

1.6 Orbits in spherical potentials

Our method for building galaxy models (often called “Schwarzschild’s method,” after Martin Schwarzschild, son of the general relativist Karl) requires that we construct an orbit library in which we store the “footprints” or probability densities of a representative set of orbits. These are then superposed to construct the luminosity distribution. Since orbits play such a major role, we discuss them in some detail

We begin by calculating orbits for the simplest systems mentioned above, those with spherically symmetric potentials $\Phi = \Phi(r)$. It is easy to get lost in the details of the calculation, so we start with an outline of the procedure we will follow:

- a) Discover that the vector angular momentum is conserved and reduce the dimensionality of the problem to motion in a plane.
- b) Split the vector equation of motion into two scalar equations of motion.
- c) Change the independent variable from time to angle.
- d) Change the dependent variable from radius to inverse radius.
- e) Multiply by the derivative of inverse radius with respect to angle and integrate, giving a differential equation for the square of the derivative of inverse radius with respect to angle. The constant of integration is proportional to the energy per unit mass divided by the angular momentum squared.
- f) Solve the differential equation for inverse radius as a function of angle.

The experience of seeing this for the first time is not unlike that of an undergraduate being led from one side of the campus to the other via basements and tunnels.

We begin with the central force equation (where we adopt the convention of expressing force $\vec{\mathbf{F}}$, energy E , and angular momentum $\vec{\mathbf{L}}$ *per unit mass*)

$$\frac{d^2}{dt^2}\vec{\mathbf{r}} = -\vec{\nabla}\Phi(r) = \vec{\mathbf{F}} = F(r)\hat{\mathbf{e}}_r. \quad (1.60)$$

Combined with the angular momentum definition

$$\vec{\mathbf{L}} \equiv \vec{\mathbf{r}} \times \vec{\mathbf{v}}, \quad (1.61)$$

gives the conservation of angular momentum in central-force systems:

$$\frac{d}{dt}\vec{\mathbf{L}} = \frac{d}{dt}\left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{r}}}{dt}\right) = \frac{d\vec{\mathbf{r}}}{dt} \times \frac{d\vec{\mathbf{r}}}{dt} + \vec{\mathbf{r}} \times \frac{d^2\vec{\mathbf{r}}}{dt^2} = 0, \quad (1.62)$$

so we immediately see that the orbit will be constrained to a plane perpendicular to $\vec{\mathbf{L}}$. Within this plane we adopt coordinates (r, ψ) with

$$L = r^2 \frac{d\psi}{dt}. \quad (1.63)$$

The equations of motion in this new coordinate system are

$$\begin{aligned}\vec{\mathbf{r}} &= r\hat{\mathbf{e}}_r \\ \frac{d}{dt}\vec{\mathbf{r}} &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\psi}\hat{\mathbf{e}}_\psi \\ \frac{d^2}{dt^2}\vec{\mathbf{r}} &= (\ddot{r} - r\dot{\psi}^2)\hat{\mathbf{e}}_r + (2\dot{r}\dot{\psi} + r\ddot{\psi})\hat{\mathbf{e}}_\psi,\end{aligned}\tag{1.64}$$

where we have used the relations

$$\frac{d}{dt}\hat{\mathbf{e}}_r = \dot{\psi}\hat{\mathbf{e}}_\psi \quad \text{and} \quad \frac{d}{dt}\hat{\mathbf{e}}_\psi = -\dot{\psi}\hat{\mathbf{e}}_r.\tag{1.65}$$

Separating the central force equation (1.60) into two scalar equations of motion gives

$$\ddot{r} - r\dot{\psi}^2 = F(r)\tag{1.66}$$

and

$$2\dot{r}\dot{\psi} + r\ddot{\psi} = 0.\tag{1.67}$$

We change the both the dependent and independent variables in the radial equation, to ψ and $u \equiv 1/r$ respectively. We then have

$$\frac{d}{dt} = \frac{L}{r^2} \frac{d}{d\psi}\tag{1.68}$$

from equation (1.63) and

$$\frac{1}{r^2} \frac{d}{d\psi} r = -\frac{d}{d\psi} u \quad .\tag{1.69}$$

The radial equation (1.66) becomes

$$\frac{d^2 u}{d\psi^2} + u = -\frac{F(u)}{L^2 u^2}.\tag{1.70}$$

We assume that $F(r)$ can be derived from a potential $F(r) = -\nabla\Phi$. We then use the common trick of turning a force equation into an energy equation by multiplying by a first derivative of position (in our case $\frac{du}{d\psi}$) and integrating with respect to the timelike variable ψ giving

$$\left(\frac{du}{d\psi}\right)^2 + u^2 + \frac{2\Phi}{L^2} = \frac{2E}{L^2},\tag{1.71}$$

where E is the conserved energy per unit mass, expressed in our original coordinates as

$$E = \frac{1}{2} \left(\frac{dr}{dt}\right)^2 + \frac{1}{2} r^2 \left(\frac{d\psi}{dt}\right)^2 + \Phi(r).\tag{1.72}$$

We can solve for the turning points in the orbit (where $\dot{r} = 0$) by setting $E = \Phi_{\text{eff}}$, where $\Phi_{\text{eff}}(r)$ is the effective potential

$$\Phi_{\text{eff}} = \frac{1}{2} \frac{L^2}{r^2} + \Phi(r).\tag{1.73}$$

To proceed further we must choose a potential. There are a wide variety of potentials used in the study of galaxies. Most of these require numerical integration of orbits. Here we consider some special cases which can be solved analytically and which serve to make interesting points about orbits.

- a) **Harmonic Potential:** This is just like a spherically symmetric harmonic oscillator, giving closed elliptical orbits centered on the origin,

$$\Phi(r) = \frac{1}{2}\Omega^2 r^2. \quad (1.74)$$

- b) **Kepler Potential:** This is the gravitational potential for a single point mass,

$$\Phi(r) = -\frac{GM}{r}. \quad (1.75)$$

Taking the derivative of this potential with respect to r and substituting into our equation for the second derivative of inverse radius with respect to angle, equation (1.70) gives

$$\frac{d^2 u}{d\psi^2} + u = \frac{GM}{L^2}. \quad (1.76)$$

which is just the equation for a simple harmonic in u with

$$u = C \cos(\psi - \psi_0) + \frac{GM}{L^2}. \quad (1.77)$$

Rearranging terms we have

$$r = \frac{a(1 - e^2)}{1 + e \cos(\psi - \psi_0)}, \quad (1.78)$$

where

$$a \equiv \frac{L^2}{GM(1 - e^2)}, \quad (1.79)$$

and e is the eccentricity. Note that there are three constants in the above equations, a , e and ψ_0 . But there are two more hidden constants, the angles giving the orientation of the orbital plane. Thus the orbit traces a one dimensional locus in six dimensional phase space. We shall loosely call such constants which reduce the dimensionality of the allowed phase space *integrals of the motion*.

- c) **Post-Newtonian Kepler Potential:** To lowest order the effects of general relativity can be approximated by adding a “post-Newtonian” correction to the Newtonian potential,

$$\Phi(r) = -\frac{GM}{r} \left(1 + \frac{k}{r}\right), \quad (1.80)$$

where $k = 2GM/c^2$ has units of length. Substituting this into our general equation for the spherically symmetric case (1.71) we find

$$\frac{d^2 u}{d\psi^2} + u \left(1 - \frac{2GMk}{L^2}\right) = \frac{GM}{L^2}. \quad (1.81)$$

This *again* gives simple harmonic motion in the radial direction, but the radial frequency is no longer unity. The radial and azimuthal oscillations now have different periods, and the orbits are no longer closed ellipses. The periastron precesses. One of the early successes of the theory of general relativity was its successful explanation of the precession of the perihelion of Mercury’s orbit. For larger k the orbit looks like the “rosette” shown in Figure 1.4.

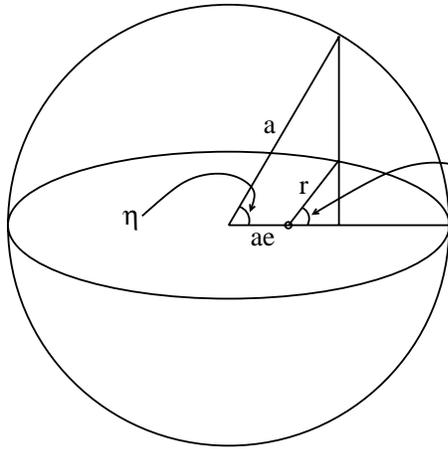


Figure 1.14: An elliptical orbit with semi-major axis a and eccentricity e . The eccentric anomaly η is defined by circumscribing a circle.

d) **Witta-Paczyński Potential:** This has the functional form

$$\Phi(r) = -\frac{GM}{r - \frac{2GM}{c^2 r}} \quad , \quad (1.82)$$

which reduces to the post-Newtonian approximation in the limit of large r . It also has the property of having an innermost stable circular orbit (ISCO) that coincides with the innermost stable orbit of a Schwarzschild black hole. It allows approximate Newtonian calculations of the dynamics of accretion disks without the apparatus of general relativity.

1.6.1 digression on the Kepler problem

The solution provided to the Kepler problem above is mathematically correct and practically useless. Observations give position or velocity as a function of time, not angle. Unfortunately there is no closed form expression for the position as a function of time. But position and time can be given parametrically in terms of an angle η called the *eccentric anomaly*. The angle ψ is called the *true anomaly*. Both angles are shown in Figure 1.14. Thus parameterized we have

$$r = a(1 - e \cos \eta) \quad \text{and} \quad (1.83)$$

$$t = \frac{P}{2\pi}(\eta - e \sin \eta), \quad (1.84)$$

where P is the orbital period given by Kepler's third law

$$P = 2\pi \sqrt{\frac{a^3}{GM}} \quad . \quad (1.85)$$

Once can solve for the true anomaly in term of the eccentric anomaly, and *vice versa* using

$$\tan \frac{\psi}{2} = \left(\frac{1+e}{1-e} \right)^{1/2} \tan \frac{\eta}{2} \quad . \quad (1.86)$$

We will see this parametric solution to Kepler's potential again when studying the gravitational expansion and collapse of the Universe.

1.6.2 digression on the virial theorem

So as not to interrupt our narrative thread we postponed a proof of the virial theorem, equation (1.40),

$$\langle v^2 \rangle_{\text{orbit}} = \langle \vec{\mathbf{r}} \cdot \vec{\nabla} \Phi \rangle_{\text{orbit}} \quad .$$

Having now given further attention to orbits we proceed with our proof. Taking the dot product of equation (1.60) with $\vec{\mathbf{r}}$ and averaging over time gives

$$\left\langle \vec{\mathbf{r}} \cdot \frac{d^2 \vec{\mathbf{r}}}{dt^2} \right\rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \vec{\mathbf{r}} \cdot \frac{d^2 \vec{\mathbf{r}}}{dt^2} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \left[\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} \Big|_0^T - \int_0^T \left(\frac{d\vec{\mathbf{r}}}{dt} \right)^2 dt \right]. \quad (1.87)$$

For bounded orbits, the long-time average of $\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt}$ is zero, so we are left with

$$-\left\langle \vec{\mathbf{r}} \cdot \frac{d^2 \vec{\mathbf{r}}}{dt^2} \right\rangle_{\text{time}} = \left\langle \left(\dot{\vec{\mathbf{r}}} \right)^2 \right\rangle_{\text{time}} = \langle \vec{\mathbf{r}} \cdot \vec{\nabla} \Phi \rangle_{\text{time}} = \langle v_c^2 \rangle_{\text{time}} \quad (1.88)$$

where the last equality holds only for cylindrically symmetric potentials. These results can be very helpful in calculating distributions of orbits in phase space, but one must be careful not to abuse the Virial Theorem. One may, for example, replace the *time* average in equation (1.88) with a *volume* average, but only if all stars and orbits are included. The result does not apply, in particular, if the average is taken over only a finite region of position space.