15.082J & 6.855J & ESD.78J

Shortest Paths 2:

Bucket implementations of Dijkstra’s Algorithm

R-Heaps

A Simple Bucket-based Scheme

Let $C = 1 + \max(c_{ij} : (i,j) \in A)$; then nC is an upper bound on the minimum length path from 1 to n.

RECALL: When we select nodes for Dijkstra's Algorithm we select them in increasing order of distance from node 1.

SIMPLE STORAGE RULE. **Create buckets from 0 to nC.**

Let BUCKET(k) = $\{i \in T: d(i) = k\}$. Buckets are sets of nodes stored as doubly linked lists. O(1) time for insertion and deletion.

Dial's Algorithm

- Whenever d(j) is updated, update the buckets so that the simple bucket scheme remains true.

- The FindMin operation looks for the minimum non-empty bucket.

- To find the minimum non-empty bucket, start where you last left off, and iteratively scan buckets with higher numbers.

Dial's Algorithm

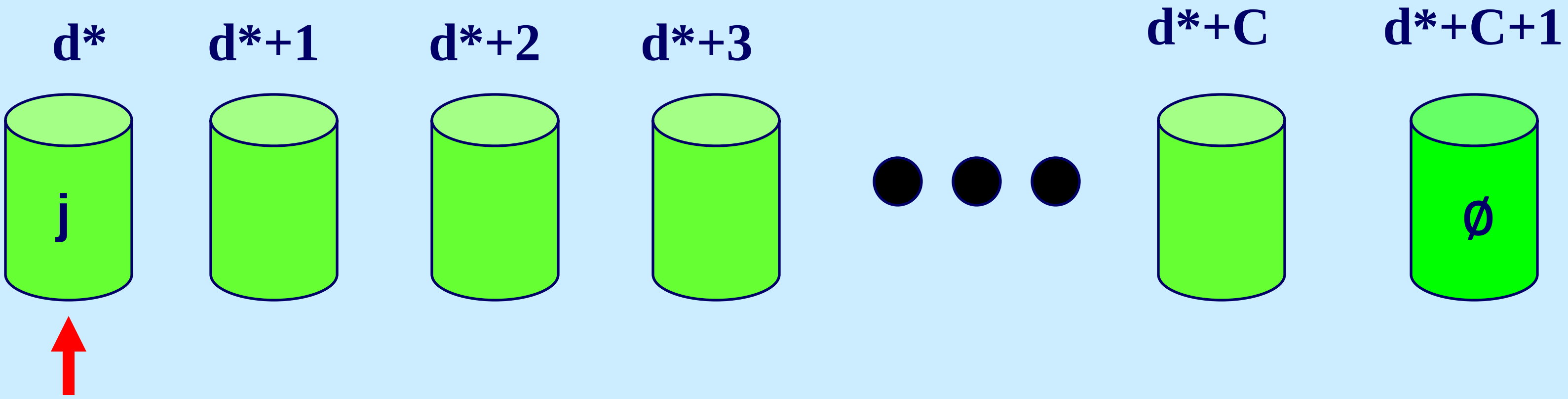
Running time for Dial's Algorithm

- $C = 1 + \max(c_{ij} : (i,j) \in A)$.
- Number of buckets needed. O(nC)
- Time to create buckets. O(nC)
- Time to update d( ) and buckets. O(m)
- Time to find min. O(nC).
- Total running time. O(m+ nC).
- This can be improved in practice; e.g., the space requirements can be reduced to O(C).

Additional comments on Dial's Algorithm

- Create buckets when needed. Stop creating buckets when each node has been stored in a bucket.

- Let $d^* = \max \{d^*(j): j \in N\}$. Then the maximum bucket ever used is at most $d^* + C$.



Suppose $j \in$ Bucket($d^* + C + 1$) after update(i).
But then $d(j) = d(i) + c_{ij} \leq d^* + C$

A 2-level bucket scheme

- Have two levels of buckets.
  - Lower buckets are labeled 0 to K-1 (e.g., K = 10)
  - Upper buckets all have a range of K. First upper bucket's range is K to 2K – 1.
  - Store node j in the bucket whose range contains d(j).

Find Min

- **FindMin consists of two subroutines**
  - **SearchLower: This procedure searches lower buckets from left to right as in Dial's algorithm. When it finds a non-empty bucket, it selects any node in the bucket.**
  - **SearchUpper: This procedure searches upper buckets from left to right. When it finds a bucket that is non-empty, it transfers its elements to lower buckets.**

FindMin

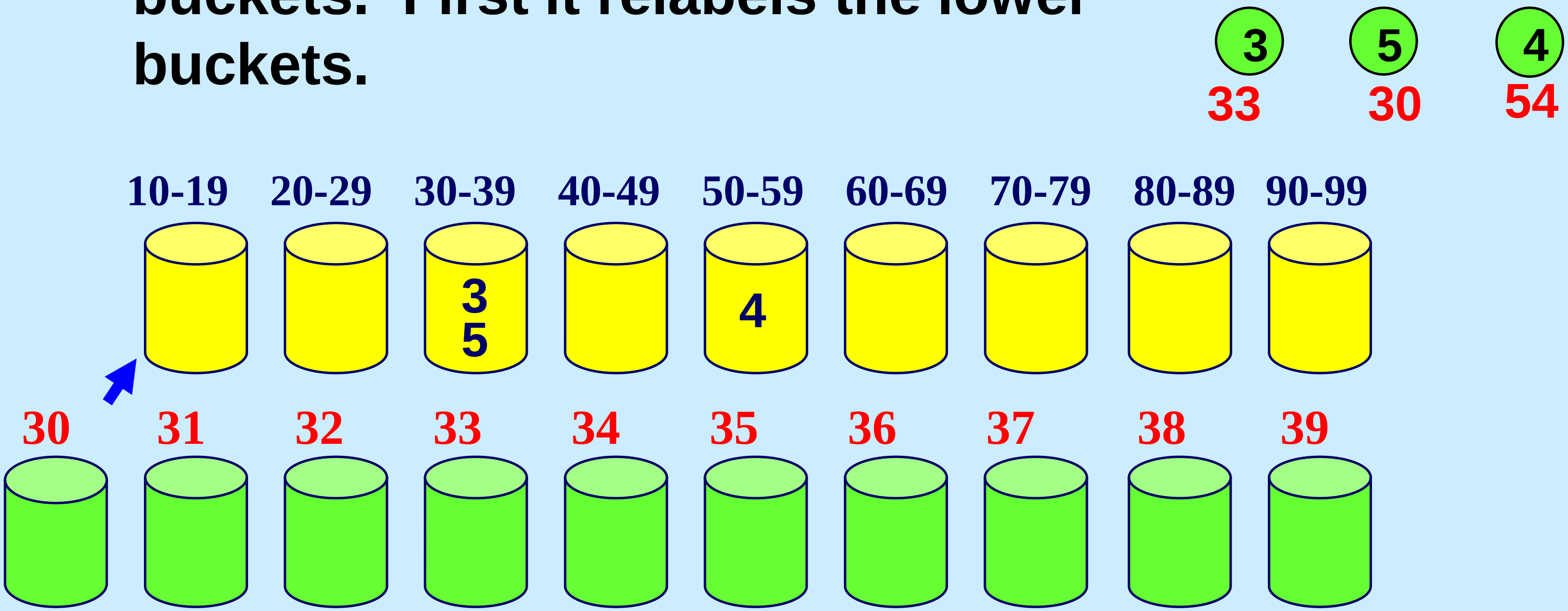
If the lower buckets are non-empty, then SearchLower;
Else, SearchUpper and then SearchLower.

More on SearchUpper

- SearchUpper is carried out when the lower buckets are all empty.

- When SearchUpper finds a non-empty bucket, it transfers its contents to lower buckets. First it relabels the lower buckets.

2-Level Bucket Algorithm



Running Time Analysis

- **Time for SearchUpper: O(nC/K)**
  - **O(1) time per bucket**
- **Number of times that the Lower Buckets are filled from the upper buckets: at most n.**
- **Total time for FindMin in SearchLower**
  - **O(nK); O(1) per bucket scanned.**
- **Total Time for scanning arcs and placing nodes in the correct buckets: O(m)**
- **Total Run Time: O(nC/K + nK + m).**
  - **Optimized when $K = C^{.5}$**
  - **$O(nC^{.5} + m)$**

More on multiple bucket levels

- Running time can be improved with three or more levels of buckets.
- Runs great in practice with two levels
- Can be improved further with buckets of range (width) 1, 1, 2, 4, 8, 16 …
  - Radix Heap Implementation

A Special Purpose Data Structure

- **RADIX HEAP**: a specialized implementation of priority queues for the shortest path problem.

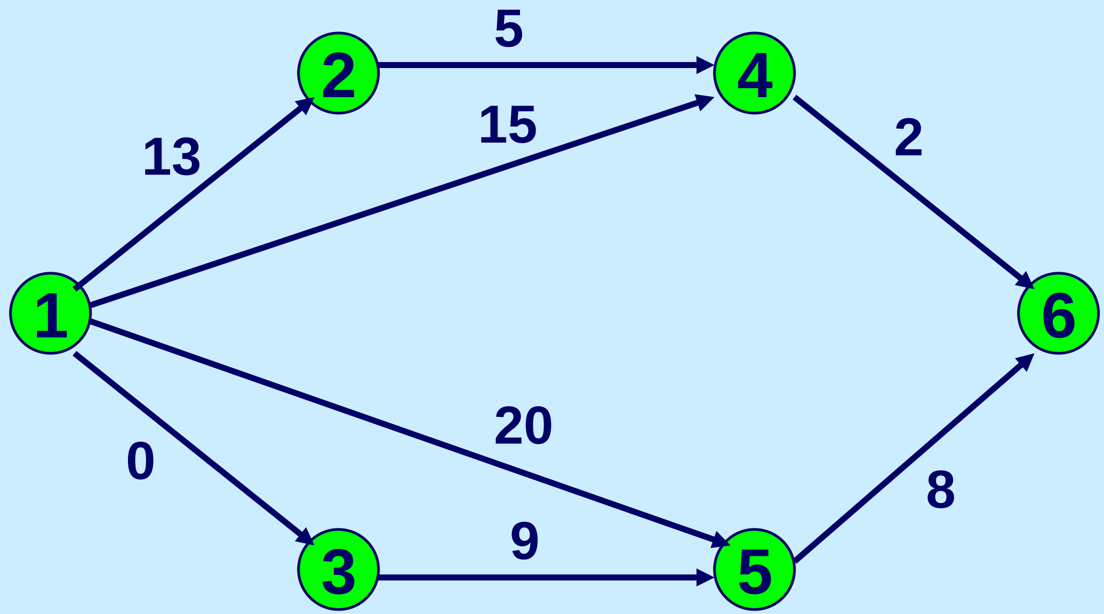
- **A USEFUL PROPERTY** (of Dijkstra's algorithm): The minimum temporary label d( ) is monotonically non-decreasing. The algorithm labels node in order of increasing distance from the origin.

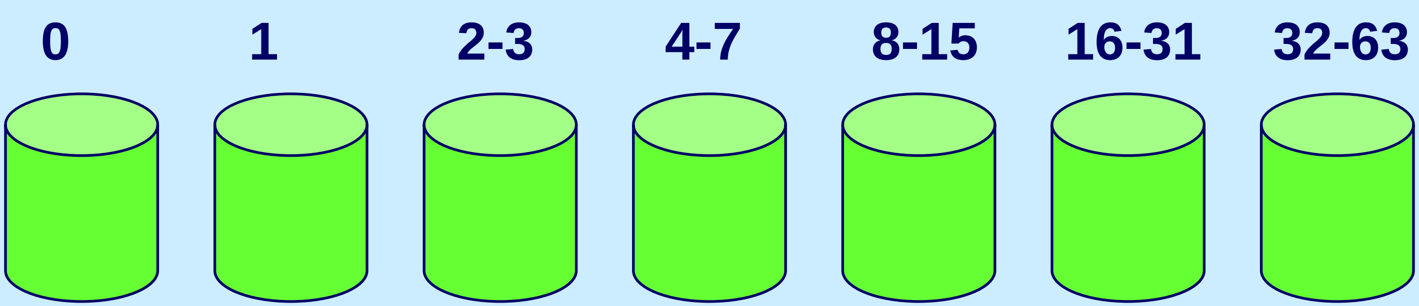
- C = 1 + max length of an arc

Radix Heap Example

Buckets:

- bucket sizes grow exponentially
- ranges change dynamically

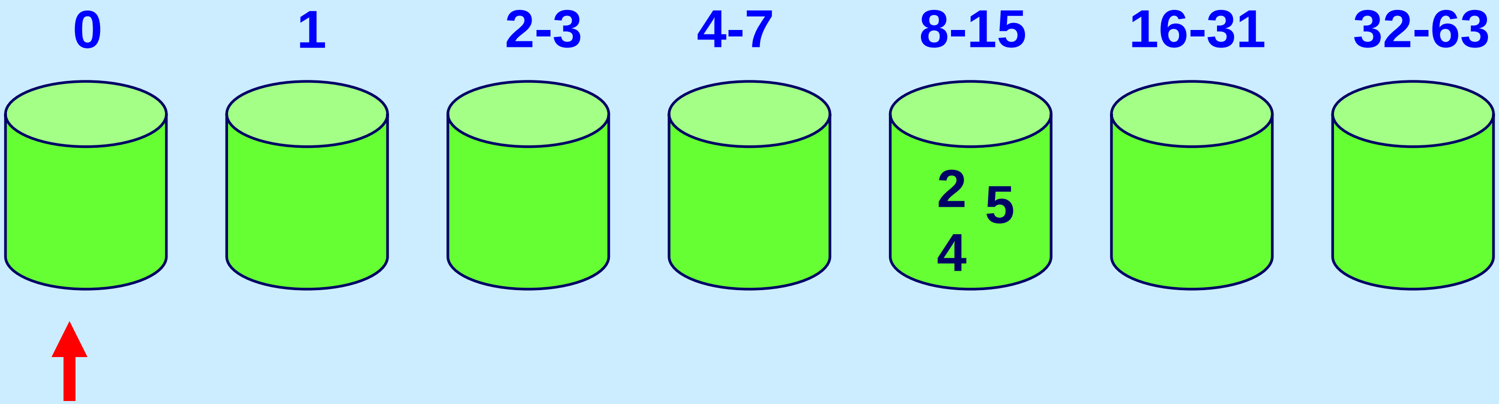




Radix Heap Animation

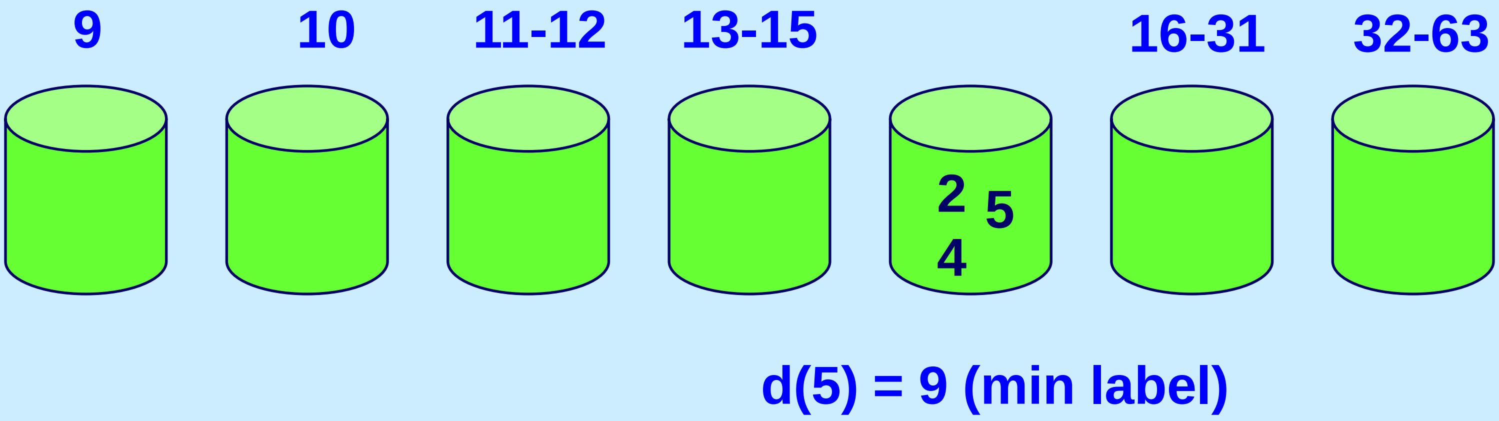
Analysis: FindMin

- **Scan from left to right until there is a non-empty bucket. If the bucket has width 1 or a single element, then select an element of the bucket.**
  - **Time per find min: O(K), where K is the number of buckets**



Analysis: Redistribute Range

- Redistribute Range: suppose that the minimum non-empty bucket is Bucket j. Determine the min distance label d* in the bucket. Then distribute the range of Bucket j into the previous j-1 buckets, starting with value d*.
  - Time per redistribute range: O(K). It takes O(1) steps per bucket.
  - Time for determining d*: see next slide.

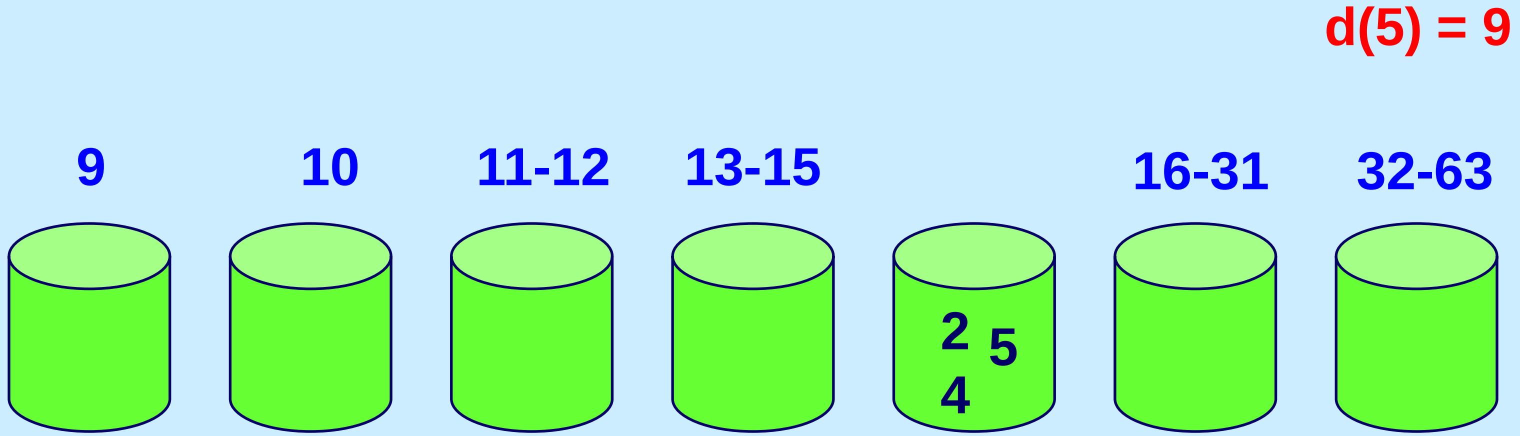


Analysis: Find min d(j) for j in bucket

- **Let b the the number of items in the minimum bucket. The time to find the min distance label of a node in the bucket is O(b).**
  - **Every item in the bucket will move to a lower index bucket after the ranges are redistributed.**
  - **Thus, the time to find d* is dominated by the time to update contents of buckets.**
  - **We analyze that next**

Analysis: Update Contents of Buckets

- When a node j needs to move buckets, it will always shift left. Determine the correct bucket by inspecting buckets one at a time.
  - O(1) whenever we need to scan the bucket to the left.
  - For node j, updating takes O(K) steps in total.



Running time analysis

- FindMin and Redistribute ranges
  - O(K) per iteration. O(nK) in total
- Find minimum d(j) in bucket
  - Dominated by time to update nodes in buckets
- Scanning arcs in Update
  - O(1) per arc.   O(m) in total.
- Updating nodes in Buckets
  - O(K) per node. O(nK) in total
- Running time: O(m + nK)
  O(m + n log nC)
- Can be improved to O(m + n log C)

Summary

- Simple bucket schemes: Dial’s Algorithm
- Double bucket schemes: Denardo and Fox’s Algorithm
- Radix Heap: A bucket based method for shortest path
  - buckets may be redistributed
  - simple implementation leads to a very good running time
  - unusual, global analysis of running time