Problem Set 7 Solution

17.881/882

November 8, 2004

1 Gibbons 2.3 (p.131)

Let us consider the three-period game first.

1.1 The Three-Period Game

The structure of the game as was described in section 2.1D (pp.68-71). Let us solve the game backwards.

1.1.1 Stage (2b)

Player 1 accepts player 2's proposal $(s_2, 1 - s_2)$ if and only if the following condition is satisfied:

$$s_2 \geq \delta_1 s$$

1.1.2 Stage (2a)

Conditional on player 1 accepting the offer, player 2 maximises his/her payoff by offering $s_2 = \delta_1 s$. Then, player 2 gets $\delta_2(1 - s_2) = \delta_2(1 - \delta_1 s)$.

Any rejected offer leads 2 to get a payoff of $\delta_2^2(1 - s)$. Player 2 is better off with a proposal that is accepted if and only if:

$$\begin{aligned} d_2 &= \delta_2(1 - \delta_1 s) - \delta_2^2(1 - s) \\ &= \delta_2[1 - \delta_2 + s(\delta_2 - \delta_1)] \geq 0 \end{aligned}$$

If $\delta_2 \geq \delta_1$, we have $d_2 \geq \delta_2[1 - \delta_2] > 0$ since $0 < \delta_2 < 1$.

If $\delta_2 < \delta_1$, we have $d_2 \geq \delta_2[1 - \delta_1] > 0$ since $0 < \delta_1, \delta_2 < 1$.

Either way, we have that player 2 prefers to have his/her proposal accepted, and offers $s_2 = \delta_1 s$

1.1.3 Stage (1b)

Player 2 accepts player 1's proposal $(s_1, 1-s_1)$ if and only if:

$$\begin{aligned} 1-s_1 &\geq \delta_2(1-s_2) \\ s_1 &\leq 1-\delta_2[1-\delta_1 s] \end{aligned}$$

1.1.4 Stage (1a)

Conditional on player 2 accepting the offer, player 1 maximises his/her payoff by offering $s_1 = 1-\delta_2[1-\delta_1 s]$.

Any rejected offer leads 1 to get a payoff of $\delta_1 s_2 = \delta_1^2 s$. Player 1 is better off with a proposal that is accepted if and only if:

$$\begin{aligned} d_1 &= 1-\delta_2[1-\delta_1 s]-\delta_1^2 s \\ &= 1-\delta_2+s\delta_1(\delta_2-\delta_1) \geq 0 \end{aligned}$$

If $\delta_2 \geq \delta_1$, we have $d_1 \geq 1-\delta_2 > 0$ since $0 < \delta_2 < 1$.

If $\delta_2 < \delta_1$, we have $d \geq (1-\delta_2+\delta_1)(1-\delta_1) > 0$ since $0 < \delta_1, \delta_2 < 1$.

Either way, we have that player 1 prefers to have his/her proposal accepted, and offers $s_1 = 1-\delta_2[1-\delta_1 s]$

The outcome of the game is that players 1 and 2 agree on the distribution $(s_1^*, 1-s_1^*) = (1-\delta_2[1-\delta_1 s], \delta_2[1-\delta_1 s])$.

1.2 The Infinite-Horizon Game

Let s be a payoff that player 1 can get in a backwards-induction of the game as a whole, and s_H the maximum value of s.Imagine using s as the third-payoff to Player 1. Player 1's first-period payoff is a function of s, namely $f(s) = 1-\delta_2[1-\delta_1 s]$. Since this function is increasing in s, $f(s_H)$ is the highest possible first-period payoff, so $f(s_H) = s_H$. Then

$$\begin{aligned} 1-\delta_2[1-\delta_1 s_H] &= s_H \\ &\iff s_H = \frac{1-\delta_2}{1-\delta_1\delta_2} \end{aligned}$$

A parallel argument shows that $f(s_L) = s_L$, where $f(s_L)$ is the lowest payoff that player 1 can achieve in any backwards-induction of the game as a whole. Therefore, the only value of s that satisfies $f(s) = s$ is $\frac{1-\delta_2}{1-\delta_1\delta_2}$. Thus $s_H = s_L = s^*$, so there is a unique backwards-induction outcome of the game as a whole: A distribution

$$(s^*, 1-s^*) = \left\{ \frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \right\}$$