15.082 and 6.855J

Lagrangian Relaxation

I never missed the opportunity to remove obstacles in the way of unity.

—Mohandas Gandhi

On bounding in optimization

In solving network flow problems, we not only solve the problem, but we provide a guarantee that we solved the problem.

Guarantees are one of the major contributions of an optimization approach.

But what can we do if a minimization problem is too hard to solve to optimality?

Sometimes, the best we can do is to offer a lower bound on the best objective value. If the bound is close to the best solution found, it is almost as good as optimizing.

Overview

Decomposition based approach.

Start with

- Easy constraints
- Complicating Constraints.

Put the complicating constraints into the objective and delete them from the constraints.

We will obtain a lower bound on the optimal solution for minimization problems.

In many situations, this bound is close to the optimal solution value.

An Example: Constrained Shortest Paths

Given: a network G = (N,A)

c_{ij} cost for arc (i,j)

t_{ij} traversal time for arc (i,j)

z* = Min $\sum_{(i,j)\in A} c_{ij} x_{ij}$

s. t.

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

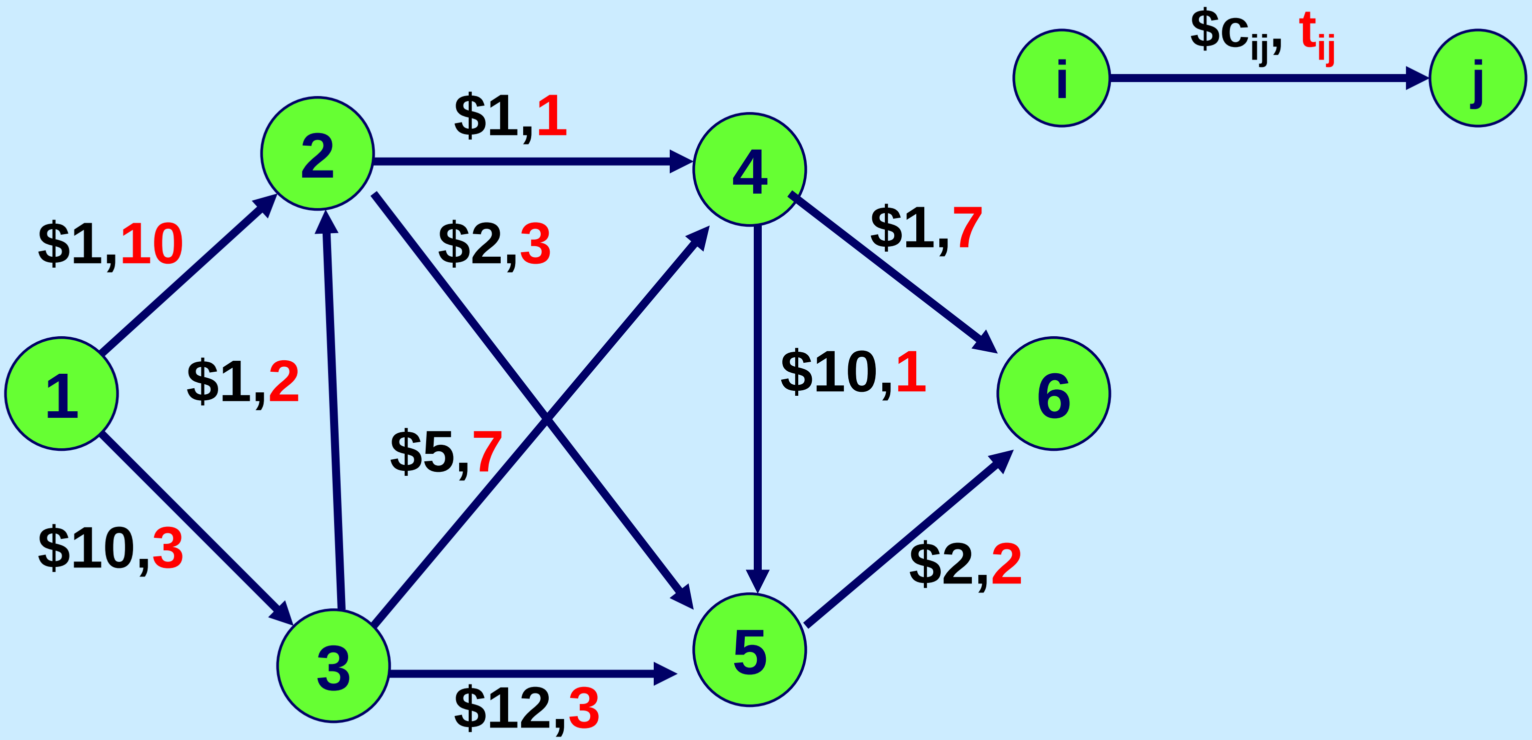
$$\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$$

Complicating constraint

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$$

Example

Find the shortest path from node 1 to node 6 with a transit time at most 10



Shortest Paths with Transit Time Restrictions

- **Shortest path problems are easy.**
- **Shortest path problems with transit time restrictions are NP-hard.**

We say that constrained optimization problem Y is a *relaxation* of problem X if Y is obtained from X by eliminating one or more constraints.

We will "relax" the complicating constraint, and then use a "heuristic" of penalizing too much transit time. We will then connect it to the theory of Lagrangian relaxations.

Shortest Paths with Transit Time Restrictions

Step 1. (A Lagrangian relaxation approach). Penalize violation of the constraint in the objective function.

z(λ) = Min $$\sum_{(i,j)\in A} c_{ij}x_{ij} + \lambda\left(\sum_{(i,j)\in A} t_{ij}x_{ij} - T\right)$$

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if i = s} \\ -1 & \text{if i = t} \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{(i,j)\in A} t_{ij}x_{ij} \le T$$ Complicating constraint

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$$

Note: z*(λ) ≤ z* ∀λ ≥ 0

Shortest Paths with Transit Time Restrictions

Step 2. Delete the complicating constraint(s) from the problem. The resulting problem is called the ***Lagrangian relaxation***.

$$L(\lambda) = \text{Min} \sum_{(i,j)\in A} \left(c_{ij} + \lambda t_{ij}\right) x_{ij} - \lambda T$$

$$\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$$

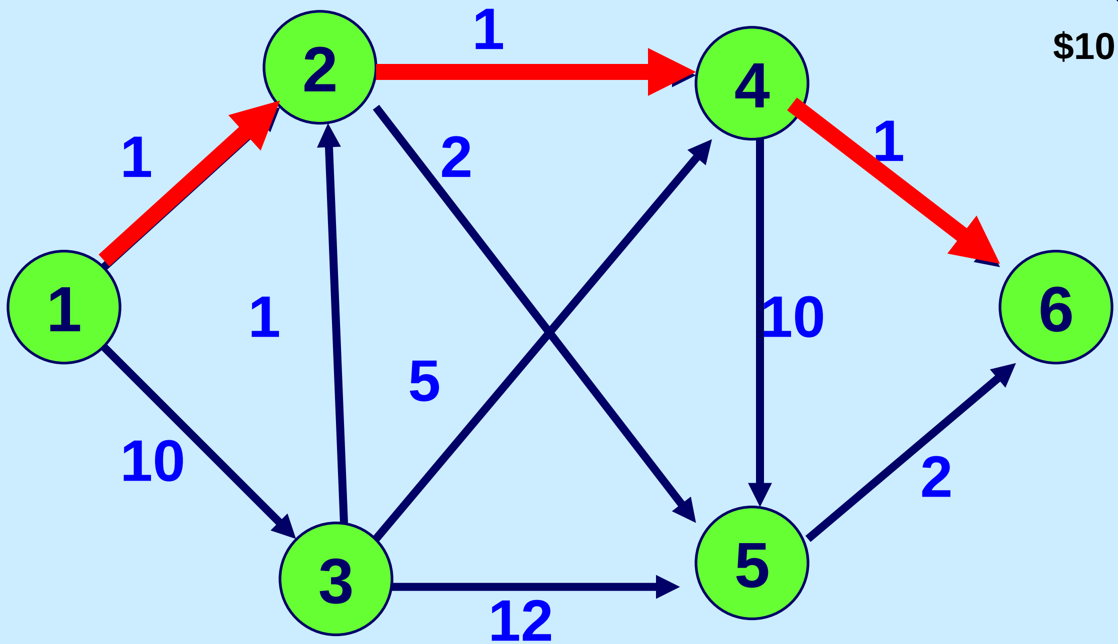
Complicating constraint

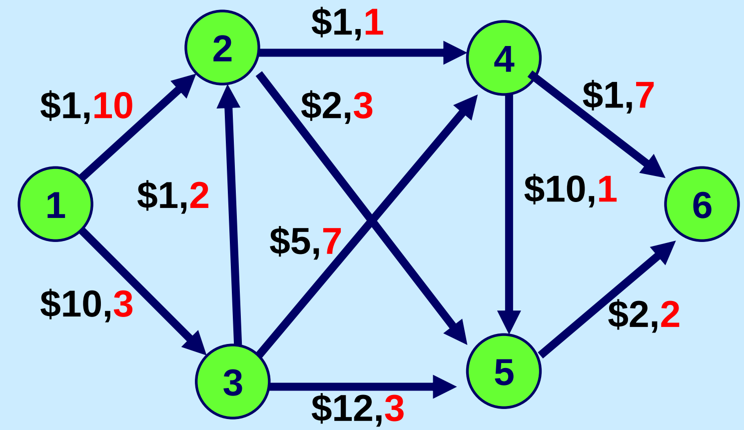
$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$$

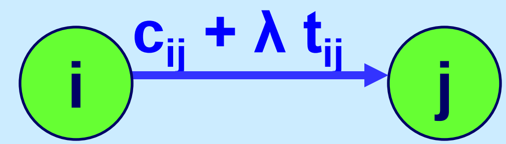
Note: L(λ) ≤ z(λ) ≤ z*   ∀λ ≥ 0

What is the effect of varying λ?

Case 1: λ = 0







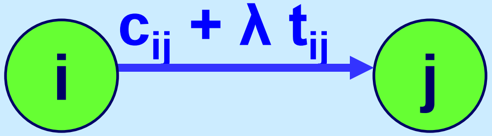
P =	c(P) =	t(P) =

Question to class

If λ = 0, the min cost path is found.

What happens to the (real) cost of the path as λ increases from 0?

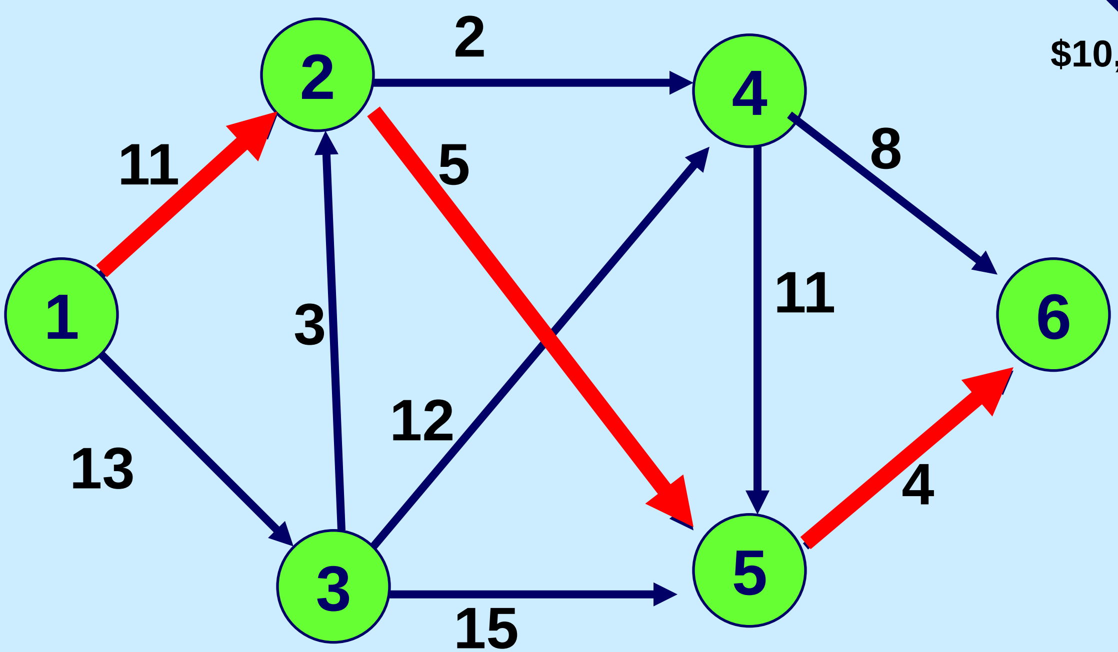
What path is determined as λ gets VERY large?

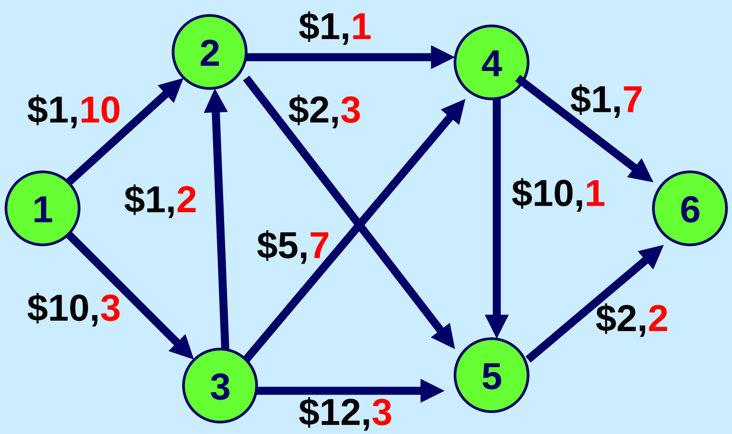


What happens to the (real) transit time of the path as λ increases from 0?

Let λ = 1

Case 2: λ = 1





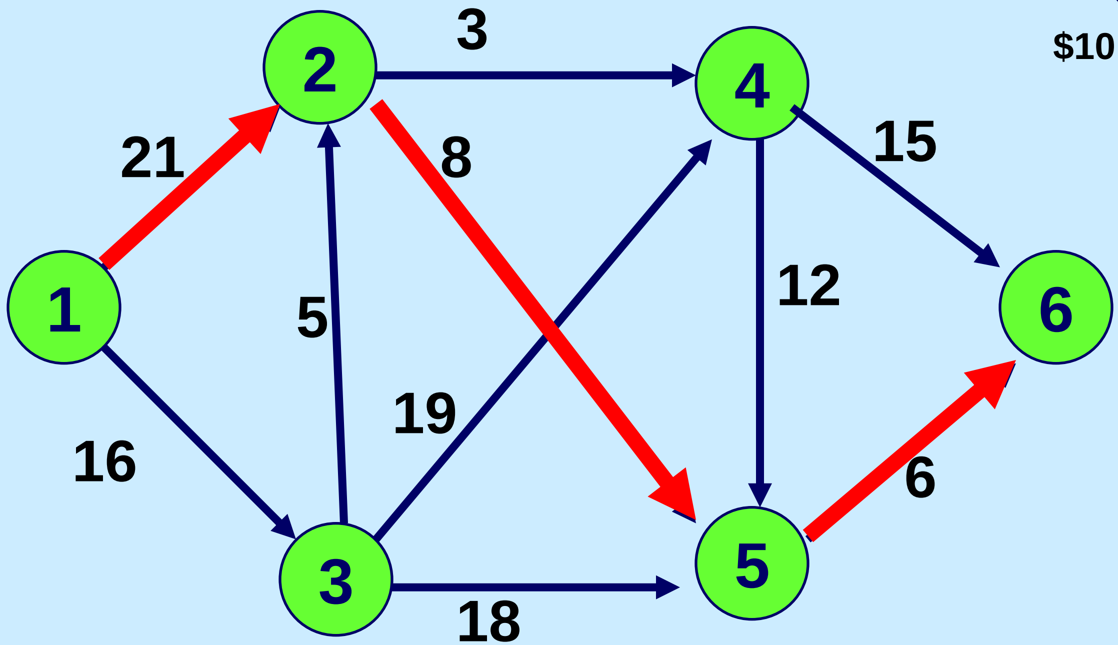
P =

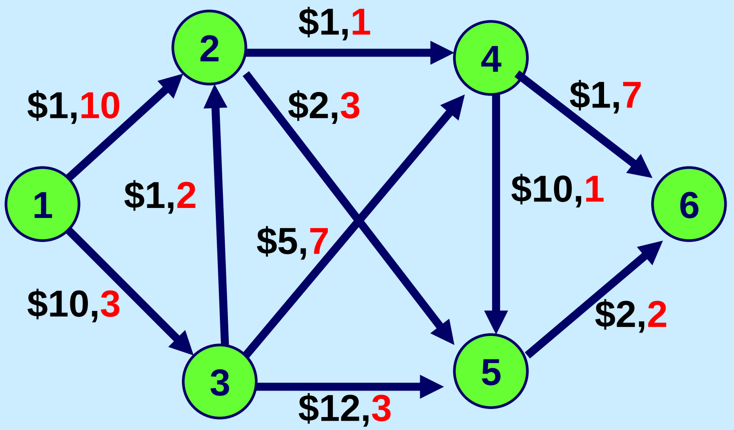
c(P) =

t(P) =

Let λ = 2

Case 3: λ = 2



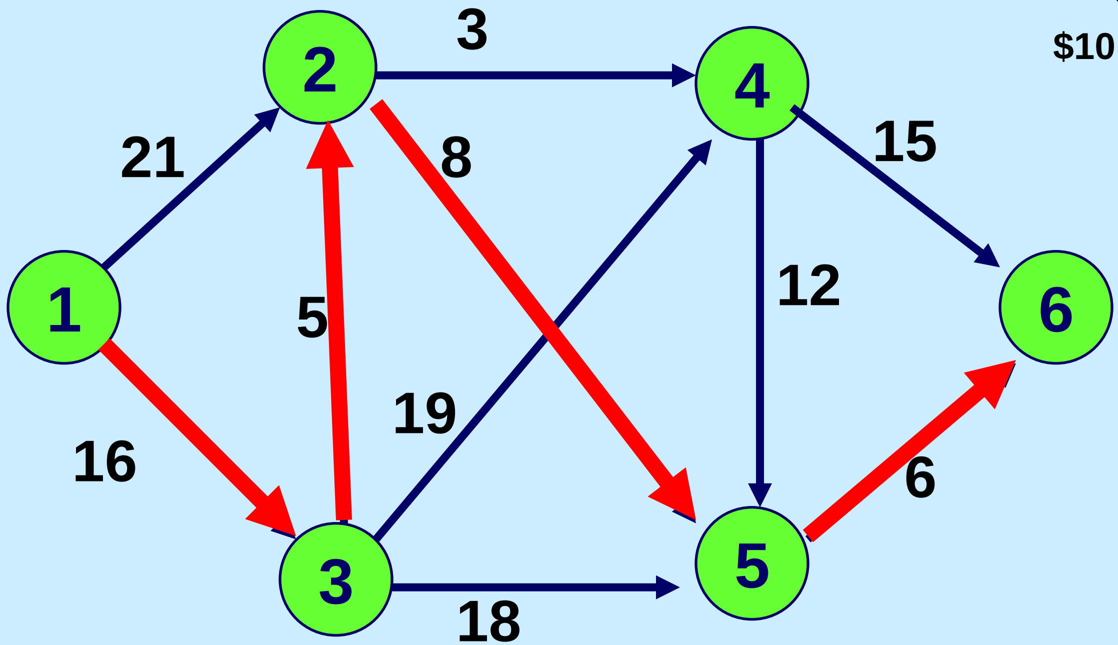


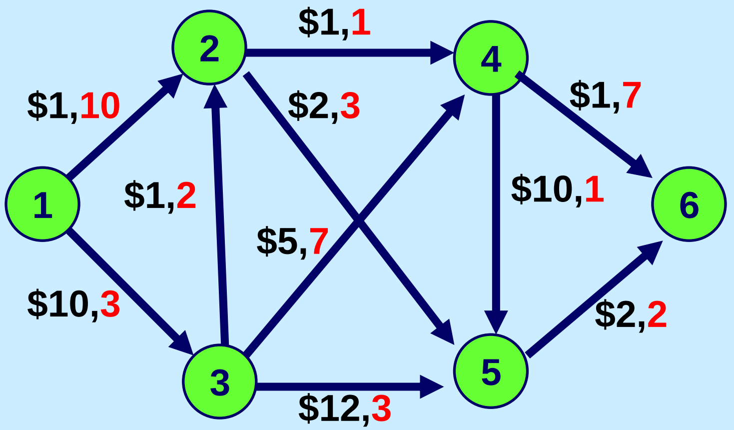
P =

c(P) =

t(P) =

And alternative shortest path when λ = 2



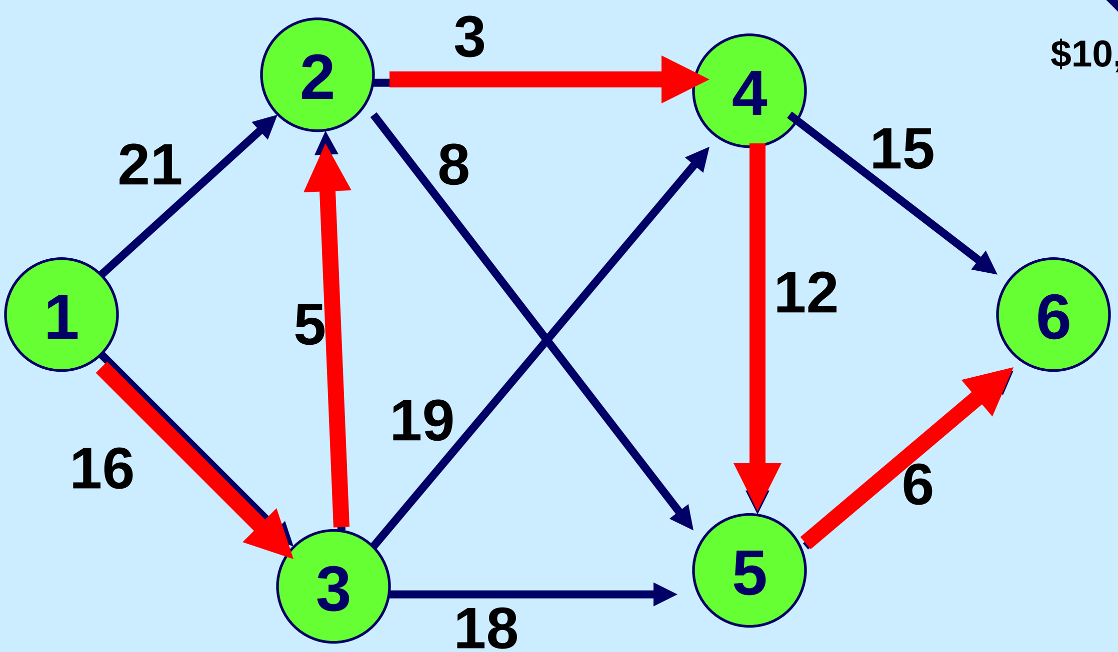


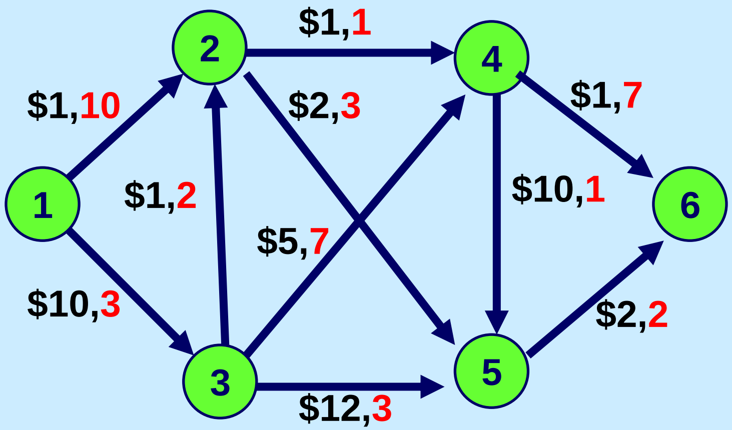
P =	c(P) =	t(P) =

6

Let λ = 5

Case 4: λ = 5





P =

c(P) =

t(P) =

A parametric analysis

Toll	modified cost	Cost	Transit Time	Modified cost -10λ A lower bound on z*
$0 \le \lambda \le 2/3$	$3 + 18\lambda$	3	18	$3 + 8\lambda$
$2/3 \le \lambda \le 2$	$5 + 15\lambda$	5	15	$5 + 3\lambda$
$2 \le \lambda \le 4.5$	$15 + 10\lambda$	15	10	15
$4.5 \le \lambda < \infty$	$24 + 8\lambda$	24	8	$24 - 2\lambda$

The best value of λ is the one that maximizes the lower bound.

Costs
Modified Cost – 10λ
Transit Times
25
20
15
10
5
0
0
5
10
λ

modified cost
120
100
80
60
40
20
0
0
2
4
6
8
10
λ

The Lagrangian Multiplier Problem

L(λ) = min $\sum_{(i,j)\in A}(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$

s.t. $\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if i = 1} \\ -1 & \text{if i = n} \\ 0 & \text{otherwise} \end{cases}$

$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$

L* = max {L(λ) : λ ≥ 0}. **Lagrangian Multiplier Problem**

Theorem. *L(λ) ≤ L* ≤ z*.*

Application to constrained shortest path

L(⌊) = min $\sum_{(i,j)\in A}(c_{ij} + \lambda t_{ij})x_{ij} - \lambda T$

Let c(P) be the cost of path P that satisfies the transit time constraint.

Corollary. For all λ, $L(\lambda) \le L^* \le z^* \le c(P)$.

If $L(\lambda') = c(P)$, then $L(\lambda') = L^* = z^* = c(P)$. In this case, P is an optimal path and λ' optimizes the Lagrangian Multiplier Problem.

More on Lagrangian relaxations

Great technique for obtaining bounds.

Questions?

1. How can one generalize the previous ideas?
2. How good are the bounds? Are there any interesting connections between Lagrangian relaxation bounds and other bounds?
3. What are some other interesting examples?

Mental Break

In 1784, there was a US state that was later merged into another state. Where was this state?

The state was called Franklin. Four years later it was merged into Tennessee.

In the US, it is called Spanish rice. What is it called in Spain?

Spanish rice is unknown in Spain. It is called "rice" in Mexico.

Why does Saudi Arabia import sand from other countries?

Their desert sand is not suitable for construction.

Mental Break

In Tokyo it is expensive to place classified ads in their newspaper. How much does a 3-line ad cost per day?

More than $3,500.

Where is the largest Gothic cathedral in the world?

New York City. It is the Cathedral of Saint John the Divine.

The Tyburn Convent is partially located in London's smallest house. How wide is the house?

Approximately 3.5 feet, or a little over 1 meter.

The Lagrangian Relaxation Technique
(Case 1: equality constraints)

$$z^* = \min \; cx$$

s.t. $Ax = b$

$x \in X$

P

$$L(\mu) = \min \; cx + \mu(Ax - b)$$

P(μ)

s.t. $x \in X$

Lemma 16.1. *For all vectors μ, L(μ) ≤ z*.*

The Lagrangian Multiplier Problem (obtaining better bounds)

$$L(\mu) = \min \; cx + \mu(Ax - b)$$

s.t. $x \in X$ P(μ)

A bound for a minimization problem is better if it is higher. The problem of finding the best bound is called the Lagrangian multiplier problem. $L^* = \max(L(\mu) : \mu \in \mathbb{R}^n)$

Lemma 16.2. *For all vectors μ, L(μ) ≤ L* ≤ z*.*

Corollary. *If x is feasible for the original problem and if L(μ) = cx, then L(μ) = L* = z* = cx. In this case x is optimal for the original problem and μ optimizes the Lagrangian multiplier problem.*

Lagrangian Relaxation and Inequality Constraints

$$z^* = \text{Min } cx \quad \text{subject to} \quad Ax \leq b, \quad x \in X. \qquad (P^*)$$

$$L(\mu) = \text{Min } cx + \mu(Ax - b) \quad \text{subject to} \quad x \in X, \qquad (P^*(\mu))$$

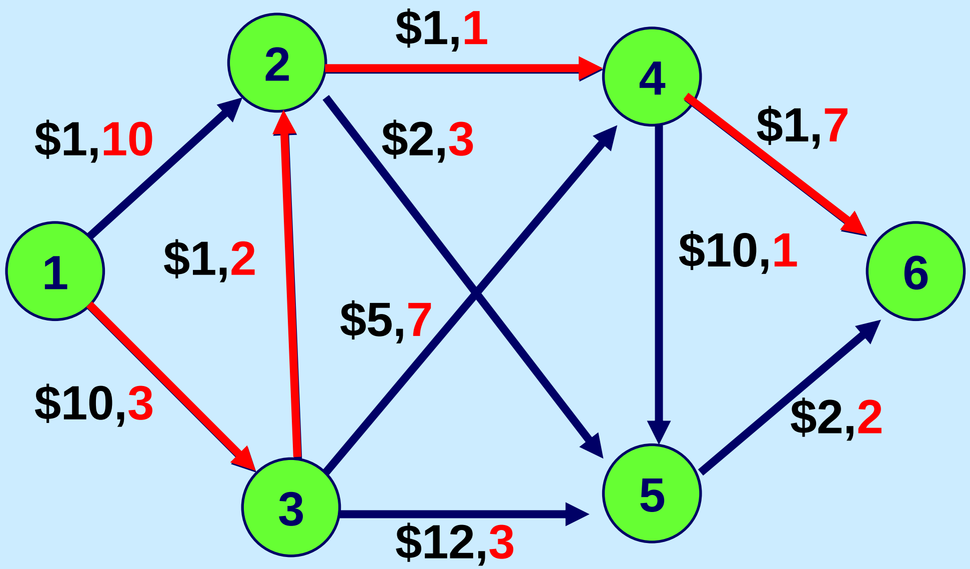
Lemma. $L(\mu) \leq z^*$ for $\mu \geq 0$.

The Lagrange Multiplier Problem: maximize $(L(\mu) : \mu \geq 0)$.

Suppose L* denotes the optimal objective value, and suppose x is feasible for P^* and $\mu \geq 0$. Then $L(\mu) \leq L^* \leq z^* \leq cx$.

A connection between Lagrangian Relaxations and LPs

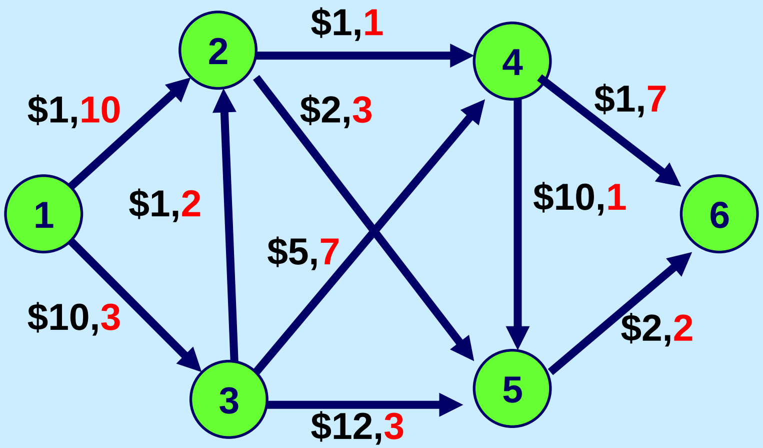
Consider the constrained shortest path problem, but with T = 13.



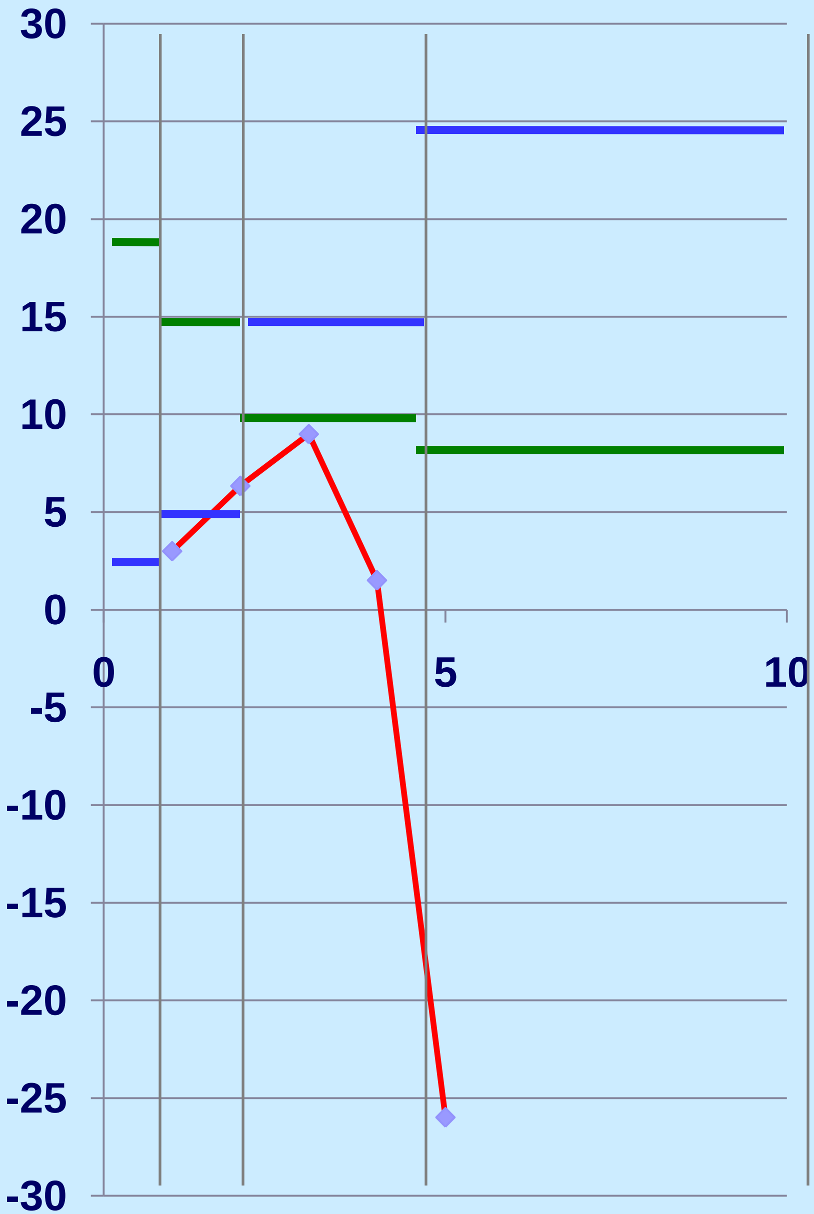
What is the min cost path with transit time at most 13?

Sometimes the Lagrangian bound isn't tight.

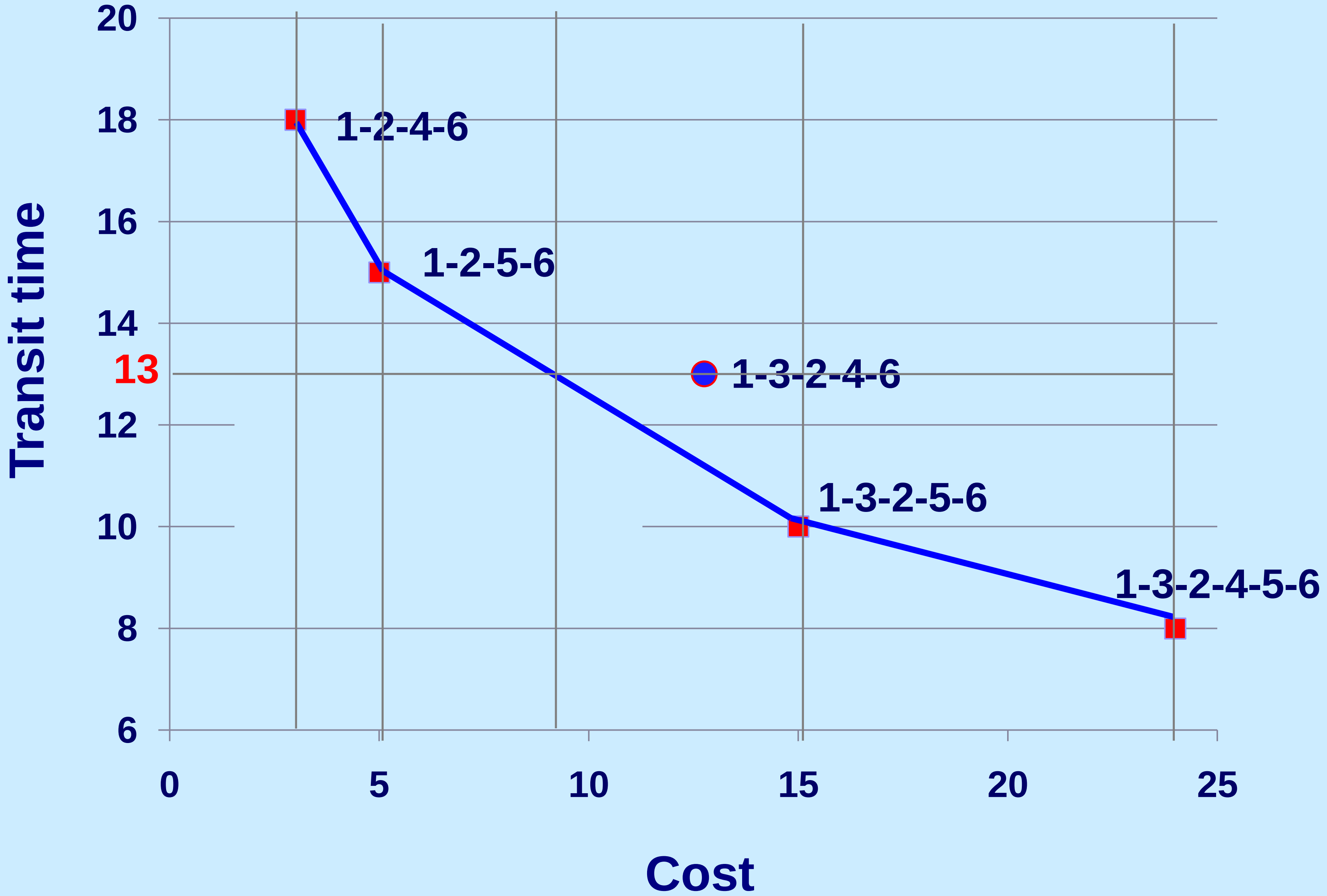
Consider the constrained shortest path problem, but with T = 13.



What is L*, the optimum solution for the lagrangian dual?



Paths obtained by parametric analysis



Application 2 of Lagrangian Relaxation.

Traveling Salesman Problem (TSP)

INPUT: n cities, denoted as 1, . . . , n

c_{ij} = travel distance from city i to city j

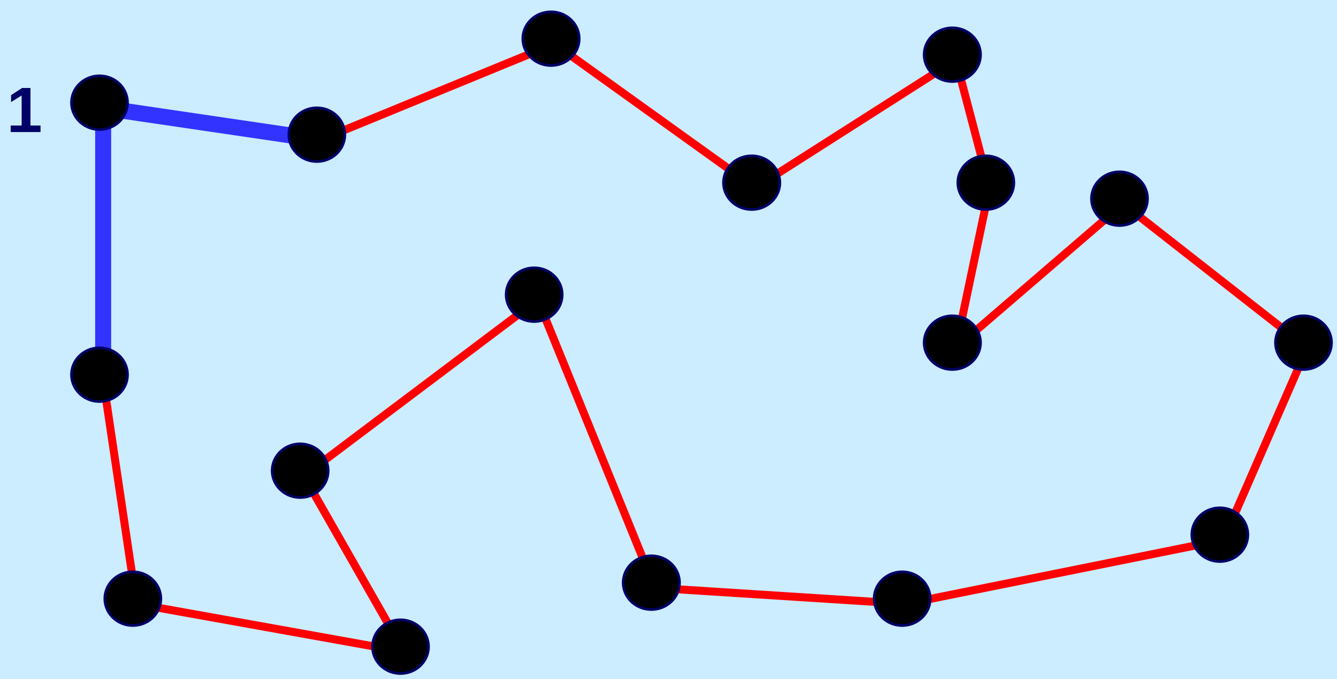
OUTPUT: A minimum distance tour.

Representing the TSP problem

A collection of arcs is a tour if

There are two arcs incident to each node

The red arcs (those not incident to node 1) form a spanning tree in G\1.



A Lagrangian Relaxation for the TSP

Let A(j) be the arcs incident to node j.

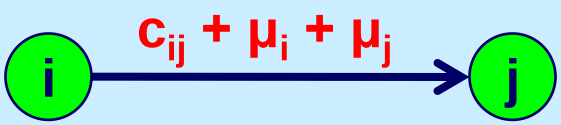
Let X denote all 1-trees, that is, there are two arcs incident to node 1, and deleting these arcs leaves a tree.

$$z^* = \min \sum_e c_e x_e$$

$$\sum_{e \in A(j)} x_e = 2 \quad \text{for each } j = 1 \text{ to } n$$

$$x \in X$$

P



$$L(\mu) = \min \sum_e c_e^\mu x_e - 2\sum_j \mu_j$$

$$x \in X$$

P(μ)

where for e = (i,j), $c_e^\mu = c_e + \mu_i + \mu_j$

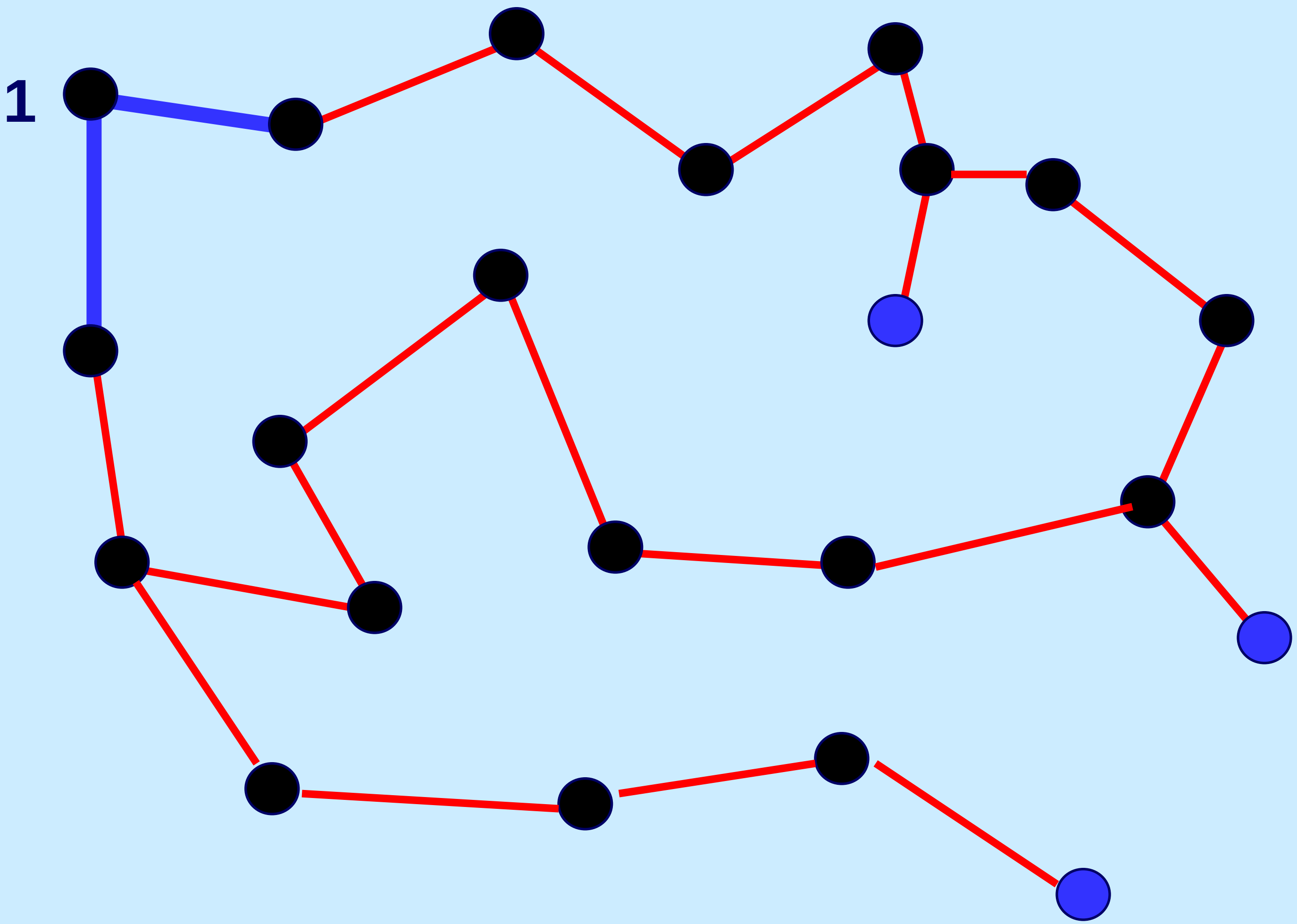
More on the TSP

This Lagrangian Relaxation was formulated by Held and Karp [1970 and 1971].

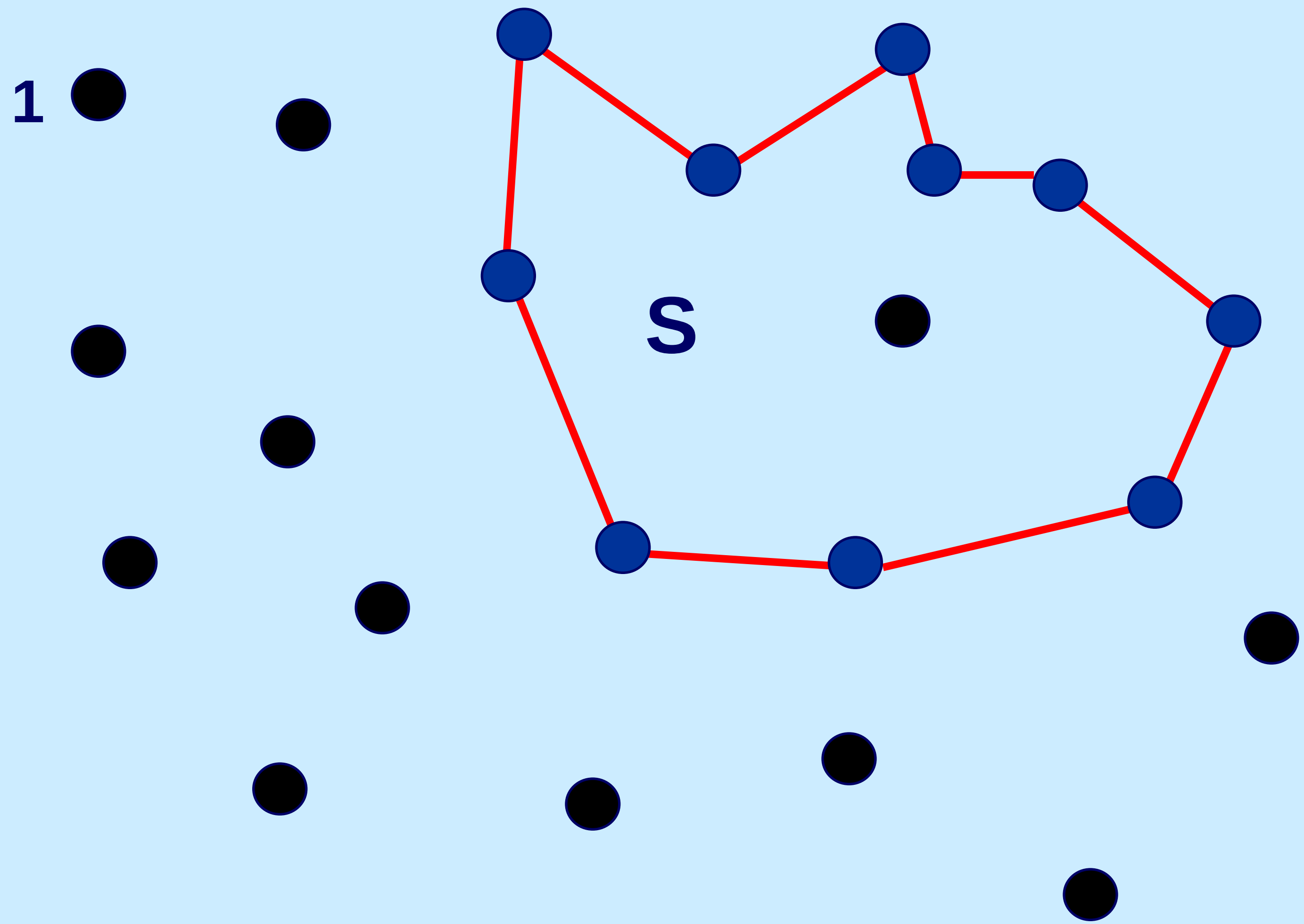
Seminal paper showing how useful Lagrangian Relaxation is in integer programming.

The solution to the Lagrange Multiplier Problem gives an excellent solution, and it tends to be "close" to a tour.

An optimal spanning tree for the Lagrangian problem L(μ*) for optimal μ* usually has few leaf nodes.



Towards a different Lagrangian Relaxation



In a tour, the number of arcs with both endpoints in S is at most |S| - 1 for |S| < n

Another Lagrangian Relaxation for the TSP

$$z^* = \min \sum_e c_e x_e$$

$$\sum_{e \in A(j)} x_e = 2 \quad \text{for each } j = 1 \text{ to } n$$

$$\sum_{e \in S} x_e \le |S| - 1 \quad \text{for each strict subset } S \text{ of } N$$

$$L(\mu) = \min \sum_e c_e^\mu x_e - 2\sum_j \mu_j$$

$$\sum_{e \in S} x_e \le |S| - 1 \quad \text{for each strict subset } S \text{ of } N$$

$$\sum_e x_e = n$$

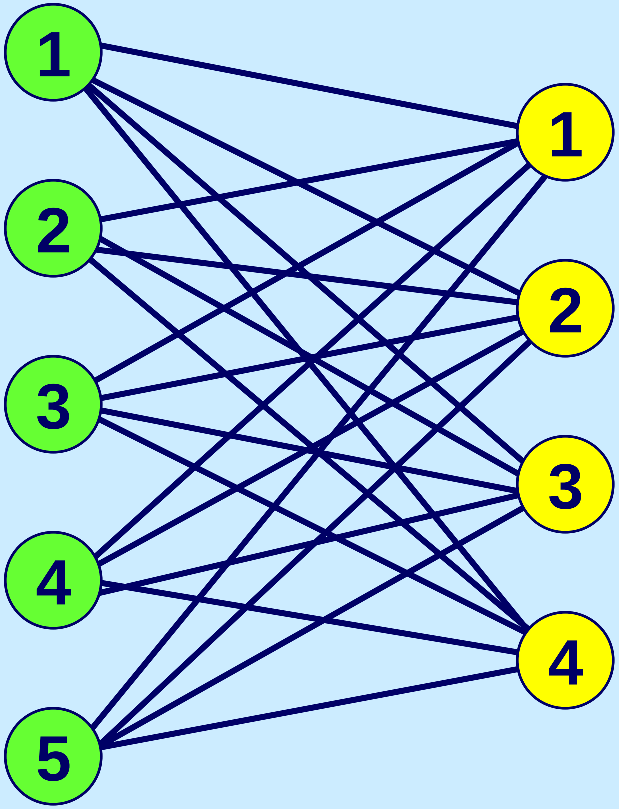
where for e = (i,j), $c_e^\mu = c_e + \mu_i + \mu_j$

A surprising fact: this relaxation gives exactly the same bound as the 1-tree relaxation for each μ.

Summary

- Constrained shortest path problem
- Lagrangian relaxations
- Lagrangian multiplier problem
- Application to TSP
- Next lecture: a little more theory. Some more applications.

Generalized assignment problem ex. 16.8 Ross and Soland [1975]



Set I of jobs

Set J of machines

a_{ij} = the amount of processing time of job i on machine j

x_{ij} = 1 if job i is processed on machine j
= 0 otherwise

Job i gets processed.

Machine j has at most d_j units of processing

Generalized assignment problem ex. 16.8 Ross and Soland [1975]

Minimize $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$ (16.10a)

$\sum_{j \in J} x_{ij} = 1$ for each $i \in I$ (16.10b)

$\sum_{i \in I} a_{ij} x_{ij} \leq d_j$ for each $j \in J$ (16.10c)

$x_{ij} \geq 0$ and integer for all $(i, j) \in A$ (16.10d)

Generalized flow with integer constraints.

Class exercise: write two different Lagrangian relaxations.

Facility Location Problem ex. 16.9 Erlenkotter 1978

Consider a set J of potential facilities

- **Opening facility $j \in J$ incurs a cost F_j.**
- **The capacity of facility j is K_j.**

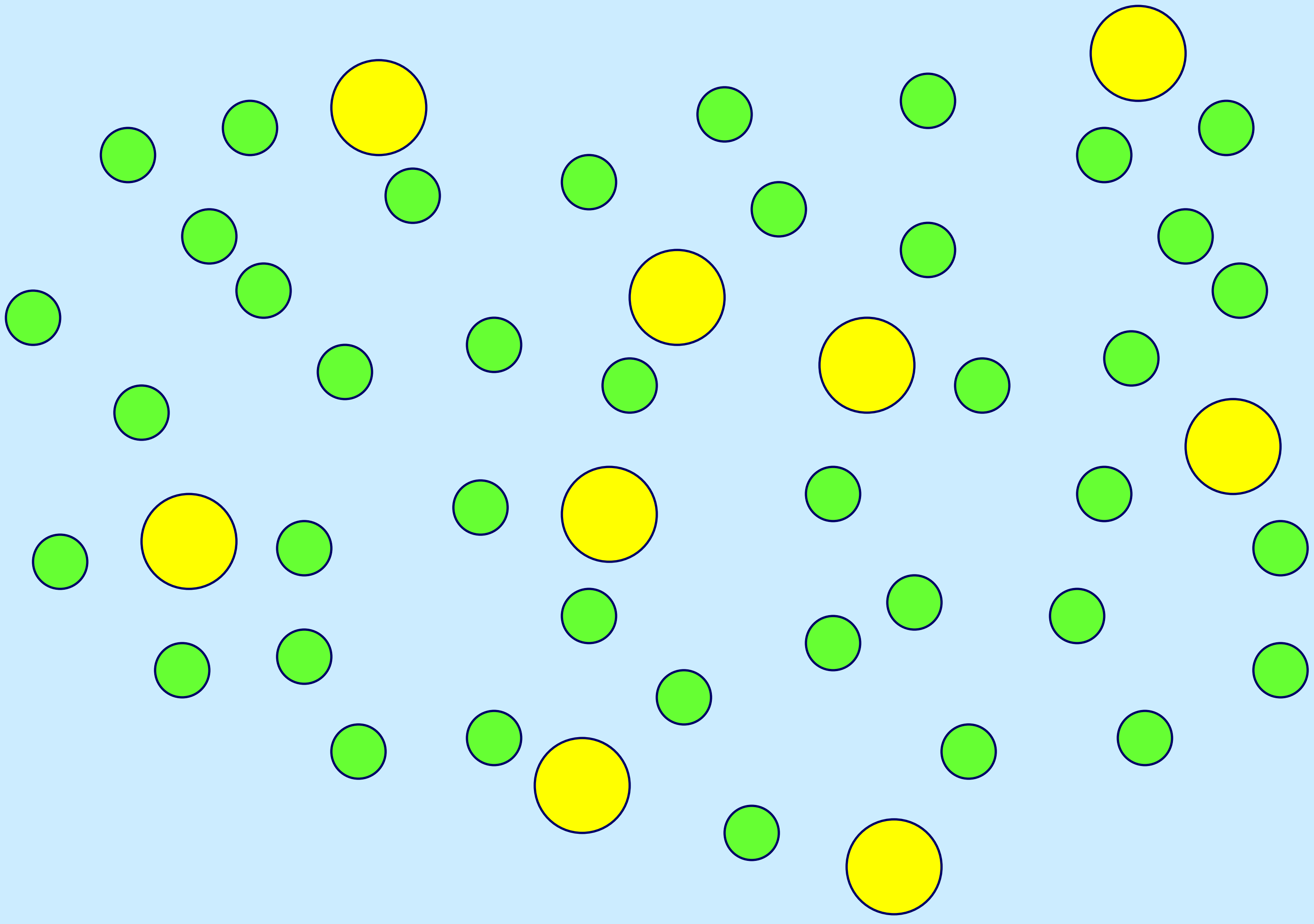
Consider a set I of customers that must be served

- **The total demand of customer i is d_i.**
- **Serving one unit of customer i's from location j costs c_{ij}.**

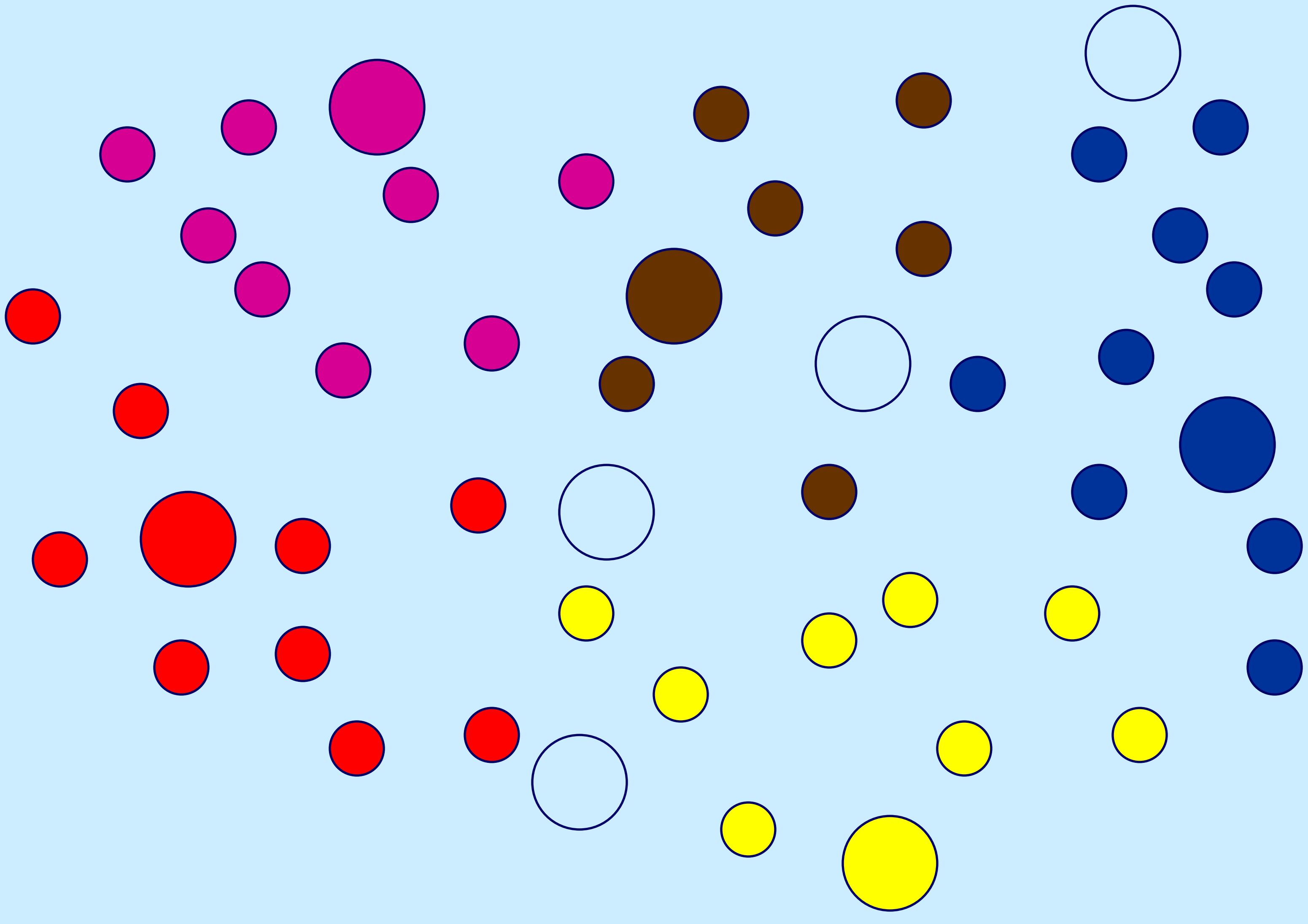
customer

potential facility

A pictorial representation



A possible solution



Class Exercise

Formulate the facility location problem as an integer program. Assume that a customer can be served by more than one facility.

Suggest a way that Lagrangian Relaxation can be used to help solve this problem.

Let x_{ij} be the amount of demand of customer i served by facility j.

Let y_j be 1 if facility j is opened, and 0 otherwise.

The facility location model

Minimize $$\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J} F_j y_j$$

subject to $$\sum_{j\in J} x_{ij} = 1 \qquad \text{for all } i \in I$$

$$\sum_{i\in I} d_i x_{ij} \le K_j y_j \qquad \text{for all } j \in J$$

$$0 \le x_{ij} \le 1 \qquad \text{for all } i \in I \text{ and } j \in J$$

$$y_j = 0 \text{ or } 1 \qquad \text{for all } j \in J$$

Summary of the Lecture

Lagrangian Relaxation

- Illustration using constrained shortest path
- Bounding principle
- Lagrangian Relaxation in a more general form
- The Lagrangian Multiplier Problem
- Lagrangian Relaxation and inequality constraints
- Very popular approach when relaxing some constraints makes the problem easy

Applications

- TSP
- Generalized assignment
- Facility Location

Next Lecture

Review of Lagrangian Relaxation

Lagrangian Relaxation for Linear Programs

Solving the Lagrangian Multiplier Problem

- **Dantzig-Wolfe decomposition**

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