

# 15.082 Final Exam

## Spring 2003

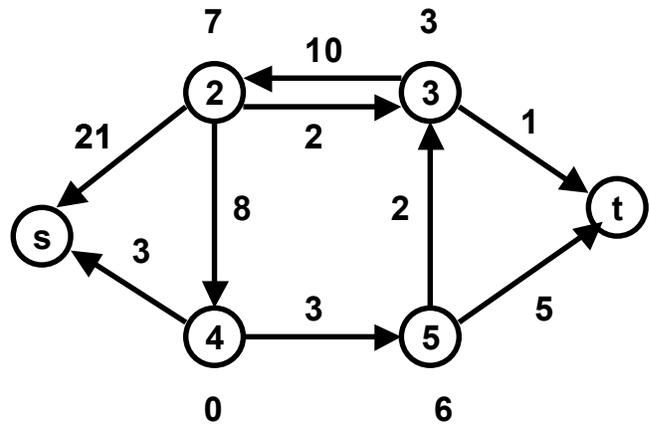
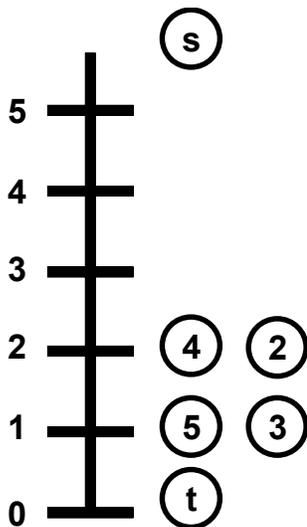
Name \_\_\_\_\_

### Instructions.

1. Please answer all questions in the exam books that are provided.
2. Please budget your time carefully. It is often a good idea to read the entire exam first, so that you can first do the questions that take the least amount of time.
3. Point values are listed on each problem.
4. Questions crossed out in red cover topics not included on Midterm 2, Fall 2010.

**Part 1. (45 points)**

1. (15 points, 2 points for parts a, b and c. 3 points for parts d, e and f.) Consider the following max flow problem given by the preflow push algorithm. The distance labels are given at left. The numbers above or below the nodes are the excesses. The numbers on the arcs are the residual capacities.

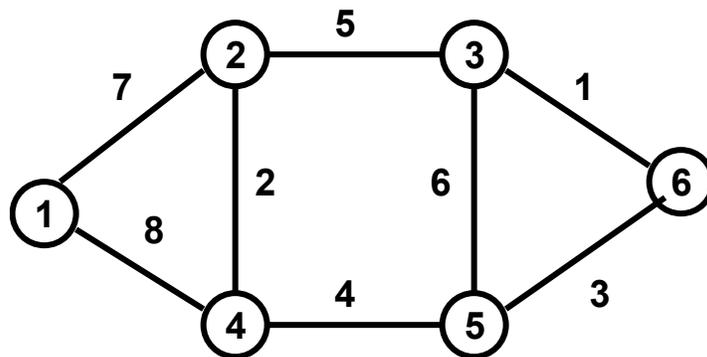


- Is the labeling of the nodes valid? Why or why not?
- What are the admissible arcs?
- Which nodes are active?
- If the highest level push algorithm is used, what is the next node selected? Perform push relabels on that node and stop after the first non-saturating push. Show intermediate steps.
- Suppose that excess scaling is used and that we are at the 8-scaling phase. So no node has an excess of more than 8, and each non-saturating push has at least 4 units of flow. According to the excess scaling algorithm given in class, which node(s) could be selected next for pushing?
- A potential function argument was used to prove the  $O(n^2m)$  time bound in class and in the text. What was the potential function? Evaluate the potential function for the current preflow.

2. (12 points) The following is a feasible flow during some iteration of a minimum cost flow problem. Assume that each lower bound on an arc flow is 0. Recall that the tail of arc  $(i, j)$  is node  $i$  and the head is node  $j$ .

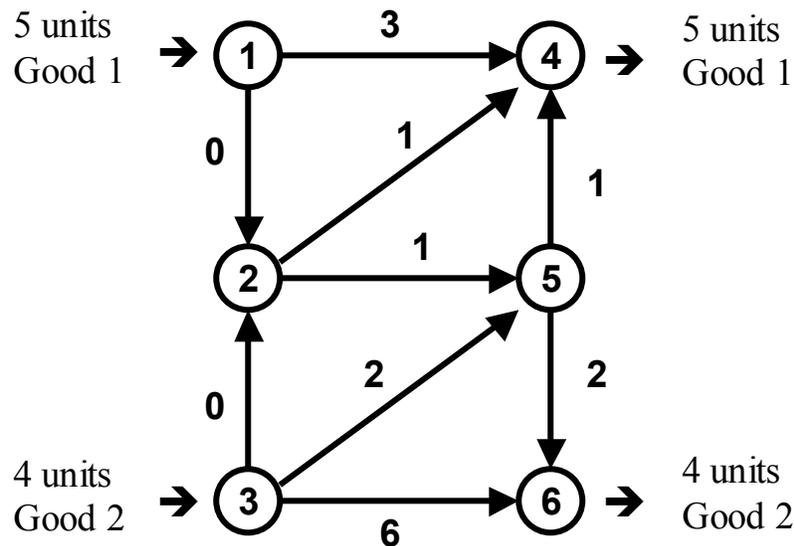
Tail	Head	Capacity	Flow	Cost
1	2	6	5	1
2	4	5	5	-2
4	3	7	4	2
1	3	7	5	2
2	3	4	4	3

- a. (5 points) Write the residual network, and find a negative cost cycle in the residual network, and state the capacity of the cycle. If there is no negative cost cycle, then say so.
- b. (2 points). Use the fact that the flow given above is feasible to determine the supply/demand  $b(j)$  of each node  $j = 1, 2, 3, 4$ .
- c. (5 points) Does the above flow correspond to a spanning tree solution? If not, why not? If so, what is the basic feasible structure, the node potentials, and the reduced costs of the non-basic arcs?
3. ~~(8 points, 4 points per part) Consider the network below for the minimum cost spanning tree problem.~~



- a. What is the minimum cost spanning tree? List the arcs of the minimum cost spanning tree in the order in which they are added by Kruskal's algorithm. You do not need to show steps of the algorithm.
- b. Consider Prim's algorithm for the minimum cost spanning tree starting from node 1. List the arcs of the minimum cost spanning tree in the order they are added to the tree by Prim's algorithm. You do not need to show the steps of the algorithm.

4. ~~(10 points, 5 points per part.) Consider the multicommodity flow problem in the diagram below.~~



The arc numbers are costs. One needs to send 5 units of commodity 1 from node 1 to node 4. One needs to send 4 units of commodity 2 from node 3 to node 6. In addition, there are bundle constraints on arcs (1, 2), (2, 5), and (5, 6).  $u_{12} = 4$ ,  $u_{25} = 6$ , and  $u_{56} = 3$ .

- Write the path formulation for this multicommodity flow problem. (This is the formulation that uses column generation, as discussed in the second lecture on multicommodity flows.)
- Suppose that the first restricted master problem consists of paths 1-4 and 3-6. What is the optimal solution to this restricted master problem, and what are the arc tolls that result? Assuming that the algorithm would add one path from each commodity, what paths would be added next to the restricted master? Please justify your answer.

**Part 2 (25 points)** Modeling and transformations

5. (10 points) Consider a simple variant of the usual maximum flow problem in a network  $G = (N, A)$  in which there is a positive lower bound  $l_{ij}$  on the flow of each arc  $(i,j)$  as well as a positive upper bound  $u_{ij}$  on the flow. Rather than send the most amount of flow from  $s$  to  $t$ , the objective in this problem is to send the least amount of flow from  $s$  to  $t$ , while satisfying the lower bound constraints and conservation of flow at all nodes other than  $s$  and  $t$ . Explain how to solve this problem by *two* applications of a maximum flow algorithm. (HINT: First explain how to determine a feasible flow for the original problem by using one application of a maximum flow algorithm in a network  $G'$  that is closely related to  $G$ .)
6. ~~(15 points, 5 points per part). Consider the minimum cost network flow problem subject to a single side constraint.~~

$$z^* = \min \sum_{(i,j) \in A} c_{ij} x_{ij} \quad (0)$$

$$\text{subject to} \quad \sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = b_i \quad \text{for } i \in N \quad (1)$$

$$\sum_{(i,j) \in A} t_{ij} x_{ij} \leq T \quad (2)$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all } (i, j) \in A \quad (3)$$

$$x \quad \text{integral} \quad (4)$$

Assume that there is a feasible solution to the above integer program.

- What is the Lagrangian problem  $L(\mu)$  if we associate a vector  $\mu$  of multipliers with (1)? Is it true that  $L(\mu) \leq z^*$  for all  $\mu$ ? State the Lagrange multiplier problem, and let  $L^*_1$  denote the optimal solution to the Lagrange multiplier problem (see part c).
- What is the Lagrangian problem  $L(\lambda)$  if we associate a scalar  $\lambda$  for the single constraint (2)? Is it true that  $L(\lambda) \leq z^*$  for all  $\lambda$ ? If not, what is true? Be careful about the sign of  $\lambda$ . State the Lagrange multiplier problem, and let  $L^*_2$  denote the optimal solution to the Lagrange multiplier problem. (see part c).
- Let  $z'$  be the optimal solution to the LP relaxation, that is the LP obtained by relaxing constraints (4) above. Can we conclude that  $z' = L^*_1$ ? Can we conclude that  $z' = L^*_2$ . Justify your answer in both cases.

Part 3. (30 points) Short answer.

7. (7 points). ~~Suppose that a generalized flow problem on a network  $G = (N, A)$  has  $m$  arcs and  $n$  nodes, and that there is at least one cycle that is not a breakeven cycle. Suppose further that there is a feasible solution. First, describe what a breakeven cycle is. Next, describe what a basis structure is and tell how many arcs are in the basis  $F$ .~~
8. (8 points) ~~In the global minimum cut algorithm, we find the minimum cut with node 1 on the  $S$  side of the cut by solving a sequence of minimum cut problems, whose total running time is the same as that of solving a single max flow problem. Suppose that the first minimum cut problem is to find a minimum 1-2 cut, that with node 2 on the  $T$  side of the  $(S, T)$  cut. What is the second cut problem solved? (Explain how the second sink node is obtained. Also explain whether the network for the second cut problem differs from the network from the first cut problem. If so, how does it differ?)~~
9. (15 points, 5 points per part.) Consider a minimum cost flow problem  $G = (N, A)$  with costs  $c$  and upper bounds  $u$ , and supply/demand vector  $b$ . Let  $x^*$  be an optimal flow, and let  $\pi$  be an optimal set of node potentials, that is  $(x^*, \pi)$  satisfies the optimality conditions. An arc  $(i, j)$  is said to be upward critical if increasing the capacity in  $(i, j)$  by one unit decreases (that is, strictly improves) the optimal objective value.
  - a. Write the optimality conditions for the minimum cost flow problem in terms of reduced costs.
  - b. Show how to determine if an arc  $(i, j)$  is upward critical by determining if there is a negative cost cycle in a graph  $G'$  that is closely related to  $G(x)$  the residual network. Explain how to modify  $G(x)$  to obtain  $G'$ .
  - c. Show how to determine if an arc  $(i, j)$  is upward critical by solving a shortest path problem on a related graph  $G''$  in which all arc costs are non-negative. Explain how to modify  $G(x)$  to obtain  $G''$ .

MIT OpenCourseWare  
<http://ocw.mit.edu>

15.082J / 6.855J / ESD.78J Network Optimization  
Fall 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.