Chapter 7 Notes - Inference for Single Samples

- You know already for a large sample, you can invoke the CLT so:

$$\bar{X} \sim N(\mu, \sigma^2).$$

  Also for a large sample, you can replace an unknown σ by s.

- You know how to do a hypothesis test for the mean, either:
  - calculate z-statistic
    $$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$
    and compare it with z_α or $z_{\alpha/2}$.
  - calculate pvalue and compare with α or $\alpha/2$.
  - calculate CI and see whether μ_0 is within it.

Let's add two more calculations.

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1) Determine n to achieve a certain width for a 2-sided confidence interval. Of course, small width $\rightarrow$ large n.

Derivation of Sample Size Calculation for CI

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \quad \textit{(Sample Size Calculation)}$$

where E is the half-width of the CI.

Example

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2) Power Calculation

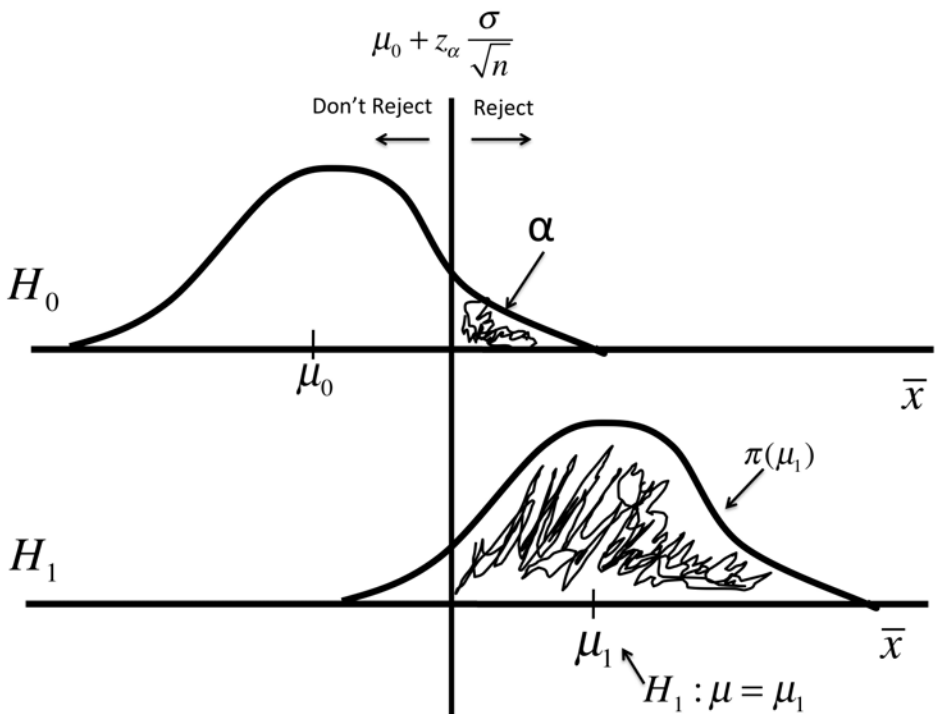
- For upper 1-sided z-tests:

$$\begin{aligned} H_0 &: \mu \leq \mu_0 \\ H_1 &: \mu > \mu_0, \text{ in fact, we'll take } \mu = \mu_1. \end{aligned}$$

  The calculation only makes sense if $\mu_1 > \mu_0$. We want to know what the power of the test is to detect mean μ_1. We'll compute power as a function of μ_1.

  Derivation of Power Calculation for Upper 1-sided z-tests

$$\pi(\mu_1) = P(\text{test rejects } H_0 \text{ in favor of } H_1 | H_1) = \Phi\left(-z_\alpha + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right).$$



Now we can consider $\pi(\mu_1)$ as a function of μ_1. Again, the alternative hypothesis only make sense if $\mu_1 > \mu_0$. As μ_1 increases, what happens to $\pi(\mu_1)$?

- For lower 1-sided tests,

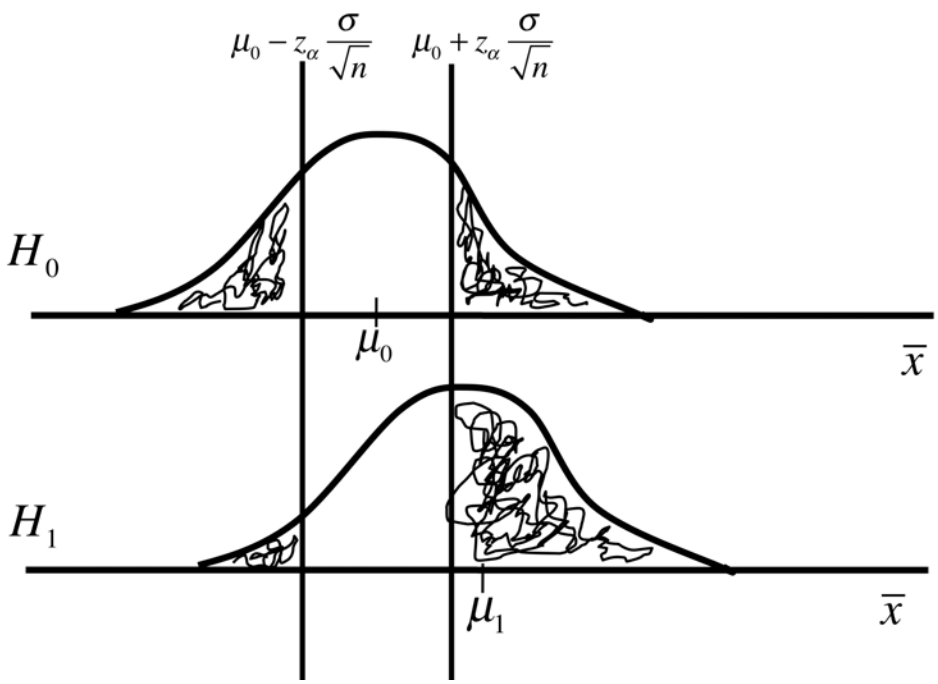
$$\pi(\mu_1) = \Phi\left(-z_\alpha + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}}\right).$$

The alternative hypothesis only makes sense when $\mu_1 < \mu_0$. As μ_1 increases (and gets closer to a μ_0), what happens to $\pi(\mu_1)$?

- For 2-sided tests,

$$\begin{aligned}\pi(\mu_1) &= \left(P \quad \bar{X} < \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\middle|\mu = \mu_1\right) + P\left(\bar{X} > \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\middle|\mu = \mu_1\right) \\ &= \Phi\left(-z_{\alpha/2} + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}}\right) + \Phi\left(-z_{\alpha/2} + \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right)\end{aligned}$$

As μ_1 changes, what happens to $\pi(\mu_1)$?



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3) Sample size calculation for power. Want to find the n required to guarantee a certain power, $1-\beta$, for an α-level z-test.

Let $\delta := \mu_1 - \mu_0$ so that $\mu_1 = \mu_0 + \delta$.

- For upper 1-sided, we have (look up at the power calculation we did for upper 1-sided):

$$\pi(\mu_1) = \pi(\mu_0 + \delta) = \Phi\left(-z_\alpha + \frac{\delta}{\sigma/\sqrt{n}}\right) = 1 - \beta.$$

Since our notation says that z_β is defined as the number where $\Phi(z_\beta) = 1 - \beta$:

$$-z_\alpha + \frac{\delta}{\sigma/\sqrt{n}} = z_\beta.$$

Now solve that for n:

$$n = \left[\frac{(z_\alpha + z_\beta)\sigma}{\delta}\right]^2.$$

- For lower 1-sided, n is the same by symmetry.
- For 2-sided, turns out one of the two terms of $\pi(\mu_1)$ can be ignored to get an approximation:

$$n \approx \left[\frac{(z_{\alpha/2} + z_\beta)\sigma}{\delta}\right]^2.$$

Remember to round up to the next integer when doing sample-size calculations!

Example

7.2 Inferences on Small Samples

If $n < 30$, we often need to use the t-distribution rather than z-distribution $N(0,1)$ since s doesn't approximate σ very well. Need $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.

The bottom line is that we replace:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad \text{by} \quad T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

for a t-test on the mean. Replace z_α by $t_{n-1,\alpha}$. Replace σ by S. There's a chart in your book on page 253 that summarizes this.

Note that the power calculation is harder for t-tests, so for this class, just say $S \approx \sigma$ and use the normal distribution power calculation. You'll get an approximation.

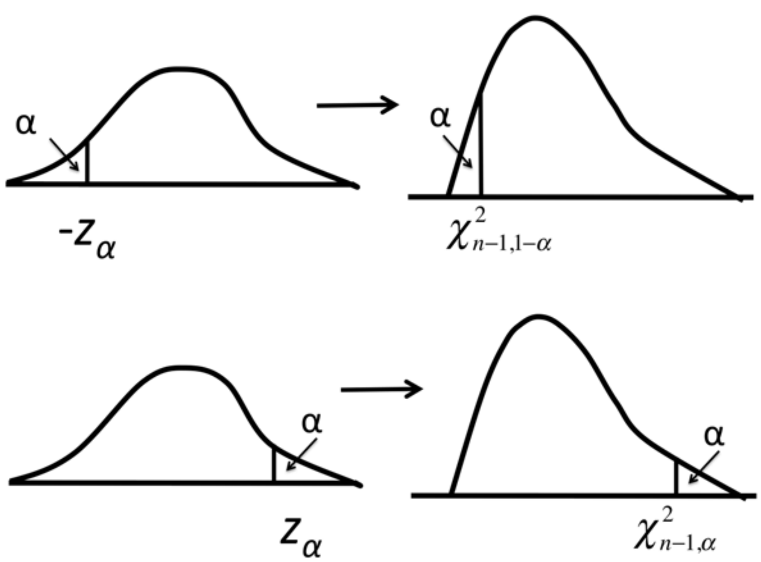
Example

7.3 Inferences on Variances

Assume $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Inferences on variance are very sensitive to this assumption, so inference only with caution!
The bottom line is that we replace:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad \text{by} \quad \chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

(and test for σ^2 not μ). Replace z_α by $\chi^2_{n-1,1-\alpha}$ and/or $\chi^2_{n-1,\alpha}$.



Hypothesis tests on variance are not quite the same as on the mean. Let's do some of the computations to show you. First, we'll compute the CI.

2-sided CI for σ^2. As usual, start with what we know:

$$1-\alpha = P\left(\chi^2_{n-1,1-\alpha/2} \underset{\substack{\Uparrow \\ (*1)}}{\leq} \frac{(n-1)S^2}{\sigma^2} \underset{\substack{\Uparrow \\ (*2)}}{\leq} \chi^2_{n-1,\alpha/2}\right) \quad \text{and remember } \chi^2 = \frac{(n-1)S^2}{\sigma^2},$$

and we want:

$$1-\alpha = P(L \leq \sigma^2 \leq U) \text{ for some } L \text{ and } U.$$

Let's solve it on the left for (*1) and on the right for (*2):

$$\sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}} \qquad \frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} \leq \sigma^2$$

Putting it together we have:

$$\begin{aligned} 1-\alpha &= P\Big[\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\Big] \\ 1-\alpha &= P\Big[L \leq \sigma^2 \leq U\Big]. \end{aligned}$$

The $100(1-\alpha)\%$ confidence interval for σ^2 is then

$$\frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}.$$

Similarly, 1-sided CI's for σ^2 are:

$$\frac{(n-1)s^2}{\chi^2_{n-1,\alpha}} \leq \sigma^2 \quad \text{and} \quad \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha}}.$$

Hypothesis tests on Variance (a chi-square test)

To test $H_0 : \sigma^2 = \sigma_0^2$ vs $H_1 : \sigma^2 \neq \sigma_0^2$, we can either:

- Compute χ^2 statistic:

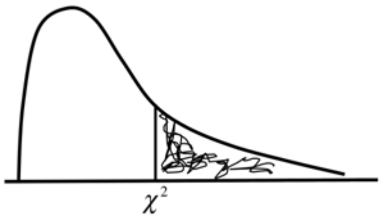
  $$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

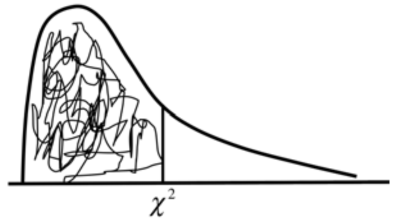
  and reject H_0 when either $\chi^2 > \chi^2_{n-1,\alpha/2}$ or $\chi^2 < \chi^2_{n-1,1-\alpha/2}$.

- Compute pvalue:

  First we calculate the probability to be as extreme in either direction:

$P_U = P(\chi^2_{n-1} \geq \chi^2)$ or $P_L = P(\chi^2_{n-1} \leq \chi^2)$





depending on which is smaller (more extreme). The probability to obtain a χ^2 at least as extreme under H_0 is:

$$2 \min(P_U, P_L).$$

This accounts for being extreme in either direction.

- Compute CI (already done)

Table 7.6 on page 257 summarizes the chi-square hypothesis test on variance.

Note that this is not the most commonly used chi-square test!

See Wikipedia: A chi-square test is any statistical hypothesis test in which the sampling distribution of the test statistic is a chi-square distribution when the null hypothesis is true...

(In this case, we have normal random variables, so the distribution of the test statistic $\frac{(n-1)S^2}{\sigma^2}$ is chi-square.)

Example

15.075J / ESD.07J Statistical Thinking and Data Analysis
Fall 2011