15.053/8 March 19, 2013

Integer Programming Formulations 2

references: *IP Formulation Guide* (on the website)
Tutorial on IP formulations.
Applied Math Programming

- announcement on meetings of teams with staff

Quote of the Day

"What chiefly characterizes creative thinking from more mundane forms are (i) willingness to accept vaguely defined problem statements and gradually structure them, (ii) continuing preoccupation with problems over a considerable period of time, and (iii) extensive background knowledge in relevant and potentially relevant areas."

-- Herbert Simon

Overview of today's lecture

- Very quick review of integer programming
- Building blocks for creating IP models
- Logical constraints
- Non-linear functions
- IP models that generalize the assignment problem or transportation problem
- Other combinatorial problems modeled as IPs

Integer Programs

Integer programs: a linear program plus the additional constraints that some or all of the variables must be integer valued.

We also permit "$x_j \in \{0,1\}$," or equivalently, "x_j is binary"

This is a shortcut for writing the constraints:

$0 \leq x_j \leq 1$ and x_j integer.

Trading for Profit (from last lecture)

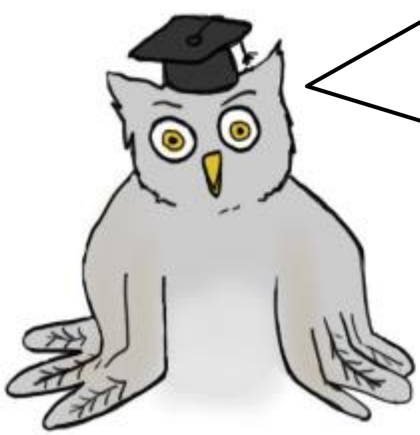
Prize	iPad 1	server 2	Brass Rat 3	Au Bon Pain 4	6.041 tutoring 5	15.053 dinner 6
Points	5	7	4	3	4	6
Utility	16	22	12	8	11	19

Budget: 14 IHTFP points.

Write Nooz's problem as an integer program.

$$\text{Let } x_i = \begin{cases} 1 & \text{if prize i is selected} \\ 0 & \text{otherwise} \end{cases}$$

Modeling logical constraints that include only two binary variables.



Modeling logical constraints with two variables can be accomplished in two steps:

Step 1. Graph the feasible region as restricted to the two variables.

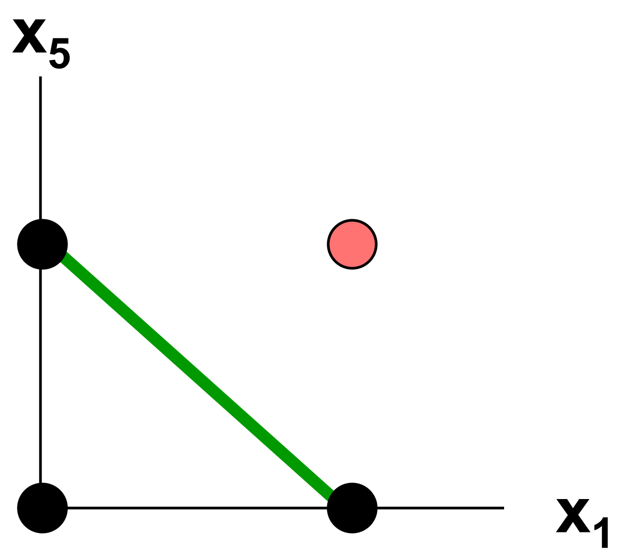
Step 2. Add linear equalities and or inequalities so that the feasible region of the IP is the same as that given in Step 1.

Logical Constraints 1

Constraint 1. **If you select the iPad, you cannot select 6.041**

$x_1 = 1$ if iPad

$x_5 = 1$ if 6.041



MIP Constraint:

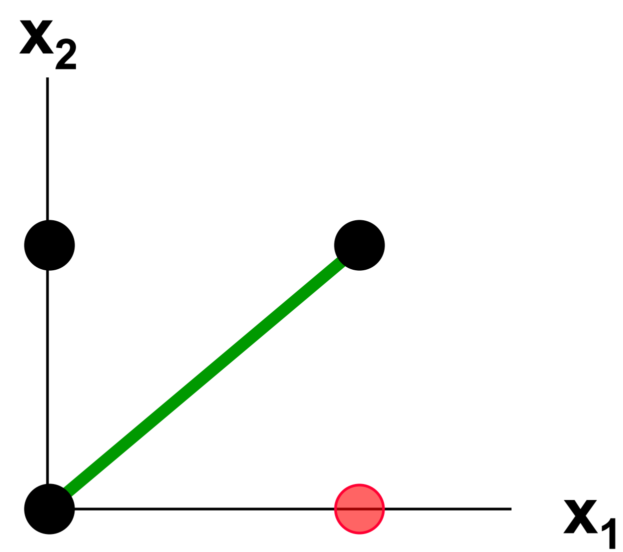
$$x_1 + x_5 \leq 1$$

Logical Constraints 2

Constraint 2. If Prize 1 is selected then Prize 2 must be selected.

x_1 = 1 if iPad

x_2 = 1 if server

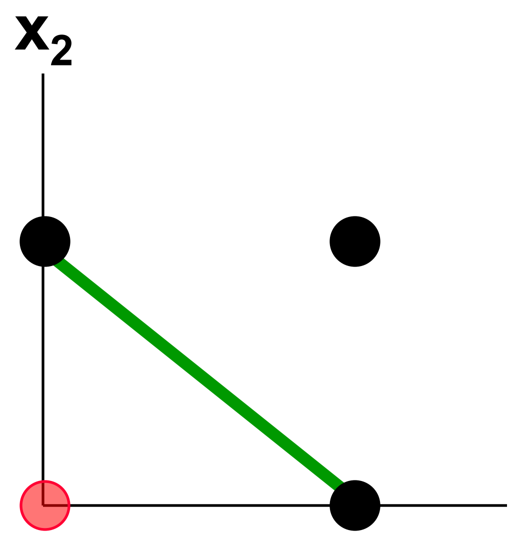


MIP Constraint: $x_1 \leq x_2$

Logical Constraints 3

$x_1 = 1$ if iPad

$x_2 = 1$ if server



Constraint 3. You must select Prize 1 or Prize 2 or both

MIP Constraint:

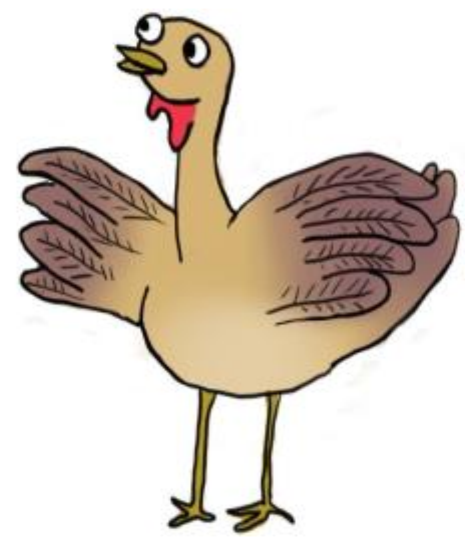
$$x_1 + x_2 \geq 1$$

Other logical constraints

Modeling logical constraints that involve non-binary variables is much harder. But we will try to make it as simple as possible.

I volunteer to be the judge of what's simple.

And I'll be the judge of what's possible.



BIG M Method for IP Formulations

- **Assume that all variables are integer valued.**
- **Assume a bound u* on coefficients and variables;**
  - **e.g.,** $x_j \leq 10{,}000$ **for all j.**
    $|a_{ij}| \leq 10{,}000$ **for all i, j.**
- **Choose M <u>really</u> large so that for every constraint i,**

$$|a_{i1}x_1 + a_{i2}\, x_2 + \ldots + a_{in}\, x_n| \leq b_i + M$$

That is, we will be able to satisfy any "≤" constraint by adding M to the RHS.

And we can satisfy any "≥" constraint by subtracting M from the RHS.

The logical constraint "$x \le 2$ or $x \ge 6$"

We formulate the logical constraint,
"$x \le 2$ or $x \ge 6$" as follows.

Choose a binary variable w so that

if $w = 1$, then $x \le 2$.

if $w = 0$, then $x \ge 6$.

$x \le 2 + M(1-w)$

$x \ge 6 - M w$

$w \in \{0,1\}$

To validate the formulation one needs to show: The logical constraints are equivalent to the IP constraints.

Suppose that (x, w) is feasible, for the IP.

If $w = 1$, then $x \le 2$.

If $w = 0$, then $x \ge 6$.

$x \le 2$ or $x \ge 6$

logical constraint

$x \le 2 + M(1-w)$

$x \ge 6 - M w$

$w \in \{0,1\}$

IP constraints

Suppose that x satisfies the logical constraints.

If $x \le 2$, then let $w = 1$. → $x \le 2$ and $x \ge 6 - M$

If $x \ge 6$, then let $w = 0$. → $x \le 2 + M$ and $x \ge 6$

In both cases, the IP constraints are satisfied.

Modeling "or constraints"

$x_1 + 2x_2 \geq 12$ or
$4x_2 - 10x_3 \leq 1$.

$x_1 + 2x_2 \geq 12 - M(1-w)$
$4x_2 - 10x_3 \leq 1 + Mw$.

Logical constraints.

IP constraints.

Suppose that x_i is bounded for all i.

Suppose that M is very large.

To show: The logical constraints are equivalent to the IP constraints.

Suppose that (x, w) is feasible, for the IP.

If w = 1, then $x_1 + 2x_2 \geq 12$

If w = 0, then $4x_2 - 10x_3 \leq 1$

Therefore, the logical constraints are satisfied.

$x_1 + 2x_2 \geq 12$ or
$4x_2 - 10x_3 \leq 1.$

$x_1 + 2x_2 \geq 12 - M(1-w)$
$4x_2 - 10x_3 \leq 1 + Mw.$

Logical constraints.

IP constraints.

Suppose that x_i is bounded for all i.

Suppose that M is very large.

To show: The logical constraints are equivalent to the IP constraints.

Suppose that x satisfies the logical constraints.

If $x_1 + 2x_2 \geq 12$, then let w = 1 → $x_1 + 2x_2 \geq 12$ AND $4x_2 - 10x_3 \leq 1 + M.$

Else $4x_2 - 10x_3 \leq 1$ then let w = 0 → $x_1 + 2x_2 \geq 12 - M$ AND $4x_2 - 10x_3 \leq 1.$

In both cases, the IP constraints are satisfied.

Mental Break

Fixed charge problems

- **Suppose that there is a linear cost of production, after the process is set up.**
- **There is a cost of setting up the production process.**
- **The process is not set up unless there is production.**

The Alchemist's Problem

In 1502, the alchemist Zor Primal has set up shop creating gold, silver, and bronze medallions to celebrate the 10th anniversary of the discovery of America. His trainee alchemist (TA) makes the medallions out of lead and pixie dust. Here is the data table.

	Gold	Silver	Bronze	Available
TA labor (days)	2	4	5	100
lead (kilos)	1	1	1	30
pixie dust (grams)	10	5	2	204
Profit ($)	52	30	20	

Zor is unable to get any of his reactions going without an expensive set up.

Cost to set up	$500	$400	$300	

Zor's problem with set up costs

Maximize $f_1(x_1) + f_2(x_2) + f_3(x_3)$

subject to

$$2x_1 + 4x_2 + 5x_3 \le 100$$
$$1x_1 + 1x_2 + 1x_3 \le 30$$
$$10x_1 + 5x_2 + 2x_3 \le 204$$
$$x_1, x_2, x_3 \ge 0 \text{ integer}$$

$$f_1(x_1) = \begin{cases} -500 + 52x_1 & \text{if } x_1 \ge 1 \\ 0 & \text{if } x_1 = 0 \end{cases}$$

Let $w_1 = \begin{cases} 1 & x_1 \ge 1 \\ 0 & x_1 = 0. \end{cases}$

$$f_2(x_2) = \begin{cases} -400 + 30x_2 & \text{if } x_2 \ge 1 \\ 0 & \text{if } x_2 = 0 \end{cases}$$

Let $w_2 = \begin{cases} 1 & x_2 \ge 1 \\ 0 & x_2 = 0. \end{cases}$

$$f_3(x_3) = \begin{cases} -300 + 20x_3 & \text{if } x_3 \ge 1 \\ 0 & \text{if } x_3 = 0 \end{cases}$$

Let $w_3 = \begin{cases} 1 & x_3 \ge 1 \\ 0 & x_3 = 0. \end{cases}$

The IP Formulation

$$w_j = \begin{cases} 1 & x_j \geq 1 \\ 0 & x_j = 0 \end{cases}$$

$$f_1(x_1) = \begin{cases} -500 + 52x_1 & \text{if } x_1 \geq 1 \\ 0 & \text{if } x_1 = 0 \end{cases}$$

$$f_2(x_2) = \begin{cases} -400 + 30x_2 & \text{if } x_2 \geq 1 \\ 0 & \text{if } x_2 = 0 \end{cases}$$

$$f_3(x_3) = \begin{cases} -300 + 20x_3 & \text{if } x_3 \geq 1 \\ 0 & \text{if } x_3 = 0 \end{cases}$$

Max $\quad -500\, w_1 + 52\, x_1 \;-\; 500\, w_2 + 30\, x_2 \;-\; 300\, w_3 + 20\, x_3$

s.t.

$$2\, x_1 + 4\, x_2 + 5\, x_3 \leq 100$$

$$1\, x_1 + 1\, x_2 + 1\, x_3 \leq 30$$

$$10\, x_1 + 5\, x_2 + 2\, x_3 \leq 204$$

$$x_1 \leq M\, w_1; \quad x_2 \leq M\, w_2; \quad x_3 \leq M\, w_3;$$

$$x_1, x_2, x_3 \geq 0 \quad \text{integer}$$

$$w_1, w_2, w_3 \in \{0,1\}.$$

The IP formulation correctly models the fixed charges.

To show:

1. **If x is feasible for the fixed charge problem, then (x, w) is feasible for the IP (w is defined on the last slide) , and the cost in the IP matches the cost of the fixed charge problem.**

2. **If (x, w) is feasible for the IP, then x is feasible for the fixed charge problem, and the IP cost is the same as the cost in the fixed charge problem.**

Suppose that x is feasible for the fixed charge problem.

If $x_i \geq 1$, then let $w_i = 1$. Otherwise $w_i = 0$.

Then (x, w) is feasible for the IP, and the objective value for the IP is the same as for the fixed charge problem.

Suppose that (x, w) is feasible for the IP.

We say that (x, w) is a *sensible* if the following is true for each i: if $x_i = 0$, then $w_i = 0$.

Remark: if (x, w) is not sensible, then it cannot be optimal.

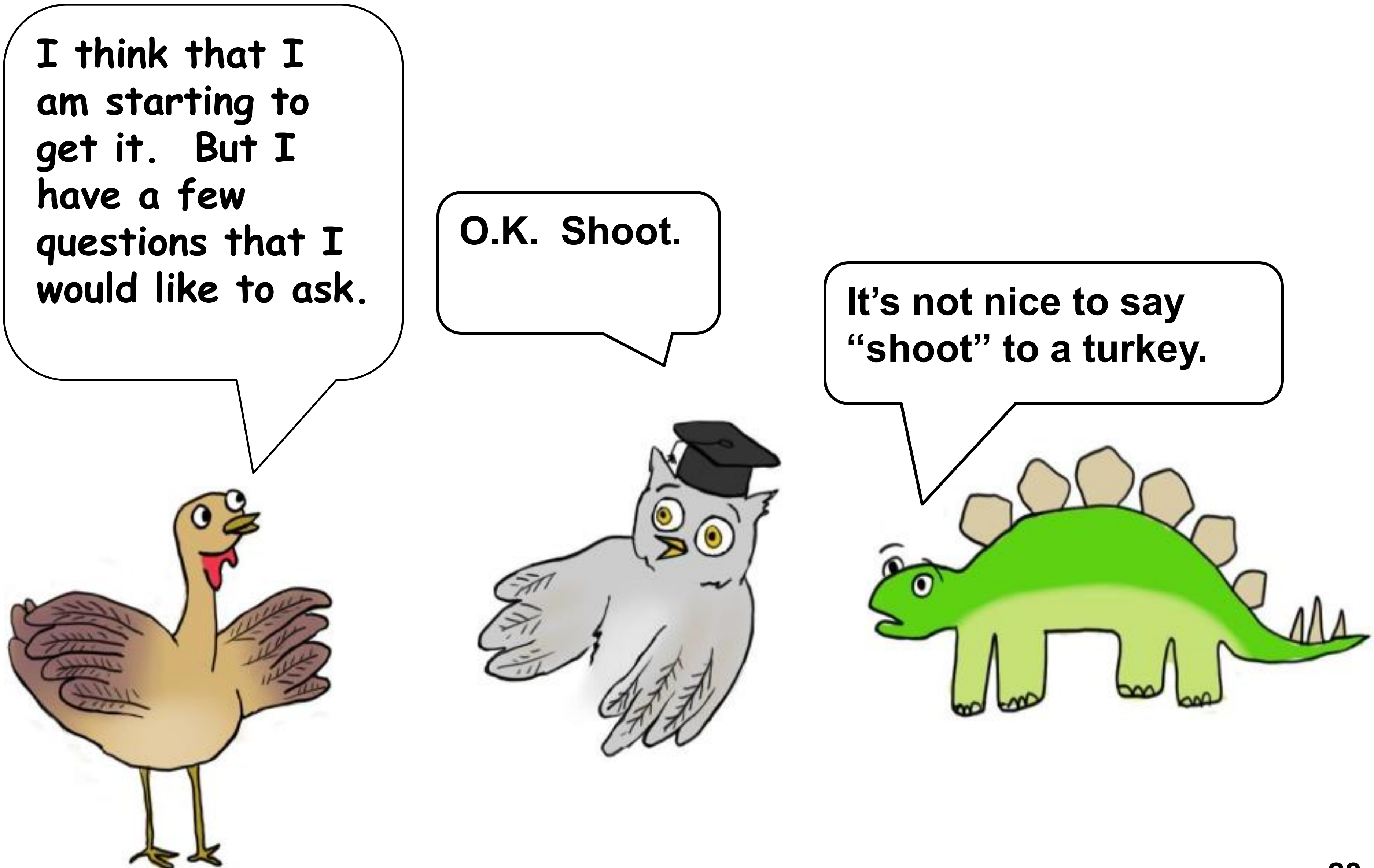
Claim. If (x, w) is feasible for the IP and if it is also sensible, then x is feasible for the fixed charge problem, and the IP cost is the same as the cost in the fixed charge problem.

x is clearly feasible for the fixed charge problem. Consider x_1.

If $x_1 \geq 1$, then $w_1 = 1$ and the cost is $-500 + 52x_1$.

If $x_1 = 0$, then $w_1 = 0$ and the cost is 0.

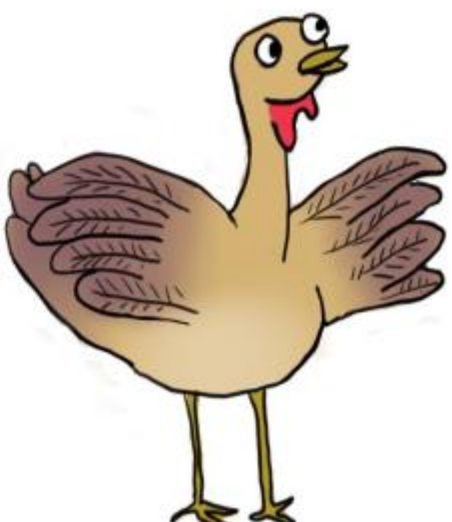
Thus, the cost of x_1 is the same for both problems. Similarly, the cost of x_2 and x_3 are the same.


I think that I am starting to get it. But I have a few questions that I would like to ask.
O.K. Shoot.
It's not nice to say "shoot" to a turkey.

$x_j \leq 30\, w_j$ for j = 1, 2, 3

First of all, I'm really unsure about what coefficient values to use. It seems very confusing.

All that really matters is that the number is sufficiently high so that for any feasible x for the fixed charge problem, one can obtain a feasible w for the IP.

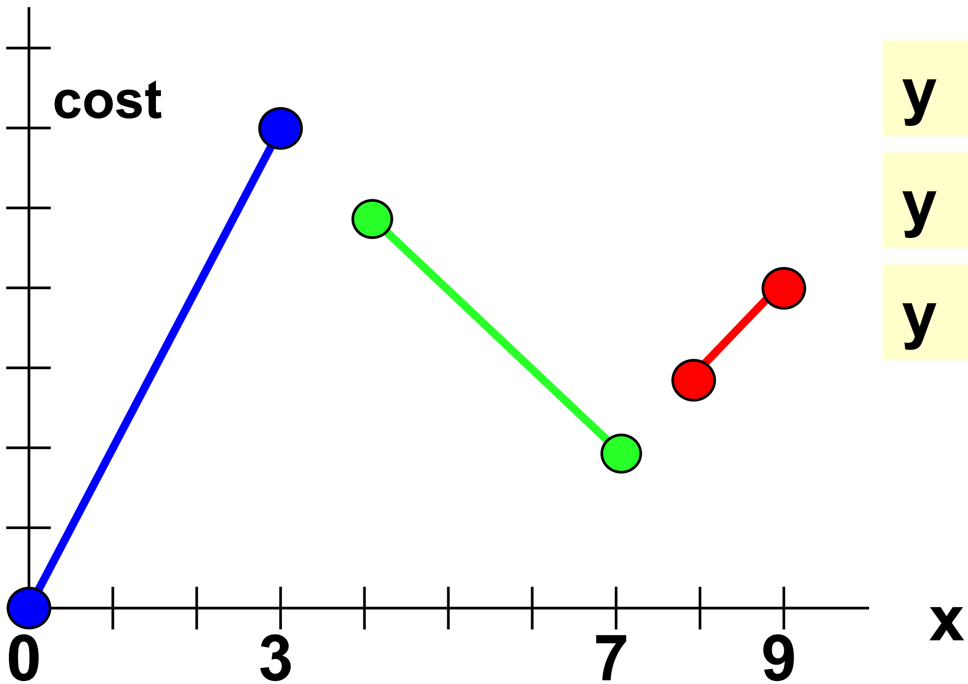


The constraint: "$x_j \leq 10\, w_j$" isn't correct because x_1 is permitted to be greater than 10 in the fixed charge problem.

On the other hand, the constraint "$x_j \leq 1000\, w_j$" is correct.

However, larger coefficients can make problems harder to solve. We say more about this in two lectures.

Modeling piecewise linear functions.

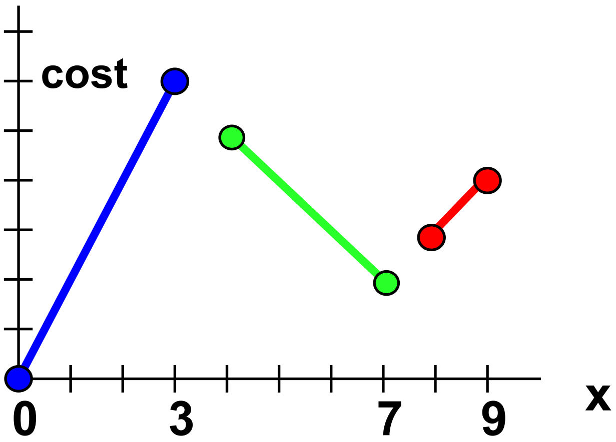


y =	$2x$	if $0 \le x \le 3$
y =	$9 - x$	if $4 \le x \le 7$
y =	$-5 + x$	if $8 \le x \le 9$

Assume that x is integer valued.

We will create an IP formulation so that the variable y is correctly modeled.

Create new binary and integer variables.



$y = 2x$ if $0 \le x \le 3$

$y = 9 - x$ if $4 \le x \le 7$

$y = -5 + x$ if $8 \le x \le 9$

x is integer valued.

$$w_1 = \begin{cases} 1 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad x_1 = \begin{cases} x & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$w_2 = \begin{cases} 1 & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases} \qquad x_2 = \begin{cases} x & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

$$w_3 = \begin{cases} 1 & 8 \le x \le 9 \\ 0 & \text{otherwise} \end{cases} \qquad x_3 = \begin{cases} x & 8 \le x \le 9 \\ 0 & \text{otherwise} \end{cases}$$

If the variables are defined as above, then

$$y = 2x_1 + (9w_2 - x_2) + (-5w_3 + x_3)$$

Add constraints

Definitions of the variables.

$w_1 = \begin{cases} 1 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$	$x_1 = \begin{cases} x & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$
$w_2 = \begin{cases} 1 & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$	$x_2 = \begin{cases} x & 4 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$
$w_3 = \begin{cases} 1 & 8 \le x \le 9 \\ 0 & \text{otherwise} \end{cases}$	$x_3 = \begin{cases} x & 8 \le x \le 9 \\ 0 & \text{otherwise} \end{cases}$

Constraints

$0 \le x_1 \le 3\, w_1$

$w_1 \in \{0, 1\}$

$4w_2 \le x_2 \le 7\, w_2$

$w_2 \in \{0, 1\}$

$8w_3 \le x_3 \le 9\, w_3$

$w_3 \in \{0, 1\}$

$w_1 + w_2 + w_3 = 1$

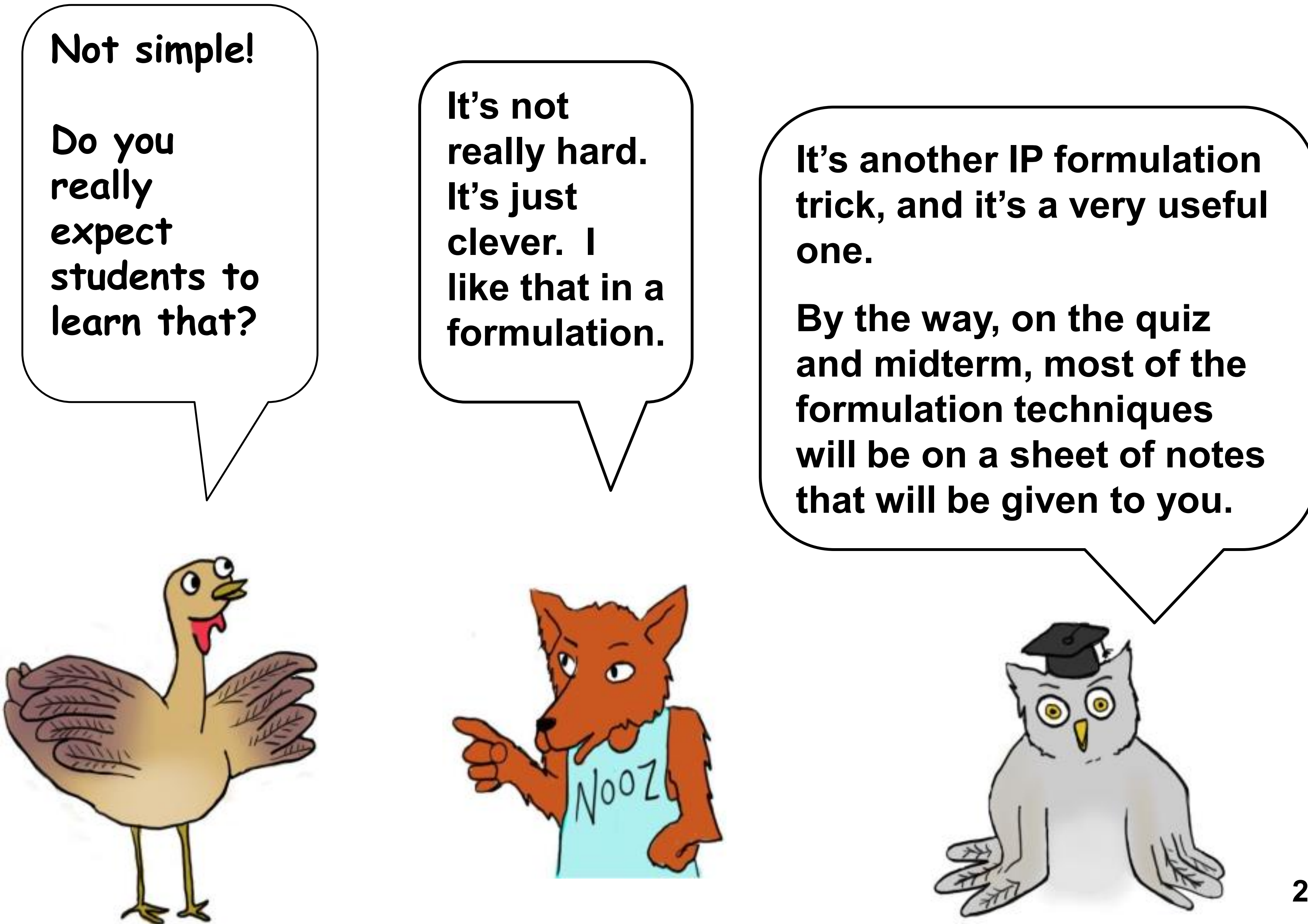
$x = x_1 + x_2 + x_3$

x_i integer $\forall\ i$

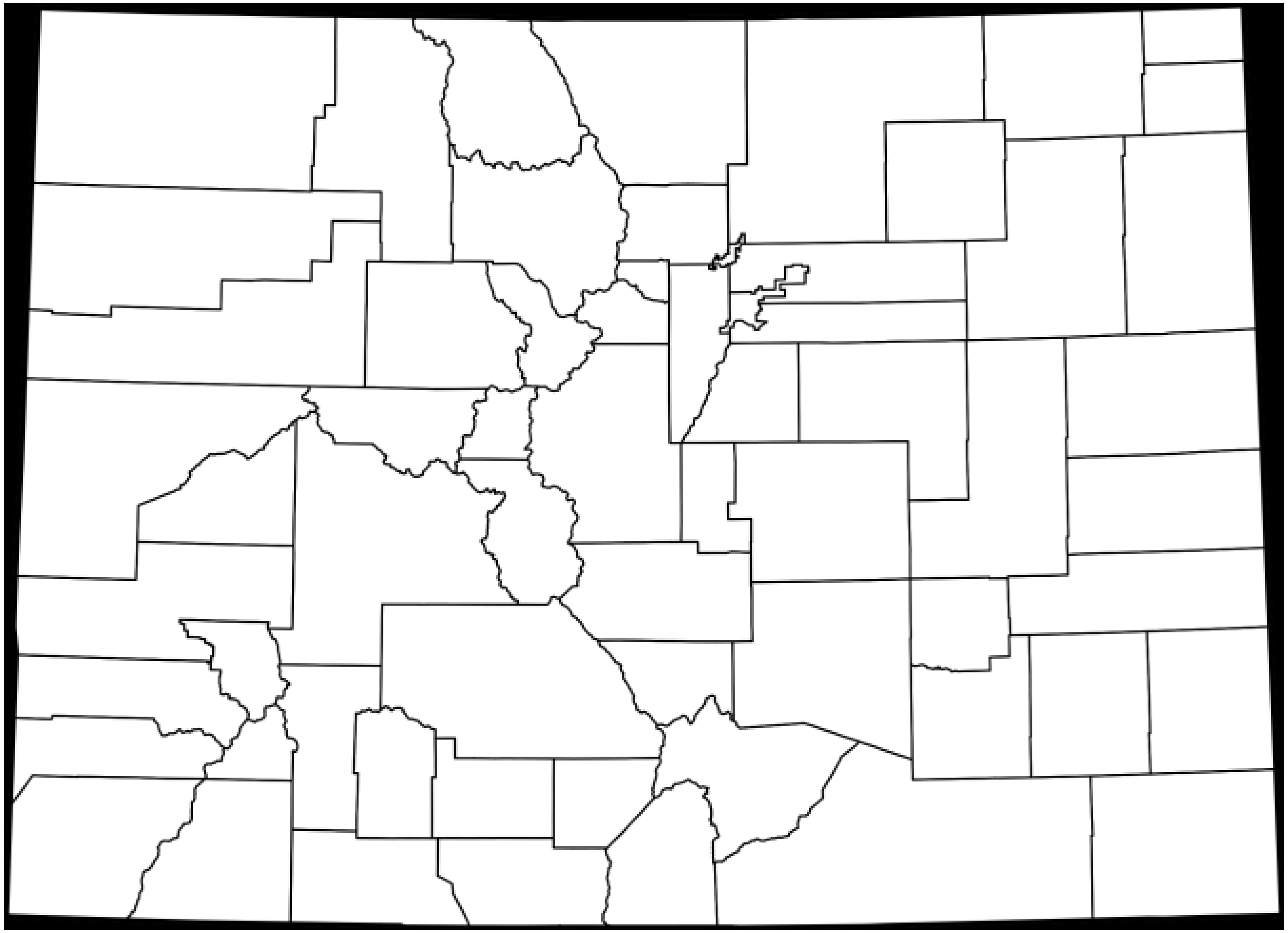
Suppose that $0 \le x \le 9$, x integer.

If (x, w) satisfies the definitions, then it also satisfies the constraints.

If (x, w) satisfies the constraints, then it also satisfies the definitions.


Not simple!
Do you really expect students to learn that?
It's not really hard. It's just clever. I like that in a formulation.
It's another IP formulation trick, and it's a very useful one.
By the way, on the quiz and midterm, most of the formulation techniques will be on a sheet of notes that will be given to you.
NOOZ

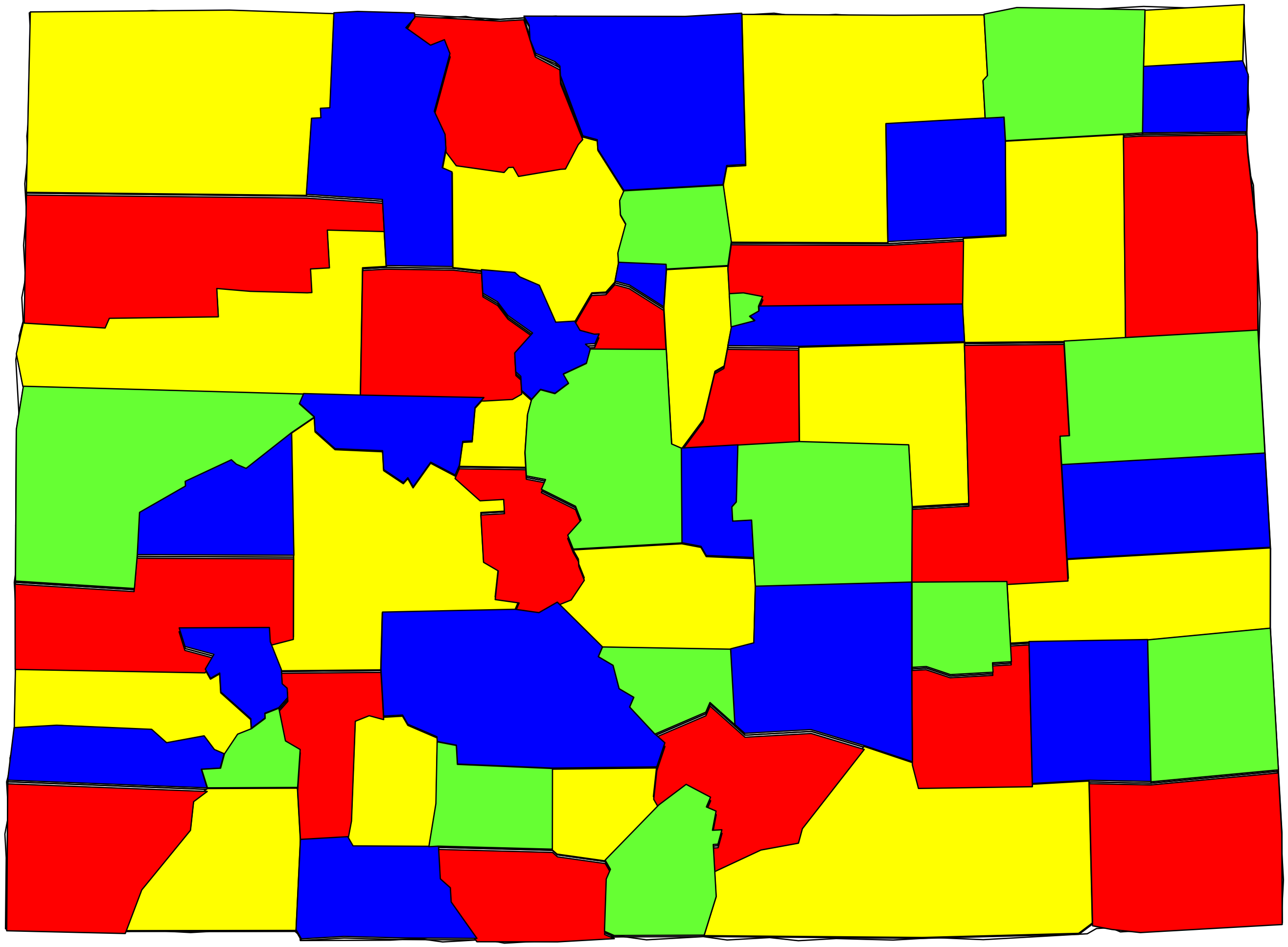
Graph Coloring



Public domain image.

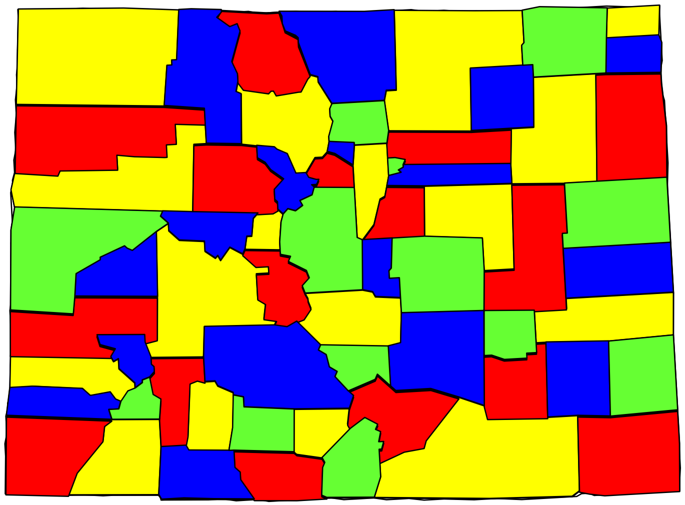
This is a map of the counties in Colorado. What is the fewest number of colors need to color all of the counties so that no counties with a common border have the same color?

Graph Coloring



Here is a four coloring of the map.

Graph Coloring Problem



G = (N, A)

N = {1, 2, 3, … n}
set of counties

A = set of arcs.
(i, j) ∈ A if counties
i and j are adjacent.

Exercise: write an integer program whose solution gives the minimum number of colors to color a map.

$$y_k = \begin{cases} 1 & \text{if color } k \text{ is used} \\ 0 & \text{if color } k \text{ is not used} \end{cases}$$

$$x_{ik} = \begin{cases} 1 & \text{if region } i \text{ is given color } k \\ 0 & \text{otherwise} \end{cases}$$

The Integer Programming Formulation

$$y_k = \begin{cases} 1 & \textbf{if color } k \textbf{ is used} \\ 0 & \textbf{if color } k \textbf{ is not used} \end{cases} \qquad x_{ik} = \begin{cases} 1 & \textbf{if region } i \textbf{ is given color } k \\ 0 & \textbf{otherwise} \end{cases}$$

Min $\sum_k y_k$ — Minimize the number of colors.

s.t

$\sum_k x_{ik} = 1 \quad \forall\ i \in N$ — Each county is given a color.

$x_{ik} + x_{jk} \leq 1$ **for** $(i, j) \in A$ **and** **for** $k = 1$ **to** 4 — If counties i and j share a common boundary, then they are not both assigned color k.

$x_{ik} \leq y_k$ **for** $i \in N$ **and** **for** $k = 1$ **to** 4 — If county i is assigned color k, then color k is used.

$$x_{ik} \in \{0, 1\} \qquad y_k \in \{0, 1\}$$

An Exam Scheduling Problem (coloring)

The University of Waterloo has to schedule 500 exams in 28 exam periods so that there are no exam conflicts.

G = (N, A)

N = {1, 2, 3, … n} set of exams. 28 periods.

A = set of arcs.
(i, j) ∈ A if a person needs to take exam i and exam j.

$$x_{ik} = \begin{cases} 1 & \text{if exam } i \text{ is assigned in period } k \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ik} + x_{jk} \leq 1 \quad \text{for } (i, j) \in A \text{ and } \forall k \in [1, 28]$$

Equivalently, can the exam conflict graph be colored with 28 colors?

Summary

- IPs can model almost any combinatorial optimization problem.

- Lots of transformation techniques.