MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.265/15.070 Fall 2013

Problem Set 1 due 9/16/2009

Problem 1. Consider the space $C[0,T]$ of continuous functions $x : [0,T] \to \mathbb{R}$, endowed with the uniform metric $\rho(x,y) = \|x-y\| = \sup_{0\le t\le T} |x(t) - y(t)|$. Construct an example of a closed bounded set $K \subset C[0,T]$ which is not compact. (A set $K \subset C[0,T]$ is bounded if there exists a large enough r such that $K \subset B(0,r)$, where 0 is a function which is identically zero on $[0,T]$).

Problem 2. Given two metric spaces $(S_1,\rho_1),(S_2,\rho_2)$ show that a function $f : S_1 \to S_2$ is continuous if and only if for every open set $O \subset S_2$, $f^{-1}(O)$ is an open subset of S_1.

Problem 3. Establish that the space $C[0,T]$ is complete with respect to $\|x-y\|$ metric and the space $D[0,T]$ is complete with respect to the Skorohod metric.

Problem 4. Problem 1 from Lecture 2. Additionally to the parts a)-c), construct an example of a random variable X with a finite mean and a number $x_0 > \mathbb{E}[X]$, such that $I(x_0) < \infty$, but $I(x) = \infty$ for all $x > x_0$. Here I is the Legendre transform of the random variable X.

Problem 5. Establish the following fact, (which we have used in proving the upper bound part of the Cramér's theorem for general closed sets F): given two strictly positive sequences $x_n, y_n > 0$, show that if $\limsup_n (1/n)\log x_n \le I, \limsup_n (1/n)\log y_n \le I$, then $\limsup_n (1/n)\log(x_n+y_n) \le I$.

Problem 6. Suppose $M(\theta) < \infty$ for all θ. Show that $I(x)$ is a strictly convex function.

Hint. Give a direct proof of convexity of I and see where inequality may turn into equality. You may use the following fact which we have established in the class: for every x there exists θ_0 such that $x = \dot{M}(\theta_0)/M(\theta_0)$.

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