Optimization Methods in Management Science

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Recitation 2

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Problem 1

This problem verifies your understanding of the geometry of Linear Programming. The questions are not trivial, therefore consider carefully your answer.

Consider the feasible region depicted in Figure 2 (note that it extends indefinitely towards the upper right part of the graph). Here is a proposition that is valid for all linear programs and will be useful for solving this problem. For any point p, let $c(p)$ be the cost of point p.
Proposition. If point p' is on the line segment joining points p and p'', then:

- $c(p') \geq \min\{c(p), c(p'')\}$, and
- $c(p') \leq \max\{c(p), c(p'')\}$.

For example, in Figure 2 we have $c(E) \geq \min\{c(D), c(F)\}$, and $c(E) \leq \max\{c(D), c(F)\}$. This follows from simple linear algebra: if p' is on the line segment between p and p'', then

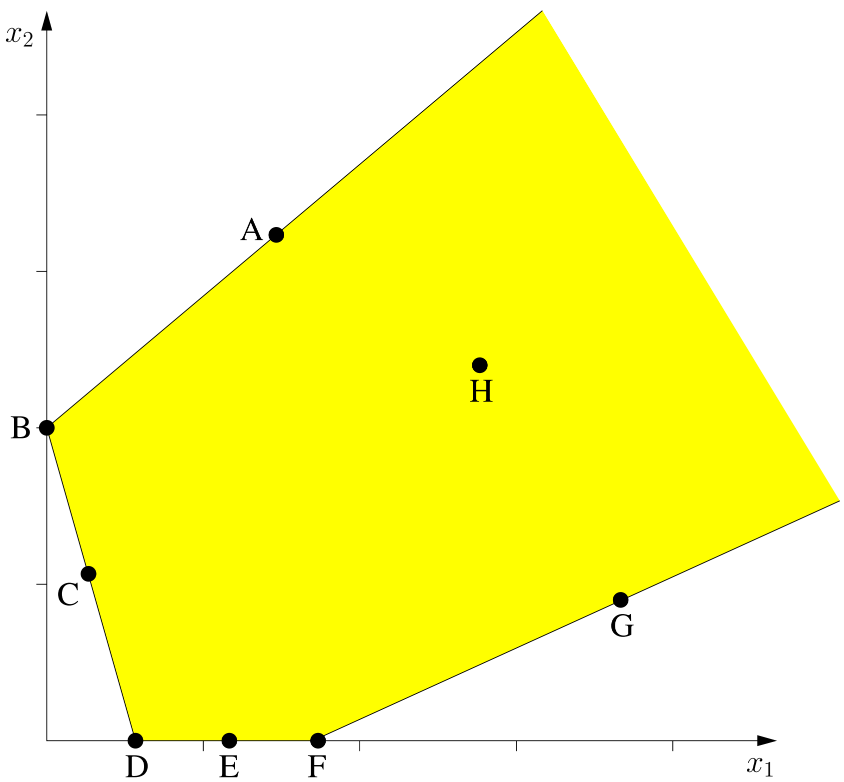

Figure 1: Feasible region discussed in Problem 3.

$p' = \lambda p + (1-\lambda)p''$ for some $0 \leq \lambda \leq 1$. The cost of a point is a linear fuction and therefore can be expressed as $c^\top p$. Then it is straightforward to verify that $c^\top p' = \lambda c^\top p + (1-\lambda)c^\top p'' \geq \lambda \min\{c(p), c(p'')\} + (1-\lambda)\min\{c(p), c(p'')\} = \min\{c(p), c(p'')\}$. The proof for the second statement is similar.

Questions. Answer the set of True/False questions below.

(a) F cannot be a unique optimum of the problem.

Solution. False.

(b) If C is an optimal solution, D is also optimal.

Solution. True.

(c) If A and B are optimal, the problem is unbounded.

Solution. False.

(d) If B and F are optimal, G is not optimal.

Solution. False.

(e) If no point among B, D and F is optimal, the problem is unbounded.

Solution. True.

(f) There exists an objective function such that the problem is infeasible.

Solution. False.

(g) If B, D and F are not optimal, the problem is infeasible.

Solution. False.

(h) D and F could simultaneously be the only optima of the problem.

Solution. False.

(i) If D and G are optimal, there is an infinite number of feasible solutions.

Solution. True.

(j) If the problem has a finite optimal objective value, G could be an optimal solution.

Solution. True.

(k) There exists an objective function such that H is optimal but A is not.

Solution. False.

(l) If H is an optimal solution, there are infinitely many optimal solutions and the limit of the objective function values is plus or minus infinity.

Solution. False.

Problem 2

Consider the feasible region defined by the following constraints:

$$\left.\begin{array}{rcl} x_1 - x_2 & \geq & -1.5 \\ x_1 - 2x_2 & \leq & 2 \\ 4x_1 + x_2 & \geq & 2 \\ x_1, x_2 & \geq & 0. \end{array}\right\} \qquad \text{(LP2)}$$

(a) Draw the feasible region of (LP2). Does this LP have an optimal solution for all possible objective functions? Why?

Solution. The feasible region is sketched in Figure 2. Because the feasible region is unbounded, this problem does not have an optimal solution for all possible objective functions.

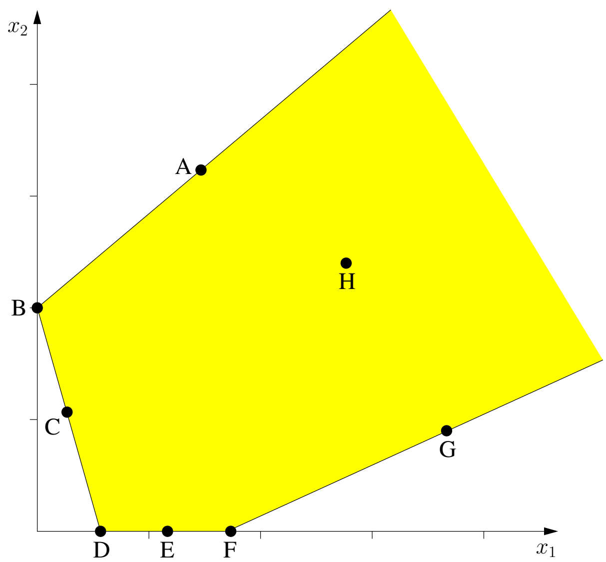

Figure 2: Feasible region for Part 2.A.

(b) Give an example of:

- An objective function such that both $(0, 0)$ and $(0, 1.5)$ are optimal (in minimization form)
- An objective function such that only the point $(0, 1.5)$ is optimal (in minimization form).
- An objective function such that (LP2) is unbounded (in maximization form).

If such an example does not exist, explain why.

Solution.

- In order for $(0, 0)$ and $(0, 1.5)$ to be optimal, the whole segment between them has to be optimal. Hence the objective function has level lines parallel to the x_2 axis. In two dimensions, the only objective function with this property is: $\min x_1$.

- The point $(0, 1.5)$ is the unique optimum for objective functions with slope between the vertical axis x_2 and the constraint $x_1 - x_2 \geq -1.5$. Hence the slope should be between 1 and $+\infty$. All functions of the form $\min x_1 - \beta x_2$ with $\beta \in (0, 1)$ have this property (note that the values $\beta = 0$ and $\beta = 1$ are excluded because $(0, 1.5)$ would be optimal but not unique).
- We are looking for objective functions that increase when moving towards the top right part of the graph. All functions of the form $\max \alpha x_1 + \beta x_2$ with $\alpha, \beta \geq 0$ have this property. There are other functions (with one negative coefficient) that yield an unbounded objective function value, but here we are just required to give an example.

Problem 3

Consider the following LP:

$$\left.\begin{array}{lrcl} \max & 10x_1 + 8x_2 - 3x_3 & & \\ \text{s.t.:} & 2x_1 + 4x_2 - 0.5x_3 & \leq & 6 \\ & -2x_1 + 6x_2 - 4.5x_3 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \\ & x_3 & & \text{free.} \end{array}\right\} \tag{LP3}$$

(a) Write (LP3) in canonical form. If you have to introduce extra variables, explain what they stand for. Compute the initial basic feasible solution and write its value for *all* of the problem's variables (regardless of whether they are present in the original formulation or introduced for the canonical form).

Solution. To put (LP3) into canonical form, we first have to substitute x_3 with two nonnegative variables: x_3^+ that represents the positive part of x_3, and x_3^- that represents its negative part, i.e. $x_3 = x_3^+ - x_3^-$. This yields:

$$\left.\begin{array}{lrcl} \max & 10x_1 + 8x_2 - 3x_3^+ + 3x_3^- & & \\ \text{s.t.:} & 2x_1 + 4x_2 - 0.5x_3^+ + 0.5x_3^- & \leq & 6 \\ & -2x_1 + 6x_2 - 4.5x_3^+ + 4.5x_3^- & \leq & 4 \\ & x_1, x_2, x_3^+, x_3^- & \geq & 0. \end{array}\right\}$$

Then we introduce nonnegative slack variables to transform the constraints into equality constraints: s_1 and s_2 represent the difference between the rhs and the lhs of the first, respectively second, constraint. We obtain the canonical form:

$$\left.\begin{array}{lrcl} \max & 10x_1 + 8x_2 - 3x_3^+ + 3x_3^- & & \\ \text{s.t.:} & 2x_1 + 4x_2 - 0.5x_3^+ + 0.5x_3^- + s_1 & = & 6 \\ & -2x_1 + 6x_2 - 4.5x_3^+ + 4.5x_3^- + s_2 & = & 4 \\ & x_1, x_2, x_3^+, x_3^-, s_1, s_2 & \geq & 0. \end{array}\right\}$$

The corresponding basic feasible solution is $s_1 = 6, s_2 = 4$. This implies that all the remaining variables x_1, x_2, x_3^+, x_3^- have value 0.

(b) Write the initial simplex tableau and perform two iterations of the simplex algorithm. Is the basic solution after two iterations optimal? Why?

Solution. Initial tableau:

Basic	x_1	x_2	x_3^+	x_3^-	s_1	s_2	Rhs
$(-z)$	10	8	-3	3			0
s_1	2	4	-0.5	0.5	1		6
s_2	-2	6	-4.5	4.5		1	4

Pivot column: x_1, pivot row: 1. First iteration:

Basic	x_1	x_2	x_3^+	x_3^-	s_1	s_2	Rhs
$(-z)$		-12	-0.5	0.5	-5		-30
x_1	1	2	-0.25	0.25	0.5		3
s_2		10	-5	5	1	1	10

Pivot column: x_3^+, pivot row: 2. Second iteration:

Basic	x_1	x_2	x_3^+	x_3^-	s_1	s_2	Rhs
$(-z)$		-13			-5.1	-0.1	-31
x_1	1	1.5			0.45	-0.05	2.5
x_3^-		2	-1	1	0.2	0.2	2

This tableau is optimal.

(c) In (LP3), replace the first constraint $2x_1 + 4x_2 - 0.5x_3 \leq 6$ with $-2x_1 + 4x_2 - 0.5x_3 \leq 6$. Write the initial simplex tableau (note that only one coefficient changes with respect to the first tableau of Part 2.B). Perform one iteration of the simplex algorithm. What happens in this case?

Solution. The amended initial tableau is:

Basic	x_1	x_2	x_3^+	x_3^-	s_1	s_2	Rhs
$(-z)$	10	8	-3	3			0
s_1	-2	4	-0.5	0.5	1		6
s_2	-2	6	-4.5	4.5		1	4

Column x_1 has a positive reduced cost and all of the coefficients are negative. Therefore the problem is unbounded: we can increase x_1 as much as we want to improve the objective function value, while still staying feasible.

Problem 4

Here we review the Simplex Algorithm in more detail, without an Excel spreadsheet to provide guidance. Be careful when carrying out the calculations.

Consider the following linear program:

$$\left.\begin{array}{lrcl}
\max & 2x_1 + 4x_2 & & \\
\text{s.t.:} & & & \\
& 0.5x_1 - 5x_2 & \leq & 12 \\
& x_1 + 2x_2 & \geq & -2 \\
& x_2 + x_3 & \geq & 4 \\
& x_1, x_2, x_3 & \geq & 0.
\end{array}\right\}$$

(a) Write the initial simplex tableau. To do so, you will have to transform the problem into canonical form. What is the initial basic feasible solution? (Hint: at least one of the original variables is basic)

Solution. We have to introduce a slack variable s_1 in the first constraint, and surplus variables s_2, s_3 in the second and third constraint. We also have to flip the second constraint, multiplying through by -1 to obtain a nonnegative rhs value. The problem in standard form is:

$$\left.\begin{array}{lrcl} \max & 2x_1 + 4x_2 & & \\ \text{s.t.:} & & & \\ & 0.5x_1 - 5x_2 + s_1 & = & 12 \\ & -x_1 - 2x_2 + s_2 & = & 2 \\ & x_2 + x_3 - s_3 & = & 4 \\ & x_1, x_2, x_3, s_1, s_2, s_3 & \geq & 0. \end{array}\right\}$$

We put this into tableau form and we obtain:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-z)$	2	4					0
s_1	0.5	-5		1			12
s_2	-1	-2			1		2
x_3		1	1			-1	4

The initial basic feasible solution is $s_1 = 12, s_2 = 2, x_3 = 4$. The remaining variables have value zero.

(b) Perform one iteration of the simplex algorithm, by pivoting in the variable with the largest reduced cost. Write down the candidate variables for pivoting out, and the corresponding value of the ratio test. Finally, report the simplex tableau after the first iteration.

Solution. We pivot in x_2 because it has the largest reduced cost (4 against 2 of x_1; x_3 has zero reduced cost). The only candidate for pivoting out is x_3 (pivot on the third row) because it is the only positive coefficient in the pivot column. The ratio test yields the value $4/1 = 4$. The tableau after the first iteration is:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Rhs
$(-z)$	2		-4			4	-16
s_1	0.5		5	1		-5	32
s_2	-1		2		1	-2	10
x_2		1	1			-1	4

(c) Would the simplex algorithm terminate after the first iteration? Why? Can you guess the optimal objective function value?

Solution. The simplex algorithm terminates after the first iteration detecting an unbounded objective function value: the column corresponding to s_3 has a positive reduced cost, but its coefficient in all rows are nonpositive. This means that we can increase s_3 as much as possible and increase the objective function value while staying within the feasible region. It follows that there is no optimal objective function value (we also accept $+\infty$ as the answer for the optimal objective function value).