15.072
Take home final exam

Given: May 15, 2006 **Due:** May 18, 2006

Note: The work must be done *individually.*

Problem 1 *A device consists of n main units, all of which must be operational for the device to be operational. Successive failure times of the main units are exponentially distributed with rate λ. There are $m + k$ additional units, m of which are active, that is their failure times have the same distribution as the main units, while the remaining k are passive and cannot fail. Failed units are sent for repair. The service time distribution is exponential with rate μ. If some of the main units fail, they are replaced by active units, and these in turn are replaced by passive units. Find the probability that the unit is operational.*

Problem 2 *Consider a closed single class queueing network with N jobs, and let $\pi^+(\boldsymbol{x} - \boldsymbol{e}_1)$ be the probability that the system is in state $\boldsymbol{x} - \boldsymbol{e}_1$ at the departure epoch of a job from node 1. Note that we do not count the departing job. Prove that $\pi^+(\boldsymbol{x} - \boldsymbol{e}_1) = \pi_{N-1}(\boldsymbol{x} - \boldsymbol{e}_1)$, where $\pi_{N-1}(\boldsymbol{x} - \boldsymbol{e}_1)$ is the probability that the state is in $\boldsymbol{x} - \boldsymbol{e}_1$ for a closed queueing network with $N - 1$ jobs.*

Problem 3 *Consider a fluid model (α, μ, P, C). Establish that every server $\sigma_j, 1 \leq j \leq J$ empties eventually in finite time. Namely, establish that for every time t and $j = 1, 2, \ldots, J$ there exists a time $\tau > t$ such that $\sum_{k \in \sigma_j} l_k(\tau) = 0$. Why does not this imply that the fluid model is stable?*

HINT. *Consider the workload $W_j(t)$ corresponding to a given server σ_j.*

Problem 4 *Consider a fluid model (α, μ, P, C) with initial fluid level $l(0) = (l_1(0), \ldots, l_N(0))$. Find a fluid solution $(l(t), u(t))$, not necessarily work-conserving, which empties in* ***shortest*** *time, and express the emptying time τ^* in terms of α, μ, P and $l(0)$. The emptying time of a fluid solution (l, u) is*

$$\tau(l, u) \triangleq \inf\{t : \|l(t)\| = \sum_{1 \leq k \leq N} l_k(t) = 0\}.$$

Thus you need to find $\inf \tau(l, u)$ over all (not necessarily work-conserving) fluid solutions (l, u).

HINT*: First obtain a lower bound on the shortest emptying time and then show that there exists an optimal solution with* constant *u which achieves this lower bound.*

Problem 5 (Extra credit) *Consider single server multiclass fluid model* (α, μ, P) *(*$J = 1$*). Suppose there is a cost* $c = (c_1, \ldots, c_N) \geq 0$ *associated with the fluid level* $l(t)$*. Given a solution* (l, u) *the associated cost is*

$$\int_0^\infty c' l(t) dt.$$

Note that the cost is finite provided that the fluid solution is stable. Find a fluid solution (l^*, u^*) *which minimizes the cost.*