15.053/8 February 7, 2013

More Linear and Non-linear Programming Models

- Optimal meal selection at McDonalds.
- A (financial) portfolio selection problem.
- Introduction to convex functions
- Workforce scheduling.

Announcements

- Optional recitations for 15.053/8 on February 8 :
  - formulations 11 AM
  - Excel Solver 2 PM
- Future (optional) recitations
- Written affirmation on problem sets

Overview of Lecture

- Goals
  - get practice in recognizing and modeling linear constraints and objectives
  - and non-linear objectives
  - to see a broader use of models in practice

Note: Read tutorials 00, 01, 02, 03 on the website.

00. Meet the characters

01 LP formulations

02. Algebraic formulations

03. Excel Solver

Quotes for today

“Reality is merely an illusion, albeit a very persistent one.”

Albert Einstein

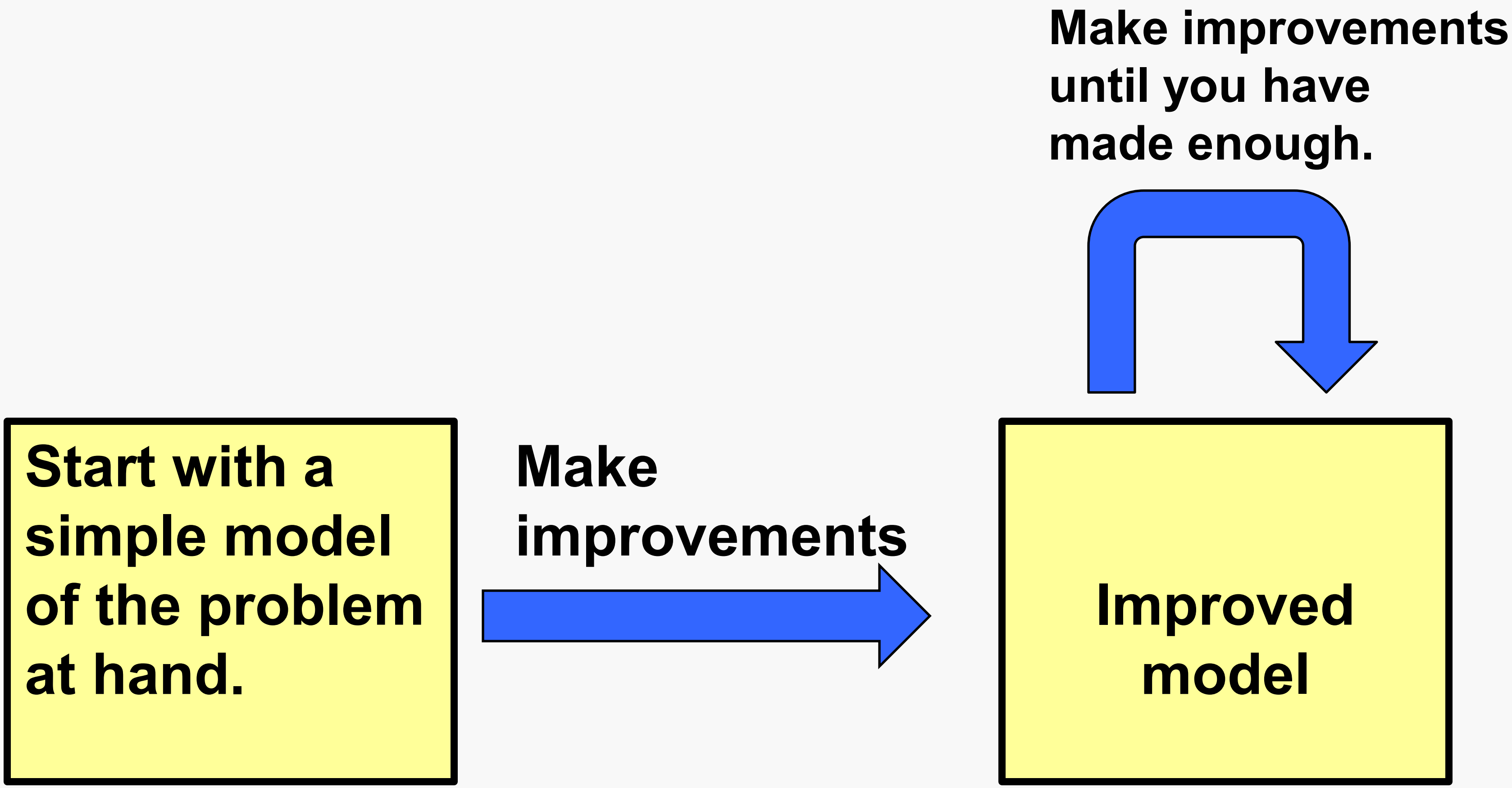
“Everything should be made as simple as possible, but not one bit simpler.”

Albert Einstein, (attributed)

Overview on modeling

- **Modeling as a mathematical skill**
- **Modeling as an art form**
- **Applications to diet problem, portfolio optimization, and workforce scheduling**

A simplified modeling process



Clicker Questions

Q1. What year are you?

1. freshman
2. sophomore
3. junior
4. senior
5. grad student

Q2. Are you taking 15.053 as

1. part of the management science major (or double major)
2. part of the management science minor
3. an elective

Q3. Do you own a clicker from Turning Technologies.

1. Yes
2. No, but I was given one for this subject.

Supersize me: 2004 documentary

- Morgan Spurlock: director and star
- 30 Day diet of McDonald's food
- His rules:
  - Eat everything on the menu at least once
  - Eat no food outside of McDonalds
  - Supersize a meal whenever offered, but only when offered.
- He averaged 5000 calories a day

Results

- gained 24.5 lbs
- suffered depression, lethargy, headaches, and low sex drive
- Day 21: heart palpitations. His internist asked him to stop what he was doing.
- Bright side
  - Oscar nomination for documentary
  - $20.6 million in box office
  - McDonalds dropped “supersizing”
- Other side: legitimate criticism of movie

Question: what would be a good diet at McDonalds?

- Suppose that we wanted to design a good 1 week diet at McDonalds. What would we do?

- What data would we need?

- Decision variables?

More on diet problem

- Objective function?
- Constraints?

A simpler problem

- Minimize the cost of a meal
  - just a few choices listed
  - between 600 and 900 calories
  - less than 50% of daily sodium
  - fewer than 40% of the calories are from fat
  - at least 30 grams of protein.
  - fractional meals permitted.

Data from McDonalds
(prices are approximate)

	Hamburger	Big Mac	McChicken	Caesar Salad with Chicken	small French fries
Total Calories	250	770	360	190	230
Fat Calories	81	360	144	45	99
Protein (grams)	31	44	14	27	3
Sodium (mg)	480	1170	800	580	160
Cost	$1.00	$3.00	$2.50	$3.00	$1.00

sodium limit: 2300 mg per day.

LP for McDonalds

$$\text{Minimize} \quad H + 3B + 2.5M + 3C + R$$

$$\text{subject to} \quad 250H + 770B + 360M + 190C + 230R - F = 0$$

$$600 \le F \le 900$$

$$81H + 360B + 144M + 45C + 99R - .4F \le 0$$

$$31H + 44B + 14M + 27C + 3R \ge 30$$

$$480H + 1770B + 800M + 580C + 160R \le 1150$$

$$H, B, M, C, R \ge 0$$

Opt LP Solution: H = 1.13 B = .41 Cost = $2.37

Opt IP Solution: H =1 R = 2 Cost = $3

Portfolio optimization

- you are managing a small ($500 million) fund of stocks of major companies.

- Information:
  - can choose from 500 stocks
  - expected returns, variances and covariances

- Sample rule:
  - no more than 2% of portfolio in any stock

Objective: average return on the investment.

BA	XON	GM
12.7	9.9	11.8

Average annual rate of return (approx)

Sample investment.

BA	XON	GM
50%	**20%**	**30%**

rate of return

= .5 * 12.7 + .2 * 9.9 + .3 * 11.8

= 11.87

BA	XON	GM
18.7	12.2	24.4

Standard deviation of annual rate of return (approx)

Stocks are very risky!

Use variance of portfolio as risk metric.

	BA	XON	GM
BA	350	50	100
XON	50	150	30
GM	100	30	600

Covariance matrix (approx)

Sample investment.

BA	XON	GM
50%	20%	30%

	$x_1 = .5$	$x_2 = .2$	$x_3 = .3$
$x_1 = .5$	350	50	100
$x_2 = .2$	50	150	30
$x_3 = .3$	100	30	600

$$\sum_{i=1}^{3} x_i a_{ij} x_j$$

Use variance to measure risk

	BA	XON	GM
BA	350	50	100
XON	50	150	30
GM	100	30	600

Covariance matrix (approx)

Sample investment.

BA	XON	GM
50%	20%	30%

$$\text{variance} = 350 \times .5^2 + 150 \times .2^2 + 600 \times .3^2$$
$$+ 2 \times 50 \times .5 \times .2$$
$$+ 2 \times 100 \times .3 \times .5$$
$$+ 2 \times 30 \times .2 \times .3$$
$$= 193.5$$

standard deviation = 13.9

The risk is almost as low as XON but the return is far better. What is the intuition?

Formulation

maximize **return**

subject to **variance of portfolio ≤ specified amount**

proportion of stock i ≤ .02

proportions ≥ 0

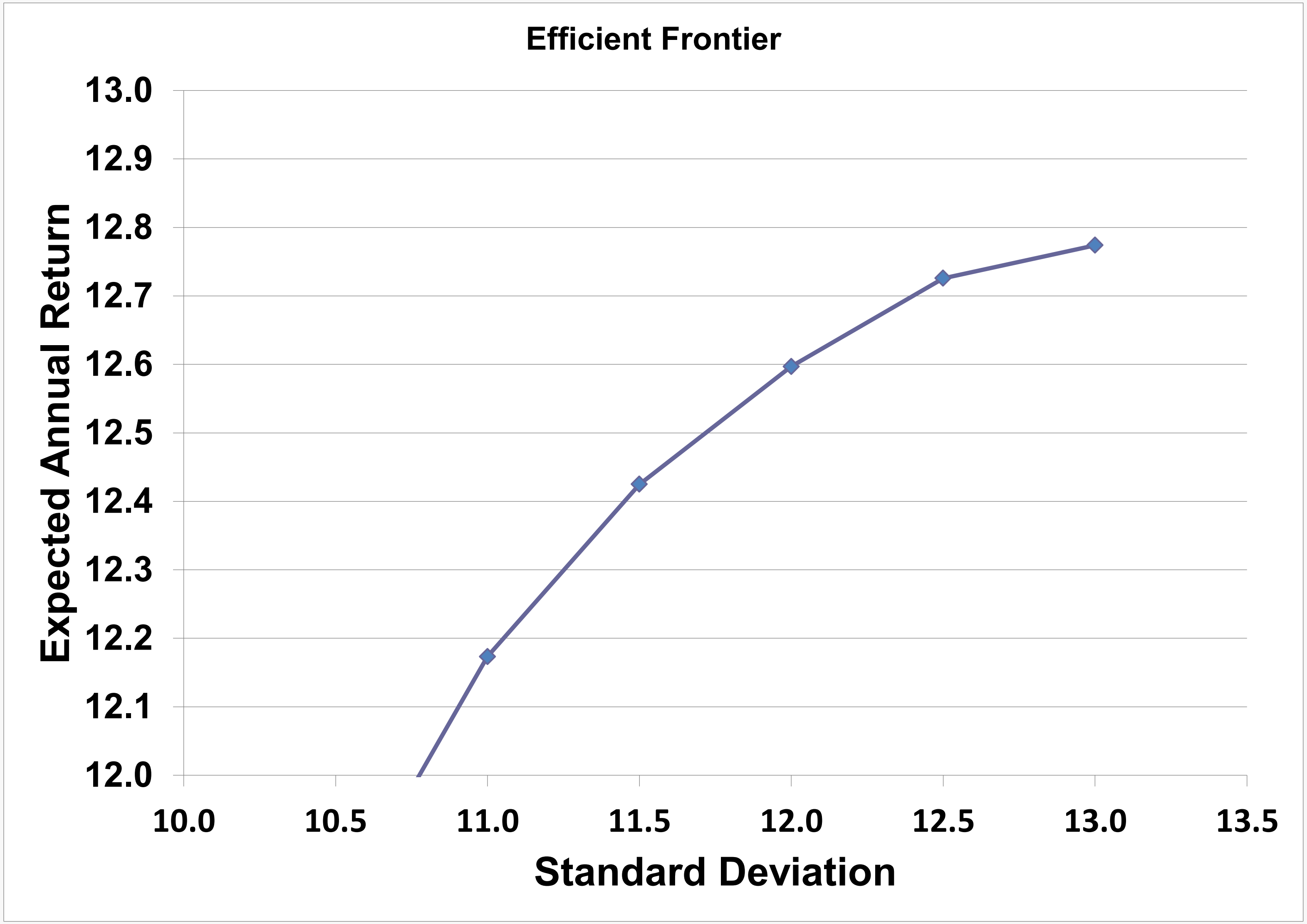
Other considerations?

from DMD, 15.060

BA	XON	GM	MCD	PG	SP
12.7	9.9	11.8	13.5	13.5	13.0

	BA	XON	GM	MCD	PG	SP
BA	363.1	47.1	103.5	179.9	107.4	110.7
XON	47.1	144.8	34.4	78.9	55.4	79.0
GM	103.5	34.4	614.8	174.9	-95.6	106.1
MCD	179.9	78.9	174.9	470.5	70.7	150.1
PG	107.4	55.4	-95.6	70.7	475.6	140.6
SP	110.7	79.0	106.1	150.1	140.6	137.1

The optimal tradeoff curve



Time for a mental break

Some cartoons on science.

Non-linear programs and convexity

- **An optimization problem with a single objective and multiple constraints.**

- **Linear programs are a special case.**

Examples of Nonlinear Objective Functions

Min $\sum_{j=1}^{7}(x_j)^2$

Max $\sum_{j=1}^{7}\frac{Cos^5(e_j)}{\sqrt{d_j}}$

Min $\sum_{j=1}^{7}x_j$

Examples of Nonlinear Constraints

$\sum_{j=1}^{7}(x_j)^2 \geq 30$

$\sum_{j=1}^{7}\frac{Cos^5(e_j)}{\sqrt{d_j}} = 13.76$

$\sum_{j=1}^{7}x_j \leq 13$

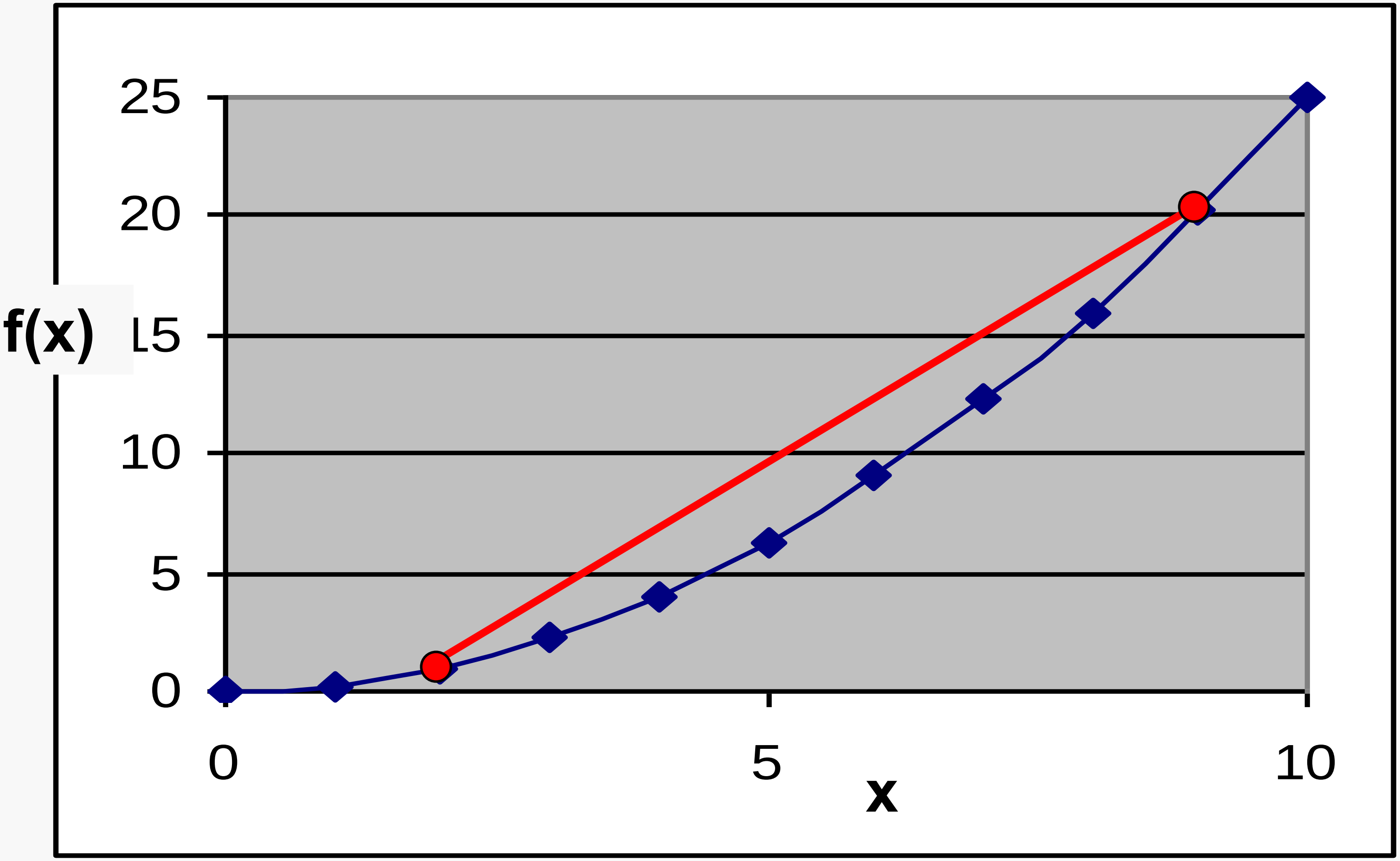
On Nonlinear Programs

- **In general, nonlinear programs are incredibly hard to solve. Sometimes they are impossible to solve.**

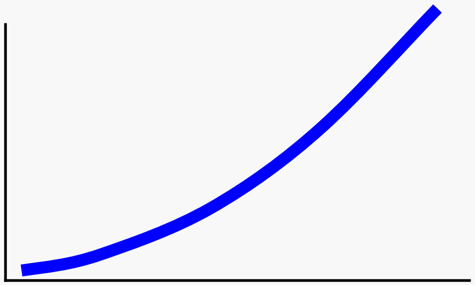
But they usually can be solved if the objective is to <u>minimize a convex function</u>, and the constraints are linear.

Convex functions of one variable

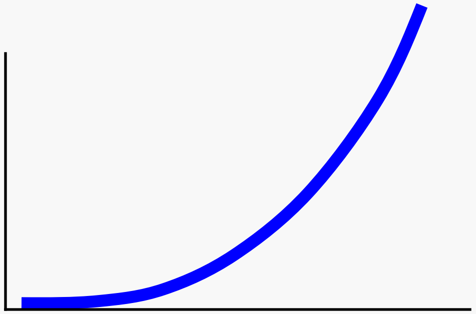
A function f(x) is ***convex*** if for all x and y, the line segment on the curve joining (x, f(x)) to (y, f(y)) lies on or above the curve.



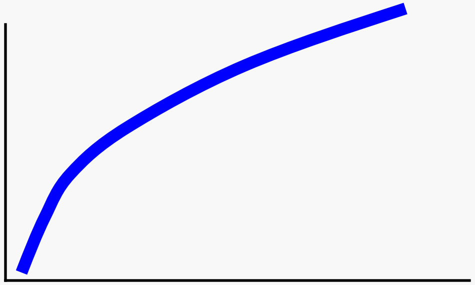
Which functions are convex?



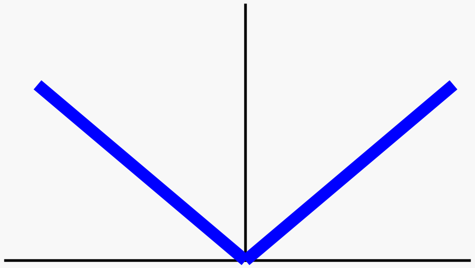
$f(x) = x^2$



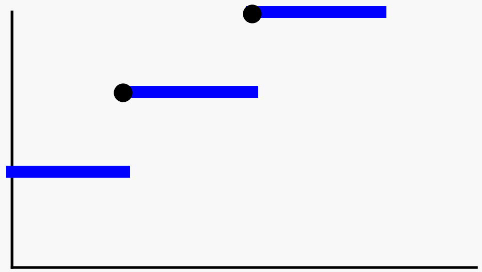
$f(x) = x^3$ for $x \geq 0$



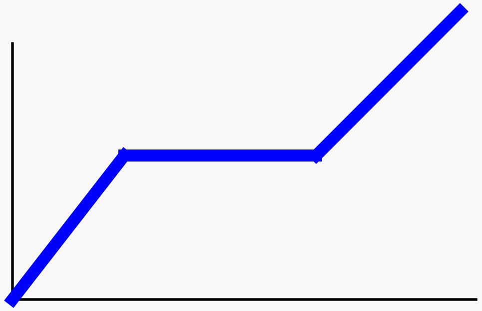
$f(x) = x^{.5}$



$f(x) = |x|$



Step Function



whatever

Yes No

And now, we return to linear programming.

Scheduling Postal Workers

- Each postal worker works for 5 consecutive days, followed by 2 days off, repeated weekly.

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

- Minimize the number of postal workers (for the time being, we will permit fractional workers on each day.)

Formulating as an LP

- Don’t look ahead.
- Let’s see if we can come up with what the decision variables should be.
- Discuss with your neighbor how one might formulate this problem as an LP.

The linear program

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

The linear program

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to

$x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$ Mon.

$x_1 + x_2 + x_5 + x_6 + x_7 \geq 13$ Tues.

$x_1 + x_2 + x_3 + x_6 + x_7 \geq 15$ Wed.

$x_1 + x_2 + x_3 + x_4 + x_7 \geq 19$ Thurs.

$x_1 + x_2 + x_3 + x_4 + x_5 \geq 14$ Fri.

$x_2 + x_3 + x_4 + x_5 + x_6 \geq 16$ Sat.

$x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$ Sun.

$x_j \geq 0$ for $j = 1$ to 7

On the selection of decision variables

- A choice of decision variables that doesn't work
  - Let y_j be the number of workers on day j.
  - No. of Workers on day j is at least d_j. (easy to formulate)
  - Each worker works 5 days on followed by 2 days off (hard).
- Conclusion: sometimes the decision variables incorporate constraints of the problem.
  - Hard to do this well, but worth keeping in mind
  - We will see more of this in integer programming.

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A Modifications of the Model

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- Suppose that there was a pay differential. The cost of each worker who works on day j is c_j. The new objective is to minimize the total cost.

What is the objective coefficient for the shift that starts on Monday for the new problem?

1. c_1
2. $c_1 + c_2 + c_3 + c_4 + c_5$
3. $c_1 + c_4 + c_5 + c_6 + c_7$

A Different Modification of the Model

- **Suppose that there is a penalty for understaffing and penalty for overstaffing. If you hire k too few workers on day j, the penalty is 5 k^2. If you hire k too many workers on day j, then the penalty is k^2. How can we model this?**

Step 1. Create new decision variables.

Let e_j = "excess workers on day j"

Let d_i = "deficit workers on day j"

Model 2

Minimize $5\sum_{i=1}^{7} d_i^2 + \sum_{i=1}^{7} e_i^2$

$$x_1 + x_4 + x_5 + x_6 + x_7 + d_1 - e_1 = 17$$
$$x_1 + x_2 + x_5 + x_6 + x_7 + d_2 - e_2 = 13$$
$$x_1 + x_2 + x_3 + x_6 + x_7 + d_3 - e_3 = 15$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + d_4 - e_4 = 19$$
$$x_1 + x_2 + x_3 + x_4 + x_5 + d_5 - e_5 = 14$$
$$x_2 + x_3 + x_4 + x_5 + x_6 + d_6 - e_6 = 16$$
$$x_3 + x_4 + x_5 + x_6 + x_7 + d_7 - e_7 = 11$$

$x_j \geq 0$, $d_j \geq 0$, $e_j \geq 0$ for $j = 1$ to 7

What is wrong with this model, other than the fact that variables should be required to be integer valued?

What is wrong with Model 2?

1. The constraints should have inequalities.
2. The constraints don't make sense.
3. The objective is incorrect. (Note: it is OK that it is nonlinear)
4. It's possible that e_j and d_j are both positive.
5. Nothing is wrong.

More Comments on Model 2.

Difficulty: **The feasible region permits feasible solutions that do not correctly model our intended constraints. Let us call these *bad feasible solutions*.**
The *good feasible solutions* are ones in which $d_1 = 0$ or $e_1 = 0$ or both. They correctly model the scenario.

Resolution: All optimal solutions are good.

Illustration of why it works:

$$10 + 10 + 0 + 0 + 0 + d_1 - e_1 = 17$$

$e_1 = 4$ and $d_1 = 1$ is a bad feasible solution.
$e_1 = 3$ and $d_1 = 0$ are good feasible solution.

For every bad feasible solution, there is a good feasible solution whose objective is better.

More on the model

- Summary: the model permits too many feasible solutions.
- All of the optimal solutions are good.
- We will see this technique more in this lecture, and in other lectures as well.

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On the practicality of these models

- In modeling in practice, one needs to capture a lot of reality (but not too much).
- Workforce scheduling is typically much more complex.
- These models are designed to help in thinking about real workforce scheduling models.

15.053 Optimization Methods in Management Science
Spring 2013