15.072
Homework Assignment 1

Problem 1 (a) **Exercise 1.3** Compare an $M/M/1$ system with arrival rate $\lambda/2$ and service rate μ, with an $M/M/2$ system with arrival rate λ and two servers each having rate μ in terms of the expected number of customers in each system.

(b) **Exercise 1.4** In a semiconductor factory a machine inspects finished products. These arrive in the machine according to a Poisson process of rate λ and are processed for a time interval which is exponentially distributed with rate μ. With probability p the parts pass the test and are ready to be used, while with probability $1-p$ they do not pass inspection and are returned to the inspection machine to be tested again.

(1) What is the ergodicity condition?

(2) Find the expected number of parts in the machine.

(c) **Exercise 1.5** The interdeparture time is the time between successive departures from a queueing system. Consider an $M/M/1$ queue with arrival rate λ and service rate μ.

(1) Derive the probability distribution of the interdeparture time from an $M/M/1$ queue in steady-state.

(2) Prove that the departure process from an $M/M/1$ system is Poisson with rate λ.

(d) **Exercise 1.9** Morning joggers enter a circular ring according to a Poisson process of rate $\lambda_k = \lambda/(k+1)$, $k \geq 0$, which qualitatively captures the phenomenon that a jogger is discouraged to join the ring if there are many people using it. If they enter, they stay in the ring for an exponentially distributed time interval with mean $1/\mu$. Find the distribution of the number of joggers in steady-state.

(e) **Exercise 1.12** Find the transient distribution of the number of customers in an $M/M/\infty$ queue.

Problem 2 Show that Coxian distribution with m stages has coefficient of variation at least $1/m$. Find a distribution for which the CV becomes $1/m$.

Problem 3 Palm-Khintchine Theorem. Special case Consider a sequence of n independent renewal processes observed at infinity. $A_j = \{\tau_1^j, \tau_1^j + \tau_2^j, \ldots, \tau_1^j + \cdots + \tau_m^j, \ldots\}$. All of them have interrenewal times τ_m^j which are i.i.d. with distribution F. We rescale all of the renewal times by a factor n (that is A_j becomes $\{n\tau_1^j, n(\tau_1^j + \tau_2^j), \ldots, n(\tau_1^j + \cdots + \tau_m^j, \ldots)\}$) and consider a superposition $\bar{A}_n = \cup_{1 \le j \le n} A_j$. Establish that $\bar{A}_n$ has in the limit a Poisson distribution:

$$\lim_{n \to \infty} \mathbb{P}(\bar{A}_n(0,t) = k) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t),$$

where $\lambda = 1/\mathbb{E}[\tau_m^j]$.