Confidence Intervals

Instead of reporting a "point estimator," that is, a single value, we want to report a confidence interval $[L, U]$ where:

$$P\{L \le \theta \le U\} = 1 - \alpha,$$

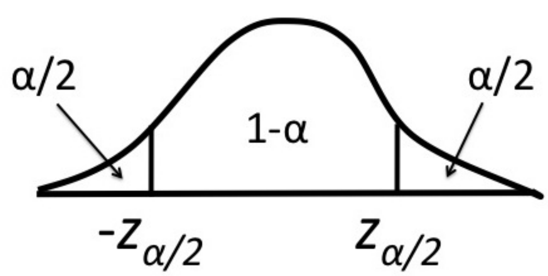
the probability of the true value θ being within $[L, U]$ is pretty large.

Here, $[L, U]$ is a 100$(1-\alpha)$% confidence interval. (Here, $[L, U]$ is two-sided meaning $L \neq -\infty$, $U \neq \infty$.)

Let's derive the confidence intervals for μ when $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ where we assume σ is known. Start with:

$$Z = \frac{(\bar{X} - \mu)}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Now, split area α between the two tails:



(Here we define z_α to solve $P(Z \ge z_\alpha) = \alpha$, meaning that α of the probability mass is on the right of z_α.)
So we know:

$$1 - \alpha = P(-z_{\alpha/2} \underset{(*1)}{\le} Z \underset{(*2)}{\le} z_{\alpha/2}) \quad \text{where } Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}},$$

and we want:

$$1 - \alpha = P(L \le \mu \le U) \text{ for some } L \text{ and } U.$$

Let's solve it on the left for (*1) and on the right for (*2):

$$-z_{\alpha/2} \le Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \qquad Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}$$

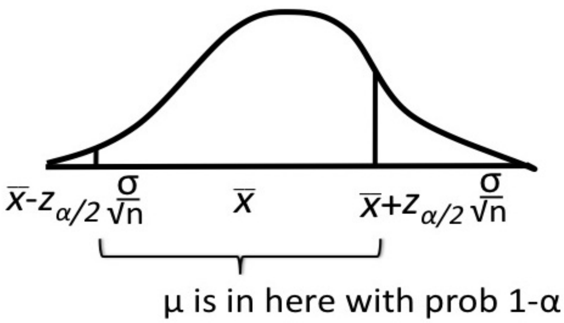
$$-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{X} - \mu \qquad \bar{X} - \mu \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\mu \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \qquad \bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu.$$

Putting it together we have:

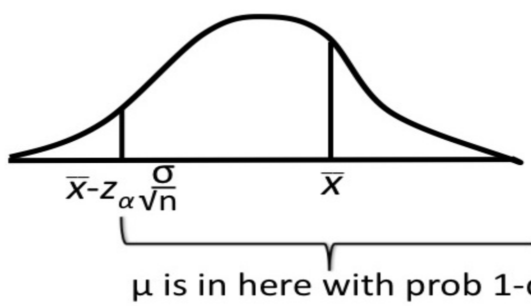
$$
\begin{aligned}
1-\alpha = \quad & P\Big[\quad \bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \quad \mu \le \quad \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\Big] \\
1-\alpha = \quad & P\Big[\quad L \le \quad \mu \le \quad U\Big].
\end{aligned}
$$

This confidence interal $[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$ is a $(1-\alpha)$ level z-interval.

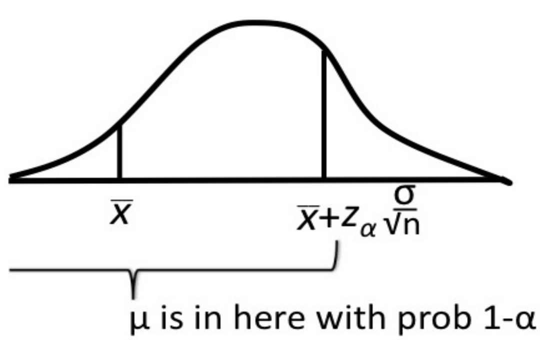


Example (confidence interval for mean revenue)

A one-sided confidence interval can be made as well, by cutting off probability α from only one side of the distribution,



that is, $\mu \ge \bar{X} - z_\alpha\frac{\sigma}{\sqrt{n}}$ or $\mu \in [\bar{X} - z_\alpha\frac{\sigma}{\sqrt{n}}, \infty)$ (Lower 1-sided) and:



$\mu \le \bar{X} + z_\alpha\frac{\sigma}{\sqrt{n}}$ or $\mu \in (-\infty, \bar{X} + z_\alpha\frac{\sigma}{\sqrt{n}}]$ (Upper 1-sided).