

# Optimization Methods in Management Science

MIT 15.053

RECITATION 8

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**At the end of this recitation, students should be able to:**

1. Derive Gomory cut from fractional rows of the Simplex tableau.
2. Have an intuition of the geometry of Gomory cuts.
3. Understand the concept of valid cover and of minimal cover for a knapsack problem.

## Problem 1

Consider the linear programming feasible region and the integer program feasible region given in the diagram to the right.

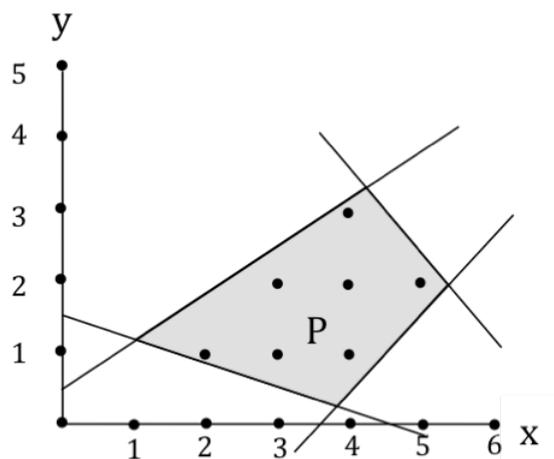


Figure 1: Sketch of the feasible region of the LP relaxation for Problem 1

Which of the following are valid inequalities for the integer program?

- (a)  $x \leq 4$
- (b)  $x \geq 2$
- (c)  $y \geq 1$
- (d)  $y \leq 6$  Yes
- (e)  $x + y \geq 3$ .

## Problem 2

Consider the feasible region of an Integer Program defined by the following constraints:

$$\left. \begin{array}{l} -3x_1 + 5x_2 \leq 12 \\ 4x_1 + 3x_2 \leq 20 \\ x_1 + x_2 \leq 11/2 \\ \forall j = 1, 2 \quad x_j \geq 0 \\ \forall j = 1, 2 \quad x_j \in \mathbb{Z}. \end{array} \right\} (IP)$$

- (a) Sketch the feasible region of the LP relaxation of the problem. Then, determine the convex hull of the feasible integer points of (IP), first graphically, and then algebraically, using your sketch to determine the inequalities that define the convex hull. (Note: the first constraint passes through the points  $(-4, 0)$  and  $(1, 3)$ , and the second constraint passes through the points  $(5, 0)$  and  $(2, 4)$ . This should allow you to draw the feasible region very quickly.)
- (b) Let  $F$  be the feasible region of (IP). (Note:  $F$  is a set of *integer* points, it is *not* the feasible region of the LP relaxation of (IP).) Consider the set  $X$  defined as:  $X = \{x_1 + x_2 \leq 11/2 : x_1, x_2 \in \mathbb{Z}\}$ . What is the relationship between  $F$  and  $X$ , in terms of inclusion?

Now determine a valid inequality for the set  $X$ . Is this inequality also valid for (IP)? Justify your answer based on the relationship between  $F$  and  $X$ . (Recall the definition of valid inequality for a set: it is an inequality that does not cut off any integer feasible point of the set.)

## Problem 3

We want to solve an integer program (which is a maximization problem) using the Gomory cutting plane technique. We first consider the linear programming relaxation with the following initial LP tableau.

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	Rhs
$(-z)$	-2	4	1				0
$s_1$	2	1	1	1			8
$s_2$	-4	4	2		1		8
$s_3$	1	2	3			1	9

After three pivots, we obtain the following optimal tableau for the LP relaxation:

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	Rhs
$(-z)$	0	0	$-2\frac{1}{3}$		$-\frac{2}{3}$	$-\frac{2}{3}$	$-11\frac{1}{3}$
$s_1$			$-1\frac{1}{2}$	1	$\frac{1}{4}$	-1	1
$x_2$		1	$1\frac{1}{6}$		$\frac{1}{12}$	$\frac{1}{3}$	$3\frac{2}{3}$
$x_1$	1		$1\frac{2}{3}$		$-\frac{1}{6}$	$\frac{1}{3}$	$1\frac{2}{3}$

- (a) Derive two Gomory cuts based on the final two constraints of the second tableau.
- (b) Does the basic feasible solution from final tableau satisfy either of the two constraints from Part (a)?

## Problem 4

Consider the knapsack set:  $K = \{x \in \{0, 1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$ . We want to derive valid inequalities for this set.

- (a) For each of the following inequalities, identify whether or not they are valid knapsack covers, and explain why.
- (a)  $x_4 + x_5 + x_6 \leq 2$
  - (b)  $x_1 + x_2 + x_6 \leq 2$
  - (c)  $x_2 + x_3 + x_6 + x_7 \leq 3$
  - (d)  $x_2 + x_4 + x_5 + x_6 \leq 3$
  - (e)  $x_1 + x_3 + x_4 + x_5 \leq 3$
  - (f)  $x_2 + x_3 + x_4 + x_5 + x_6 \leq 4$
- (b) For all valid knapsack covers in Part (a), identify whether or not they are minimal.
- (c) (Advanced material: understanding this question is *not* required for the Midterm.)

Consider the knapsack cover  $C = \{2, 3, 4, 5\}$  that yields the inequality  $x_2 + x_3 + x_4 + x_5 \leq 3$ . Do you see a way of enlarging the cover while keeping the same right-hand side of the inequality? That is: can you find a valid inequality of the form  $x_2 + x_3 + x_4 + x_5 + [\dots] \leq 3$ , where the  $[\dots]$  stands for one or more variables? If you can find such an inequality (called an *extended* cover), do you think that it is stronger or weaker than the initial cover? Justify your answer.

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