MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.265/15.070 Fall 2013
Problem Set 4 due 11/13/2013

Problem 1. Consider the longest increasing subsequence problem. Denoting by L_n the length of a longest increasing subsequence of a random permutation, as we have discussed in the lecture, it is known that the limit $\lim_n E[L_n]/\sqrt{n} = \alpha$ exists. Namely $\mathbb{E}[L_n]$ is asymptotically $\alpha\sqrt{n} + o(\sqrt{n})$ We used Talagrand's concentration inequality to obtain concentration of L_n around its median m_n. Show that in fact it must be the case that $\lim_n m_n/(\alpha\sqrt{n})$. Namely the median is also asymptotically $\alpha\sqrt{n} + o(\sqrt{n})$.

Problem 2. Given a graph G with the node set V and edge set E, a set of nodes $I \subset V$ is called an independent set if there is no edge between any two nodes in I. Let Z be the total number of independent sets in G (which is at least one since we assume that the empty set is an independent set) and at most $2^{|V|}$. Let $G = G(n, dn)$ be the Erdös-Rényi graph and let Z_n be the corresponding (random) number of independent sets in Erdös-Rényi . Establish the following concentration inequality for $\log Z_n$ around its expectation $\mathbb{E}[\log Z_n]$:

$$\mathbb{P}(\log Z_n \geq \mathbb{E}[\log Z_n] + t) \leq 2\exp\left(-\frac{t^2}{Cn}\right),$$

for some constant C which does not depend on n.

Problem 3. Exercise 1 lecture 15.

Problem 4. Suppose $X^n \in \mathcal{L}_2$ is a sequence of processes converging to process $X \in \mathcal{L}_2$ in the sense $\mathbb{E}[\int_0^t (X_s^n - X_s)^2 ds] \to 0$ as $n \to \infty$. Recall that as a part of the proof of Proposition 4 in lecture 16 we needed to show that $\mathbb{E}[\int_0^t (X_s^n - X_s)^2 ds]$ is uniformly bounded, namely $\sup_n \mathbb{E}[\int_0^t (X_s^n + X_s)^2 ds] < \infty$. Establish this fact.

Problem 5. The goal of this exercise is to show that much of Ito calculus can be generalized to integration with respect to an arbitrary continuous square integrable martingales.

Let $M_t \in \mathcal{M}_{2,c}$ and $X_t \in \mathcal{L}_2$.

(a) Define $\int_0^t X_s dM_s$ for simple processes. Show that the resulting process is a martingale and establish Ito isometry for it.

(b) Given an arbitrary $X \in \mathcal{L}_2$, show that if X^n is a sequence of simple processes such that $\lim_n \mathbb{E}[\int_0^t (X_s^n - X_s)^2 d\langle M_s \rangle] = 0$, then the sequence $\int_0^t X_s^n dM_s$ is Cauchy for every t in the $\mathbb{L}_2$ sense. Use this to define $\int_0^t X_s dM_s$ for any process $X \in \mathcal{L}_2$ and establish Ito isometry for the Ito integral. Here the integration $d\langle M_s \rangle$ is understood in the Stieltjes sense. You do not need to prove existence of processes X_t^n satisfying the requirement above (unless you would like to).

15.070J / 6.265J Advanced Stochastic Processes
Fall 2013