



RECITATION NOTES #7

The Basics of Game Theory

Friday - November 5, 2004

OUTLINE OF TODAY'S RECITATION

- 1. Game theory definitions:** definitions of the most important terms in Game Theory
- 2. The Cournot Model:** what happens when two firms compete simultaneously on the quantity of output they produce of a homogeneous good.
- 3. The Stackelberg Model:** what happens when two firms compete sequentially on the quantity of output they produce of a homogeneous good.
- 4. The Bertrand Model:** what happens when two firms compete simultaneously on the price of a homogenous good.
- 5. Numeric Examples:** applying these concepts to exercises.

1. GAME THEORY DEFINITIONS

- 1.1 Dominant strategy**
- 1.2 Nash Equilibrium**
- 1.3 Maximin strategy**

1.1 Dominant Strategy

A player will have a dominant strategy if its choice is optimal regardless of what the opponent does. A strategy is dominant if it pays at least as much as any other strategy regardless the action of the other player.

1.2 Nash Equilibrium

In a Nash Equilibrium, each firm is doing the best it can, given what its competitors are doing. Nash equilibria are usually non-cooperative outcomes. Each firm chooses the strategy to maximize its profits given its opponent's actions. At the equilibrium, there is no incentive to change strategies, since you cannot improve payoffs.

1.3 Maximin Strategy

It is the strategy that minimizes a player's worst possible outcome. This strategy may be used by players concerned about their opponent's rationality.

2. THE COURNOT MODEL

2.1 Definition

2.2 Optimization in Cournot equilibrium

2.1 Definition

The Cournot model is a one period game, in which two firms produce an undifferentiated product with a known demand curve. The two firms compete by choosing their respective level of output **simultaneously**. Each firm chooses Q assuming their opponents' output is fixed.

2.2 Optimization in a Cournot game

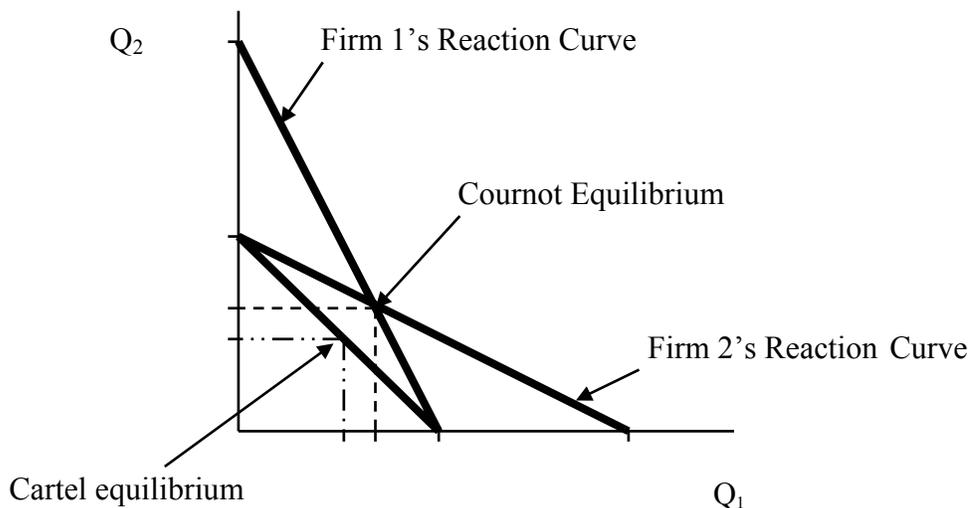
In a Cournot game, equilibrium is reached when each firm correctly assumes the opponents' output and chooses a level of output Q that maximize its own profits. There is no incentive for either firm to change from this equilibrium.

So, given a market demand of: $Q(P)$ and production levels by two producers of $Q = Q_1 + Q_2$, then in order to maximize profits, each company has to:

1. Calculate its Marginal Revenue as a function of Q_1 and Q_2 (using the equation $Q=Q_1+Q_2$)
2. Set this Marginal Revenue equal to the Marginal Cost
3. Solve for its Quantity. By doing so, the optimal level of quantity for each company will be given by an equation that is a function of the other firm's quantity. This equation is called **reaction curve** and illustrates the optimal level of quantity of each firm, given the other produced quantities:

$$Q_1^* = f(Q_2) \text{ and } Q_2^* = f(Q_1)$$

Graphically:



NOTE: A Cournot equilibrium is an example of a Nash equilibrium.

3. THE STACKELBERG MODEL

3.1 Definition

3.2 Optimizing in the Stackelberg model

3.1 Definition

This is a one period game, where two firms offer an undifferentiated product with known demand. Firms have to compete by choosing the amount of output Q_1 and Q_2 to produce, but **one of the two firms goes first**.

Firm 2 can observe what the Firm 1 has chosen for Q_1 , and choose Q_2 accordingly to maximize its profits. Furthermore, Firm 1 knows that Firm 2 will pursue this strategy since it can rely on the other firm's economic rationality.

3.2 Optimizing in the Stackelberg model

In a Stackelberg model, equilibrium is reached when Firm 1 pre-emptively expands output and secures larger profits. Hence the term "**first mover advantage**". In fact, Firm 2 is forced to curtail output given that the leader (firm 1) has already produced a large output ("As I produce more, you react by producing less").

4. THE BERTRAND MODEL

4.1 Definition

4.2 Optimizing in the Bertrand model

4.1 Definition

If two firms sell a homogenous good and have to choose the optimal level of prices (P_1 and P_2), for these goods simultaneously, then we can use the Bertrand Model to analyze the situation.

4.2 Optimizing in the Bertrand model

With homogenous goods, consumers will clearly buy the good from the firm that offers the lower price. Each firm therefore has an incentive to undercut the opponent's price by an infinitesimal amount and capture all the demand.

As a result, equilibrium is reached when each firm sets:

$$P=MC$$

since each firm knows it will undercut the opponent as long as it is economically viable to do so (in other words, they will undercut the opponent so long as lowering price does not imply incurring a loss. Always keep in mind that we are assuming rational behavior for the companies!)

NOTE: If the products were differentiated the equilibrium would be reached for prices above MC and firms would be able to make profits (a lower price would not move the whole demand!). Also, if this game was sequential and the goods were differentiated, the second mover would have a significant advantage over the first mover. **Also note that for firms it is much better for firms to compete on quantity, as in a Cournot model, than on price.** This is because when

firms compete on price, prices will be driven down to marginal costs and consequently economic profits will decrease.

5. NUMERIC EXAMPLES

5.1 Examples of Game theory

5.2 Examples of Cournot games

5.3 Example of a Stackelberg game

5.1 Game theory examples

5.1.1 US – Japan Trade Relations

This is problem 7 from Chapter 13 in P&R. *The two countries are considering policies to open or close their import markets. The payoff matrix is shown below [US payoff, Japan payoff].*

	Japan Open	Closed
US Open	10,10	5,5
Closed	-100,5	1,1

- a. *Assume that each country knows the payoff matrix and believes the other country will act in its own interest. Does either country have a dominant strategy? What will be the equilibrium policies if each country acts rationally to maximize its welfare?*

Dominant Strategy

- Japan: it will choose to open its market regardless of what the U.S. does (it is always the profit maximizing decision).
- U.S.: it will always choose to open its market regardless of what Japan does.

Equilibrium

- The equilibrium dictated by the dominant strategies is for both countries to open their import markets. This is a Nash equilibrium. Why?

- b. *Now let's assume that Japan is not certain that the US will behave rationally. In particular, Japan is concerned that US politicians may want to penalize Japan even if that does not maximize US welfare. How might this affect Japan's choice of strategy? How might this change the equilibrium?*

The question asks for Japan's maximin strategy (the strategy that maximizes Japan's outcome assuming that the US behaves irrationally). Japan's maximin strategy, it turns out, will continue to be to open its import sector. With an open Japanese import market, the US will be worse off, and hence, the situation will eventually move back to the open-open scenario.

5.1.2 Computers

This is exercise 3 of Chapter 13 in P&R. *Two computer firms, A and B, are planning to market network systems for office information management. Each firm can develop either a fast, high-quality system (H), or a slower, low-quality system (L). Market research indicates that the resulting profits to each firm for the alternative strategies are given by the following payoff matrix: [A payoff, B payoff]*

Firm A	Firm B	
	H	L
H	30,30	50,35
L	40,60	20,20

- a. *Does any player have a dominant strategy? What is the Nash Equilibrium*
- From the payoff matrix, neither player has a dominant strategy. There are two Nash equilibria as marked above.
- b. *What will be the outcome if both pursue maxi-min strategies? Does the maxi-min output make sense (Would competitors choose it)?*
- Let's start with Firm A. The maximin strategy seeks to avoid the worst possible outcome (Low / Low, or \$20). Firm A would choose to produce High quality systems to avoid the worst case scenario.
 - Similarly, Firm B will also avoid its worst scenario (Low / Low, or \$20). It will also choose to produce High quality systems.
 - The outcome if both firms pursue a maxi-min strategy would be High-High. This is sub-optimal when compared to the possible Nash Equilibrium. However, as this is a simultaneous game, coordination between the players might be difficult. This is particularly important since no player has a dominant strategy.
 - The maxi-min strategy might make sense in simultaneous games where coordination is difficult to achieve.
- c. *What will the outcome be if A commits first? If B commits first?*
- If A commits first: A knows what B's response will be in the case that it plays either high or low. A would then choose the outcome that yields the highest payoff (produce High, forcing B to produce Low).
 - If B commits first: Similarly, B knows what A's response will be in the case that it plays either high or low. B would then choose the outcome that yields the highest payoff (produce High, forcing A to produce Low).
 - This outcome is similar to the Stackelberg Model; the initial mover has an advantage.

d. *A head start costs money – Consider a two-stage game where, in Stage 1, each firm can pay an amount to become a first mover. Which firm will pay the most? Which firm will end up as a first mover?*

- Firm B is willing to spend \$25 (60-35) to be in the high column. Firm A is willing to spend a maximum of \$10 (50-40) to be in the high row. So, firm B is likely to outspend firm A. In fact, firm B should be able to offer \$11 and gain the first mover position.

5.1.3 Commitment and Credibility

You can find an in-depth analysis of this situation on page 481 of P&R. *This is a sequential game in which Race Car Motors is the leader. It will decide what kind of cars to build, and Far Out Engines will then decide what kind of engines to produce. The following payoff matrix shows the possible outcomes of this game. [Far Out Engines, Race Car Motors]*

	Race Car Motors	
Far Out Engines	Small cars	Big cars
Small engines	3,6	3,0
Big Engines	1,1	8,3

Race Car Motors will do best by deciding to produce small cars. It knows that in response to this, Far Out Engines will produce small engines. Far Out Engines, however, would much prefer the outcome in the lower right-hand corner of the matrix. Can it induce Race Car Motors to produce big cars instead of small ones?

- Far Out Engines can threaten to produce big engines. This threat, however, is not credible since once Race Car chooses to produce small cars there is not incentive for Far Out to produce big engines.
- One thing that Far Out Engines might do to make its threat credible is by irreversibly reducing its payoffs from producing small engines. For example, it can shut down its small engine production capacity.
- What is the new equilibrium if Far Out Engines shuts down its small engine production capacity?

	Race Car Motors	
Far Out Engines	Small cars	Big cars
Small engines	0,6	0,0
Big Engines	1,1	8,3

5.2 Cournot Equilibrium Examples

5.2.1 Example seen in class yesterday

Let's assume two firms face a demand curve of:

$$P = 60 - Q$$

And both have marginal costs equal to zero ($MC_1 = MC_2 = 0$)

Total production in the industry will be $Q_{TOT} = Q_1 + Q_2$

In order to maximize profits, firm 1 will have to calculate the following:

First it will set $MR = MC$

Where MR is calculated as follows:

$$\begin{aligned} R_1 = PQ_1 &= (60 - Q)Q_1 \\ &= 60Q_1 - (Q_1 + Q_2)Q_1 \\ &= 60Q_1 - (Q_1)^2 - Q_2Q_1 \end{aligned}$$

and:

$$MR_1 = dR_1/dQ_1 = 60 - 2Q_1 - Q_2$$

Setting $MR_1 = MC = 0$, the first company is able to calculate its reaction curve:

$$Q_1 = 30 - \frac{1}{2} Q_2$$

Similarly, the reaction curve for the second firm considering the symmetry in the cost structure, will be:

$$Q_2 = 30 - \frac{1}{2} Q_1$$

Equilibrium will therefore be reached when the two reaction curves are equal (because the reaction curves are exactly symmetrical), or when:

$$Q_1 = Q_2$$

The Equilibrium level therefore is: $Q_1 = Q_2 = 20$

The total quantity produced will be: $Q = Q_1 + Q_2 = 40$

The optimal price will be: $P = 60 - Q = 20$

And the firm's profits will be: $\Pi_1 = \Pi_2 = 20 \cdot 20 = 400$

5.2.2 The T-shirt market in the Cournot Equilibrium

You are the CEO of a firm that is producing T-shirts for the Latin C-Function (the year's "Best C-Function" and a Best of Boston Hall-of-Famer according to Boston Magazine). Assume you are operating as a duopolist. Both you and your competitor must decide now how many T-shirts to produce. (There is some lead-time involved in producing the T-shirts.) Once you have decided how many shirts to produce, you cannot revise your decision and you cannot produce a second set of shirts (i.e. you only play the "game" once). Assume that both you and your competitor's T-shirts are similar (i.e. undifferentiated products).

There is a market study (accessible to you and your competitor) that analyses the demand for these T-shirts. The market demand is given by the following equation:

$$Q=100-P$$

$$\text{i.e. } P=100-Q = 100 - (Q_1+Q_2) = 100 - Q_1 - Q_2$$

You and your competitor both have constant marginal cost equal to $MC_1=MC_2=10$

How many T-shirts should you produce?

In the Cournot model, each firm assumes its competitor's output is fixed and chooses output to maximize its own profits. Given this assumption, we can calculate each firm's reaction function. (Think of the best response function as the optimal quantity for Firm 1, Q_1 , given that Firm 2 chooses quantity Q_2) To do so, we first have to calculate Firm 1's marginal revenue function MR_1

$$TR_1 = P * Q_1 = (100 - Q_1 - Q_2) * Q_1$$

$$MR_1 = dTR_1 / dQ_1 = 100 - 2*Q_1 - Q_2$$

In order to maximize Firm 1's profits, set MR_1 equal to MC_1

$$\Leftrightarrow 100 - 2 * Q_1 - Q_2 = 10$$

$$\Leftrightarrow Q_1^* = 45 - 0.5 * Q_2$$

This equation is **Firm 1's reaction's curve**; it lets you calculate the optimum Q_1 given any particular level of Q_2 . So, if Firm 1 believes that its opponent will produce $Q_2 = 22.5$ units, Firm 1 should produce a Q_1 equal to 33.75. If Firm 1 assumes that $Q_2 = 45$, then it should produce $Q_1 = 22.5$.

Similarly, you can derive **Firm 2's reaction curve** (it is symmetric, given identical costs and demand)

$$Q_2^* = 45 - 0.5 * Q_1$$

You know from class that the Cournot Equilibrium (which is also a Nash Equilibrium) is given by the intersection of the two reaction functions. You have two equations with two unknowns and can solve for Q_1^* and Q_2^* .

You start with your best-response function for Firm 1:

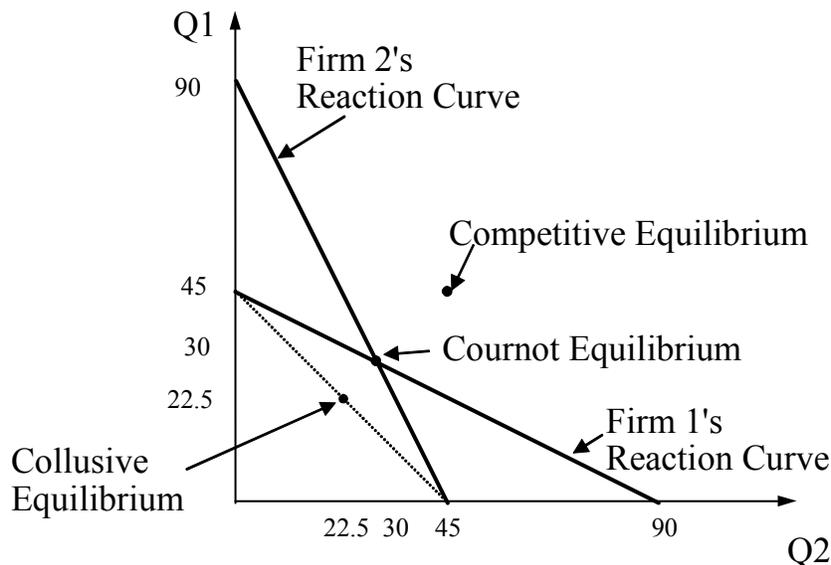
$$Q_1^* = 45 - 0.5 * Q_2$$

And plug in your best-response function for Firm 2 :

$$Q_1^* = 45 - 0.5 * (45 - 0.5 * Q_1)$$

$$\Leftrightarrow Q_1^* = 30$$

Substitute back into the equation for $Q_2^* \Rightarrow Q_2^* = 30$



At the Cournot Equilibrium each firm is doing the best given its opponent's output decision. There is no incentive to produce more or less since both firms are maximizing their profits (given the other firm's output).

Now price and profits are equal to:

$$P = 100 - (30 + 30)$$

$$\Rightarrow \underline{P^* = 40}$$

$$\pi_1 = TR_1 - TC_1 = P^*Q_1^* - 10Q_1 = 40 \cdot 30 - 10 \cdot 30 = \underline{900}$$

By symmetry $\pi_2 = \underline{900}$

You are still not convinced that producing 30 units is the optimal decision. Let's examine the following payoff matrix. The payoff matrix provides you with the profits of Firm 1 and Firm 2 for a given combination of output Q_1 and Q_2 . Here we have chosen four different output levels for each firm and calculated the resulting profits. We leave it up to you to confirm these numbers. (Note: The payoff matrix concept is commonly used tool in game theory. You will have a chance of revisiting this concept in a number of examples during the next few weeks).

Some general conventions: Firm 1's decisions are in the first column (in this case Firm 1's quantity decisions). Firm 2's decisions are in the first row. Firm 1's payoff is on the left of each cell and Firm 2's payoff is on the right. The underline shows the optimum decisions for a firm given a decision of the other Firm (e.g. Firm 2 chooses 22.5 so the best decision for Firm 1 would be 33.75 which yields a profit of 1139.1).

		Output Firm 2			
		22.5	30	33.75	45
Output Firm 1	22.5	1012.5, 1012.5	843.8, 1125.0	759.4, <u>1139.1</u>	<u>506.3</u> , 1012.5
	30	1125.0, 843.8	<u>900.0</u> , <u>900.0</u>	<u>787.5</u> , 885.9	450.0, 675.0
	33.75	<u>1139.1</u> , 759.4	885.9, 787.5	759.4, 759.4	379.7, 506.3
	45	1012.5, <u>506.3</u>	675.0, 450.0	506.3, 379.7	0.0, 0.0

 Cournot Equilibrium (Nash Equilibrium)

How can we figure out the best output decision from this payoff matrix? Let's start with the assumption that you expect your competitor to produce an output of 22.5 units. Given this assumption should you also produce 22.5 units? No, in this case you make an even higher profit by producing 33.75 units ($\pi_1 = 1139$). Knowing this, why should your competitor produce 22.5? Assuming that you will produce 33.75, she will produce 30 units to maximize her profits. Assuming that your competitor will produce 30 units you also will produce 30 units. The Cournot Equilibrium $Q_1 = 30, Q_2 = 30$ is the only rational outcome.

5.3 Example of a Stackelberg Game

Let's consider again the previous example of the T-shirt industry. Let's also assume that you are the first firm to choose output. Now you can take your opponent's reaction function into account. Firm 2's output decision will be determined by your output. You already know your opponent's reaction function from the previous example:

$$Q_2^* = 45 - 0.5 * Q_1 \quad \text{Eqn. (I)}$$

With this information you can calculate your marginal revenue function:

$$TR_1 = P * Q_1 = (100 - (Q_1 + Q_2)) * Q_1$$

Substitute Eqn (I) – This substitution is necessary because you want to take into account your competitor's reaction if you set quantity first. If you set a quantity = Q_1 , your competitor will set a quantity given by Eqn I.

$$\begin{aligned} TR_1 &= (100 - (Q_1 + 45 - 0.5 * Q_1)) * Q_1 \\ &= 55 * Q_1 - 0.5 * Q_1^2 \end{aligned}$$

$$\Rightarrow MR_1 = dTR_1 / dQ_1 = 55 - Q_1$$

To maximize your profits set MR_1 equal to MC_1 :

$$\Rightarrow 55 - Q_1 = 10$$

$$\Rightarrow \underline{Q_1 = 45}$$

Plugging this number back into Eqn. (I), Firm 2 will produce **$Q_2 = 22.5$ units**. The profits are **$\pi_1 = 1012.5$** and **$\pi_2 = 506$** . You actually do better by setting your output level first. You have a first mover advantage. By fixing your output first you can preempt the market. Assuming that you committed yourself to produce 45 T-shirts, your competitor has no other means than to accept this fact and it's forced to produce less assuming he wants to maximize his profits).