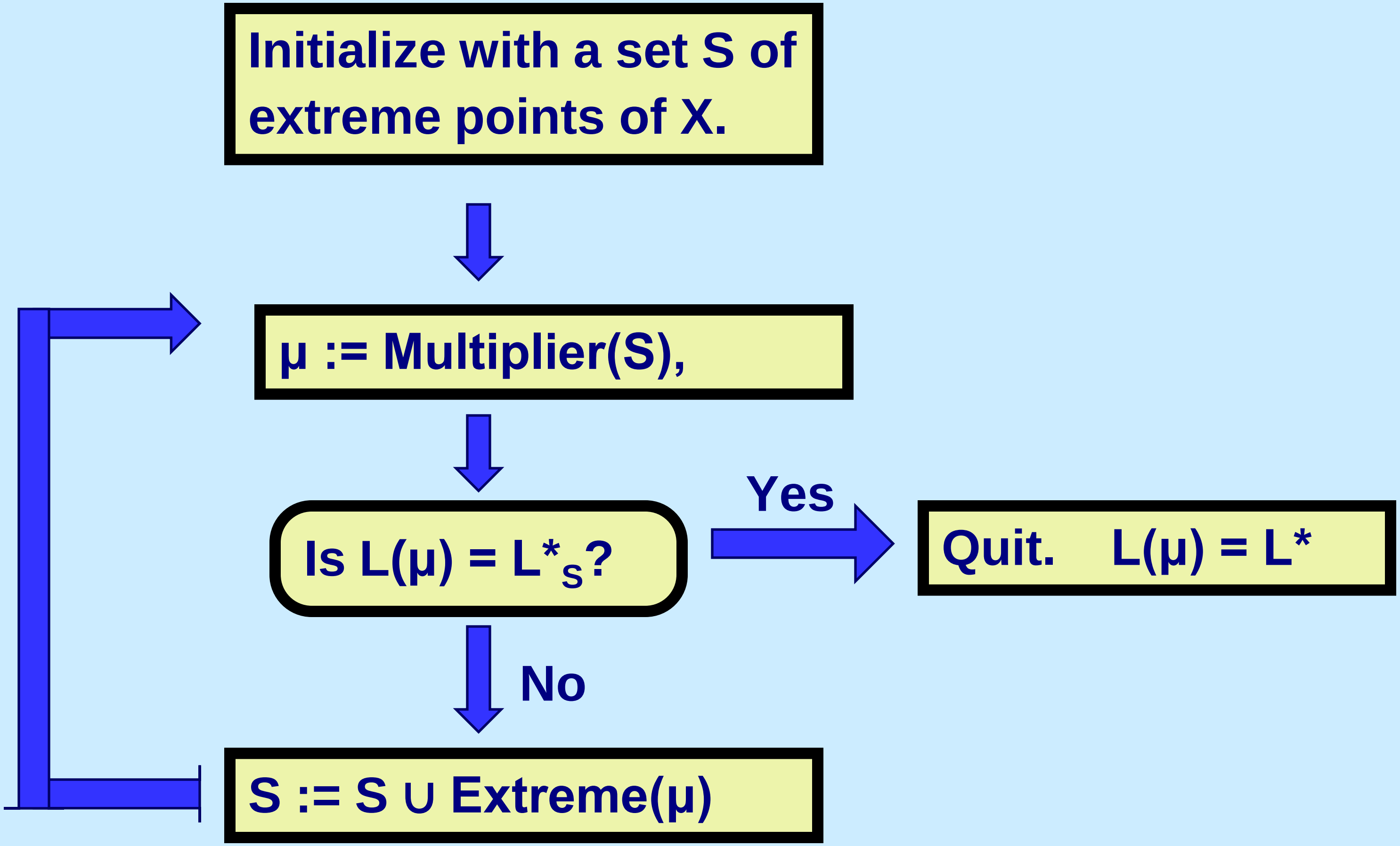
$$L^*_S(\mu) = \max \; w$$
$$\text{s.t.} \quad w \le (c + \mu A)x^k - \mu b \text{ for all } k \in S$$

$$L^*_S = \max \; w$$
$$\text{s.t.} \quad w \le (c + \mu A)x^k - \mu b \text{ for all } k \in S$$
$$\mu \in R^n$$

Constraint Generation for Finding L*

Suppose that Extreme(μ) optimizes the Lagrangian.

Let Multiplier(S) be the value of μ that optimizes $L_S(\mu)$.



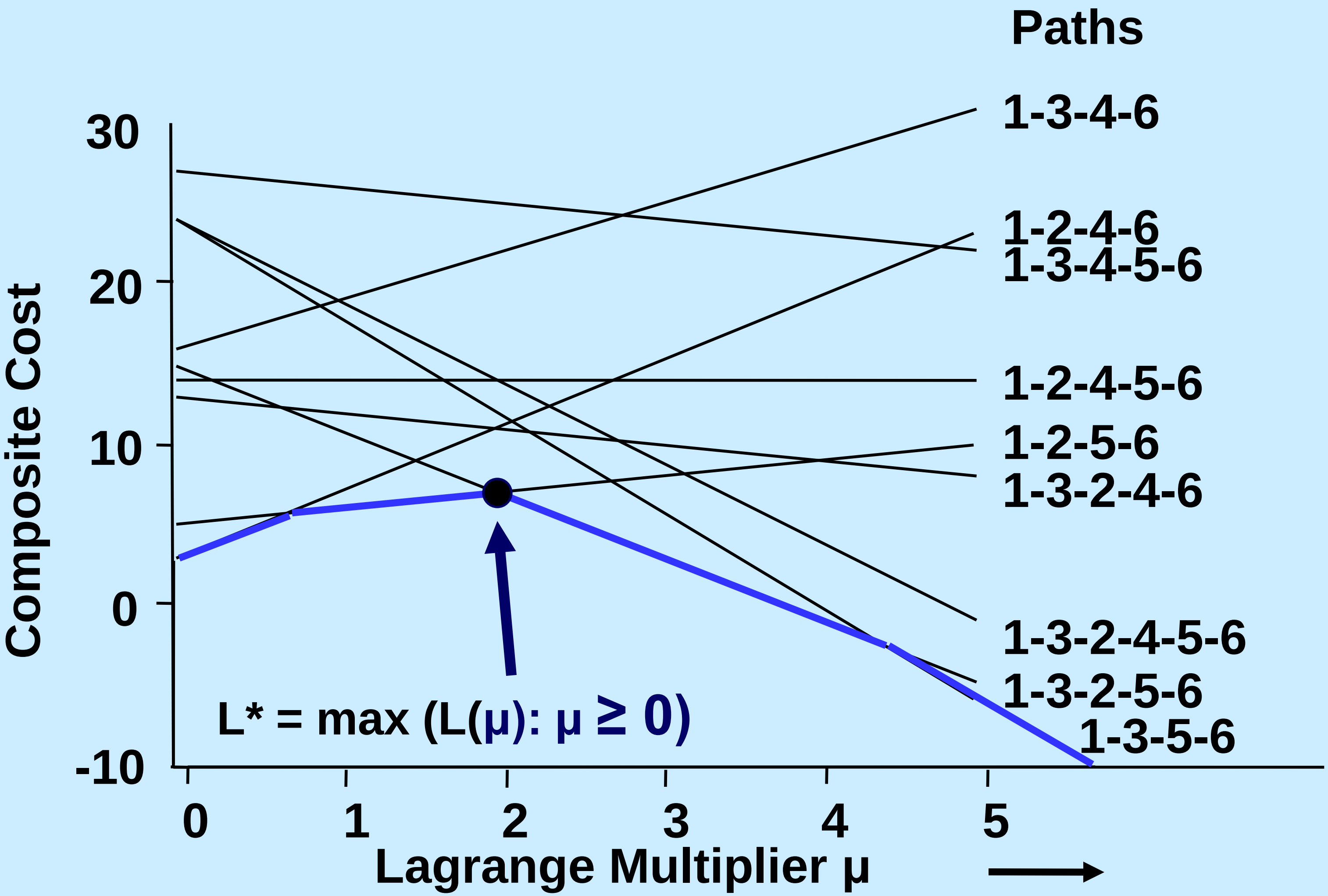
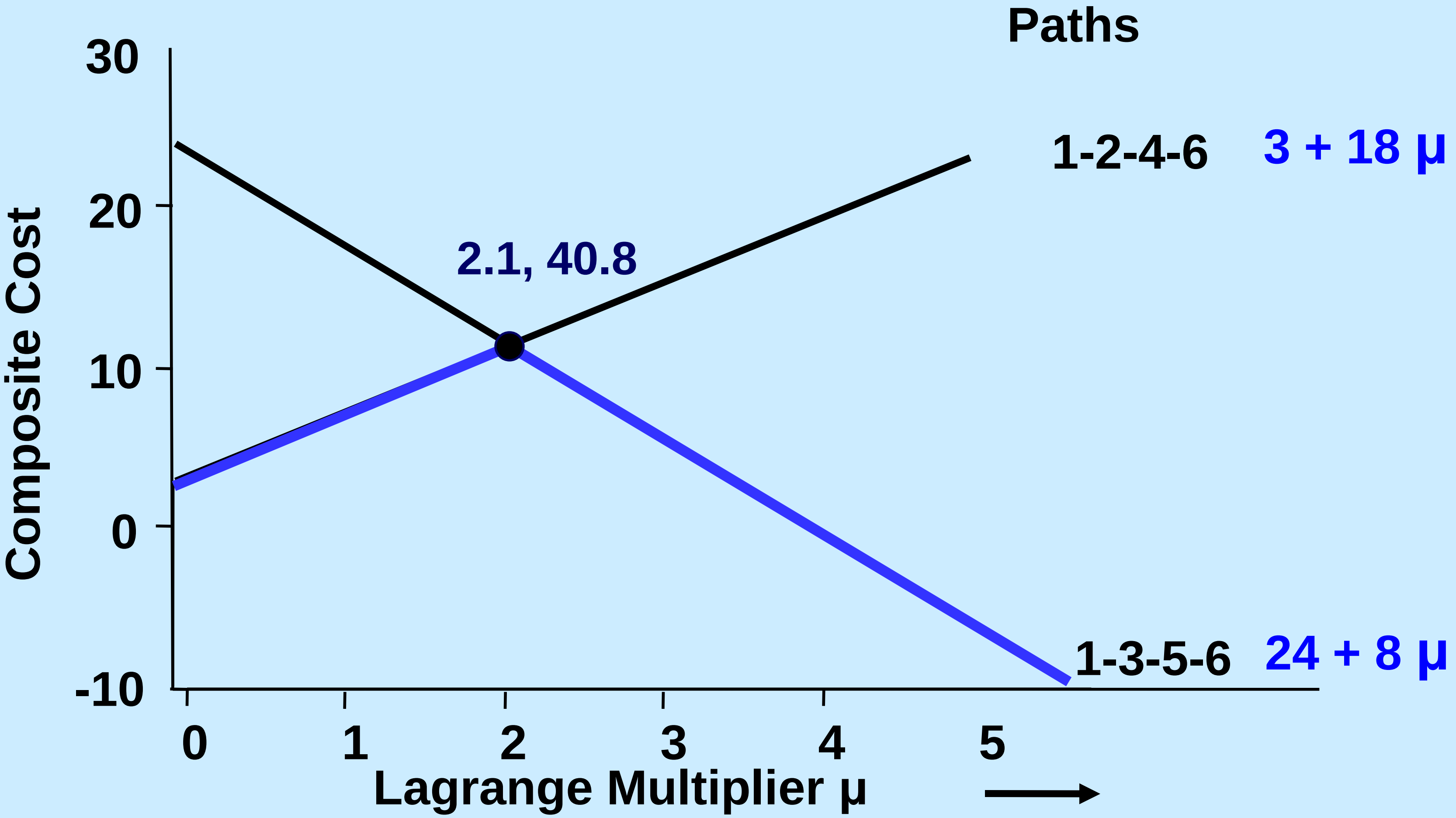
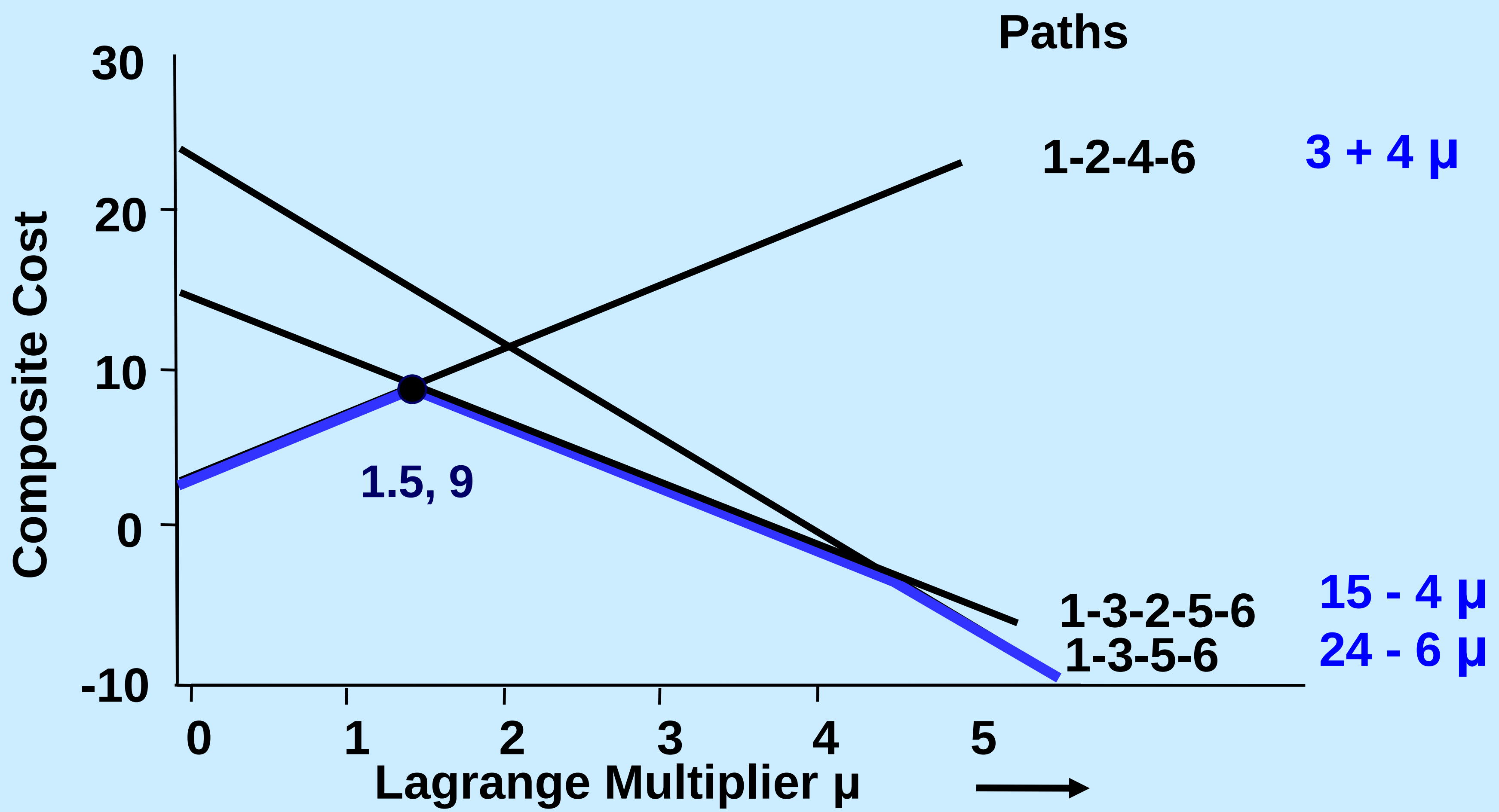


Figure 16.3 The Lagrangian function for T = 14.

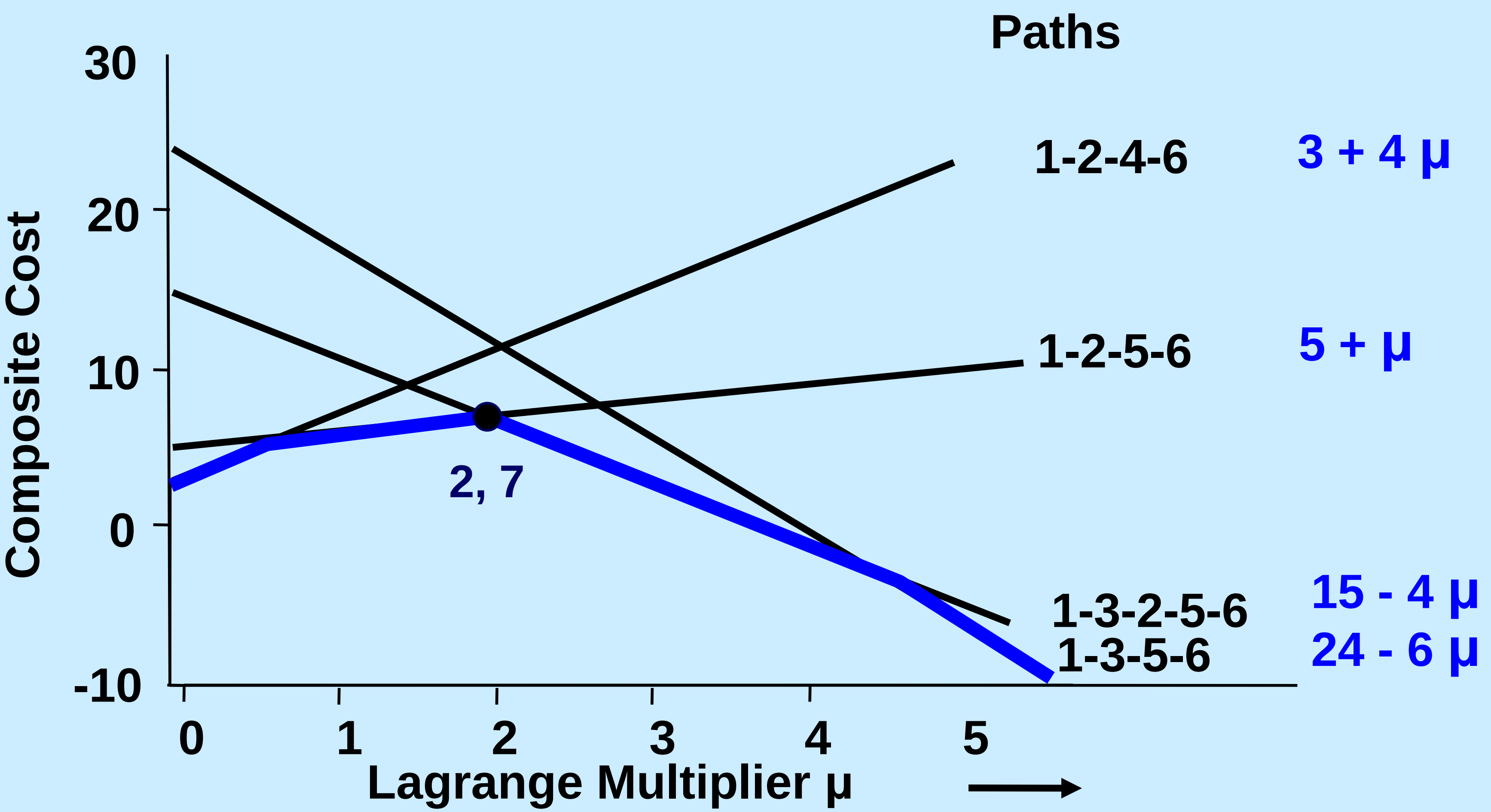
We start with the paths 1-2-4-6, and 1-3-5-6 which are optimal for L(0) and L(μ).



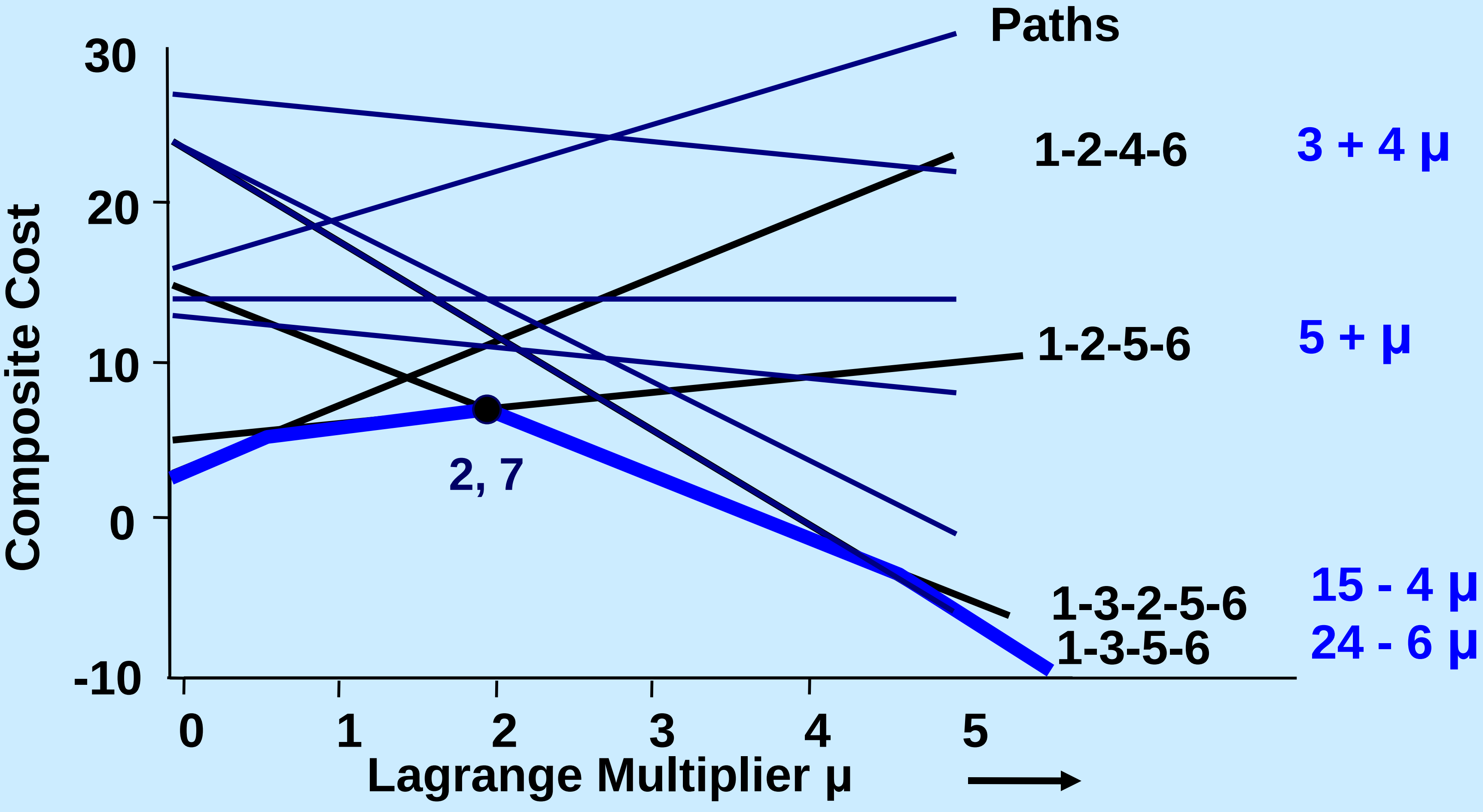
Add Path 1-3-2-5-6 and reoptimize



Add Path 1-2-5-6 and reoptimize



There are no new paths to add.
μ* is optimal for the multiplier problem



Subgradient optimization

Another major solution technique for solving the Lagrange Multiplier Problem is subgradient optimization.

Based on ideas from non-linear programming.

It converges (often slowly) to the optimum.

See the textbook for more information.

Application	*Embedded Network Structure*
Networks with side constraints	minimum cost flows shortest paths
Traveling Salesman Problem	assignment problem minimum cost spanning tree
Vehicle routing	assignment problem variant of min cost spanning tree
Network design	shortest paths
Two-duty operator scheduling	shortest paths minimum cost flows
Multi-time production planning	shortest paths / DPs minimum cost flows

Interpreting L*

1. For LP's, L* is the optimum value for the LP

2. Relation of L* to optimizing over a convex hull

Lagrangian Relaxation applied to LPs

$$z^* = \min \; cx \quad \text{s.t.} \quad Ax = b, \quad Dx = d, \quad x \geq 0$$

LP

$$L(\mu) = \min \; cx + \mu(Ax - b) \quad \text{s.t.} \quad Dx = d, \quad x \geq 0$$

LP(μ)

$$L^* = \max \; L(\mu) \quad \text{s.t.} \quad \mu \in R^n$$

LMP

Theorem 16.6 If $-\infty < z^* < \infty$, then $L^* = z^*$.

On the Lagrange Multiplier Problem

Theorem 16.6 If $-\infty < z^* < \infty$, then $L^* = z^*$.

Does this mean that solving the Lagrange Multiplier Problem solves the original LP?

No! It just means that the two optimum objective values are the same.

Sometimes it is MUCH easier to solve the Lagrangian problem, and getting an approximation to L^* is also fast.

Property 16.7

1. The set H(X) is a polyhedron, that is, it can be expressed as $H(X) = \{x : Ax \leq b\}$ for some matrix A and vector b.
2. Each extreme point of H(X) is in X. If we minimize $\{cx : x \in H(X)\}$, the optimum solution lies in X.
3. Suppose $X \subseteq Y = \{x : Dx \leq c \text{ and } x \geq 0\}$.
   Then $H(X) \subseteq Y$.

Relationships concerning LPs

$$z^* = \min \; cx \quad \text{s.t} \quad Ax = b, \quad x \in X$$

Original Problem

$$v^* = \min \; cx \quad \text{s.t} \quad Ax = b, \quad x \in H(X)$$

X replaced by H(X)

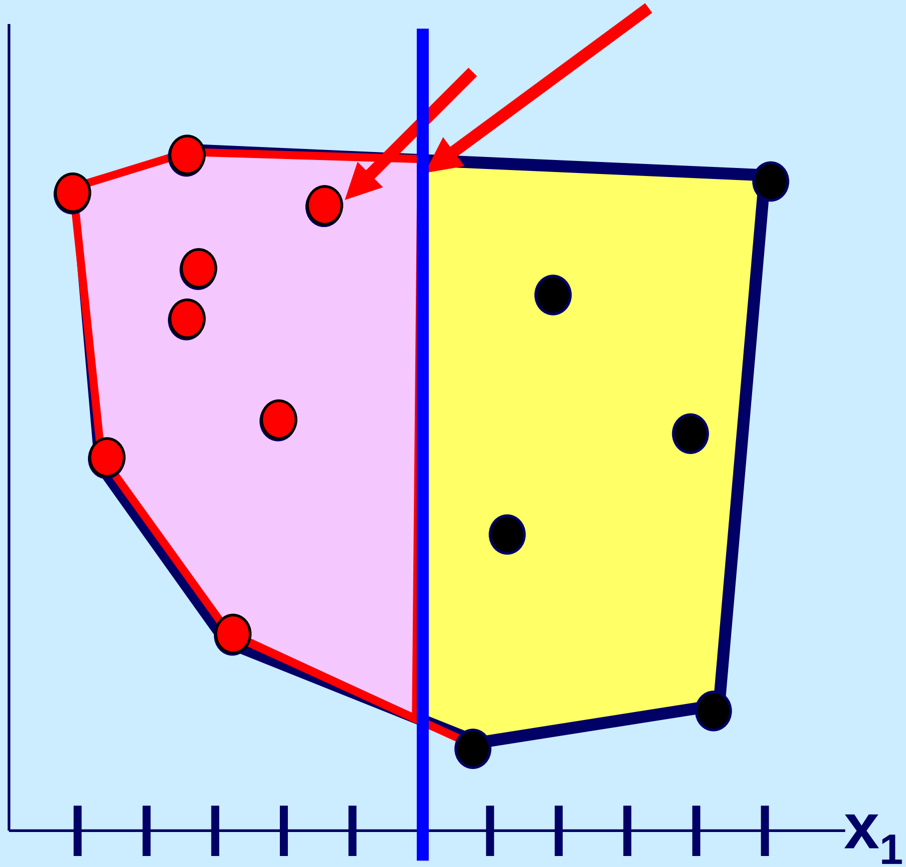
$$L(\mu) = \min \; cx + \mu(Ax - b) \quad \text{s.t} \quad x \in X$$

Lagrangian

$$v(\mu) = \min \; cx + \mu(Ax - b) \quad \text{s.t} \quad x \in H(X)$$

X replaced by H(X)

$$L(\mu) = v(\mu) \leq v^* \leq z^*$$



max x_1

s.t.

$x_1 \leq 6$

$x \in X$

is different from

max x_1

s.t.

$x_1 \leq 6$

$x \in H(X)$

Relationships concerning LPs

$$z^* = \min \; cx \quad \text{s.t} \quad Ax = b, \quad x \in X$$

$$v^* = \min \; cx \quad \text{s.t} \quad Ax = b, \quad x \in H(X)$$

$$L(\mu) = \min \; cx + \mu(Ax - b) \quad \text{s.t} \quad x \in X$$

$$v(\mu) = \min \; cx + \mu(Ax - b) \quad \text{s.t} \quad x \in H(X)$$

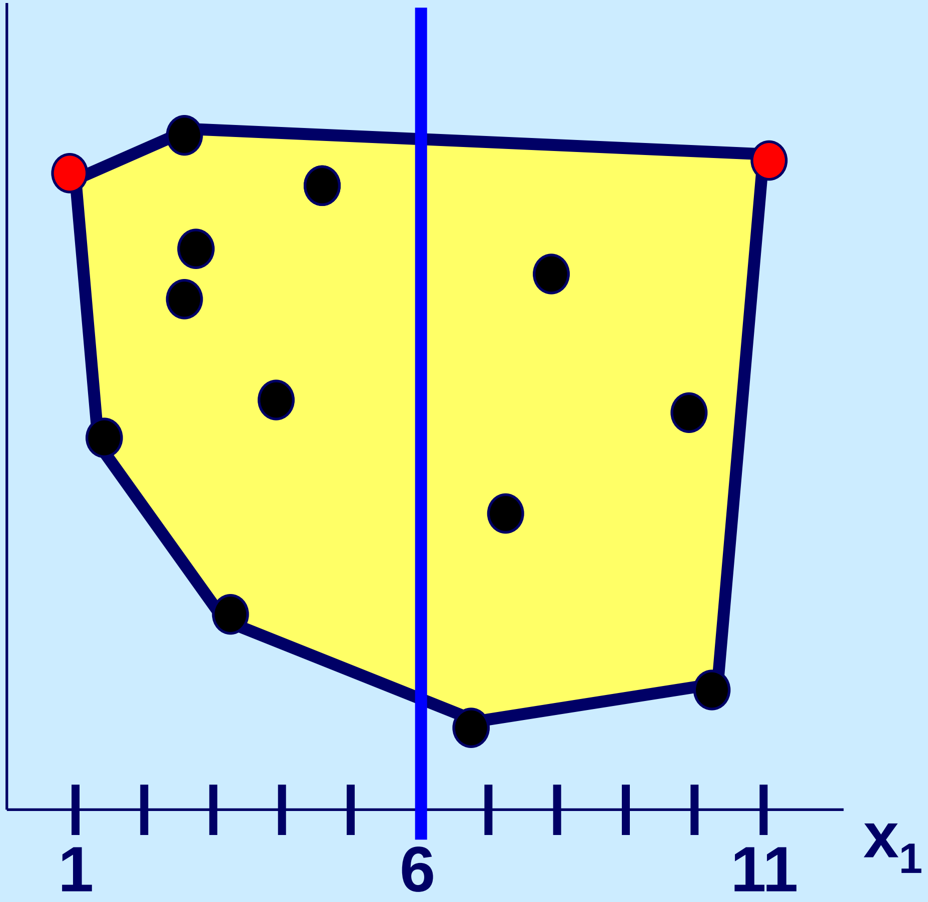
$$L^* = \max \{L(\mu) : \mu \in R^n\}$$

$$v^* = \max \{v(\mu) : \mu \in R^n\}$$

$$L(\mu) = v(\mu) \;\leq\; L^* = v^* \;\leq\; z^*$$

Theorem 16.8. $L^* = v^*$.

Illustration



$\min \; -x_1$

s.t.

$x_1 \leq 6$

$x \in H(X)$

Lagrangian

$\min \; -x_1 + \mu(x_1 - 6)$

$= (\mu-1)x_1 - 6\mu$

s.t. $x \in X$

$L(\mu) = (\mu-1) - 6\mu = -5\mu - 1$ if $\mu \geq 1$

$L(\mu) = 11(\mu-1) - 6\mu = 5\mu - 11$ if $\mu \leq 1$

$L^* = -6$

Integrality Property

Suppose X = {x : Dx = q , x ≥ 0, x integer}.

We say that X satisfies if the *integrality property* if the following LP has integer solutions for all d

Min	**dx**
s.t	**x ∈ X**

Fact: The LP region for min cost flow problems has the integrality property.

Integrality Property

Let $X = \{x : Dx = q \quad x \geq 0, x \text{ integer}\}$.

$$z^* = \text{Min } cx \quad \text{s.t } Ax = b, \; x \in X$$

$$z_{LP} = \min cx \quad \text{s.t } Ax = b, \; Dx = q, \; x \geq 0$$

$$L(\mu) = \text{Min } cx + \mu(Ax - b) \quad \text{s.t } x \in X$$

$$L^* = \text{Max } L(\mu) \quad \text{s.t } \mu \in R^n$$

Theorem 16.10. If X has the integrality property, then $z_{LP} = L^*$.

Proof of Integrality Property

Suppose $X = \{x : Dx = q \quad x \geq 0, x \text{ integer}\}$ has the integrality property.

$$\begin{aligned} z_{LP} = \ &\min \quad cx \\ &\text{s.t} \quad Ax = b \\ &\qquad\ \ Dx = q \\ &\qquad\ \ x \geq 0 \end{aligned}$$

$$\begin{aligned} L(\mu) = \ &\text{Min} \quad cx + \mu(Ax - b) \\ &\text{s.t} \quad x \in X \end{aligned}$$

$$\begin{aligned} L(\mu) = \ &\text{Min} \quad cx + \mu(Ax - b) \\ &\text{s.t} \quad Dx = q \\ &\qquad\ \ x \geq 0 \end{aligned}$$

$$\begin{aligned} L^* = \ &\text{Max} \quad L(\mu) \\ &\text{s.t} \quad \mu \in R^n \end{aligned}$$

$z_{LP} = L^*$ by Theorem 16.6.

Example: Generalized Assignment

Minimize $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$ (16.10a)

$\sum_{j \in J} x_{ij} = 1$ for each $i \in I$ (16.10b)

$\sum_{i \in I} a_{ij} x_{ij} \le d_j$ for each $j \in J$ (16.10c)

$x_{ij} \ge 0$ and integer for all $(i, j) \in A$ (16.10d)

If we relax (16.10c), the bound for the Lagrangian multiplier problem is the same as the bound for the LP relaxation.

If we relax (16.10b), the LP does not satisfy the integrality property, and we should get a better bound than z^0.

Summary

- A decomposition approach for Lagrangian Relaxations
- Relating Lagrangian Relaxations to LPs