

1.10 Jeans' equation in spherical coordinates

We start by writing down the collisionless Boltzmann equation (1.119) in spherical coordinates (r, θ, ϕ) :

$$\frac{\partial f}{\partial t} + \dot{r} \frac{\partial f}{\partial r} + \dot{\theta} \frac{\partial f}{\partial \theta} + \dot{\phi} \frac{\partial f}{\partial \phi} + \dot{v}_r \frac{\partial f}{\partial v_r} + \dot{v}_\theta \frac{\partial f}{\partial v_\theta} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} = 0, \quad (1.136)$$

where the time derivatives of the coordinates may be expressed in terms the velocity components,

$$\begin{aligned} \dot{r} &= v_r \quad , \\ \dot{\theta} &= \frac{v_\theta}{r} \quad \text{and} \\ \dot{\phi} &= \frac{v_\phi}{r \sin \theta} \quad . \end{aligned} \quad (1.137)$$

Lagrange's equations give the components of the acceleration,

$$\begin{aligned} \dot{v}_r &= \frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial \Phi}{\partial r} \quad , \\ \dot{v}_\theta &= \frac{v_\phi^2 \cot \theta - v_r v_\theta}{r} - \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \quad \text{and} \\ \dot{v}_\phi &= \frac{-v_\phi v_r - v_\phi v_\theta \cot \theta}{r} - \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \quad . \end{aligned} \quad (1.138)$$

(Continued on next page.)

One then substitutes these into the CBE (1.136), multiplies by the powers of v_r , v_θ and v_ϕ , and integrates over velocity space to obtain moment equations. These give one Jeans equation for each component of the potential gradient (force term). The radial Jeans equation is

$$\begin{aligned} \nu \frac{\partial \bar{v}_r}{\partial t} + \nu \left(\bar{v}_r \frac{\partial \bar{v}_r}{\partial r} + \frac{\bar{v}_\theta}{r} \frac{\partial \bar{v}_r}{\partial \theta} + \frac{\bar{v}_\phi}{r \sin \theta} \frac{\partial \bar{v}_r}{\partial \phi} \right) + \frac{\partial}{\partial r} (\nu \sigma_{rr}^2) + \frac{1}{r} \frac{\partial}{\partial \theta} (\nu \sigma_{r\theta}^2) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\nu \sigma_{r\phi}^2) \\ + \frac{\nu}{r} [2\sigma_{rr}^2 - (\sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2 + \bar{v}_\theta^2 + \bar{v}_\phi^2) + \sigma_{r\theta}^2 \cot \theta] = -\nu \frac{\partial \Phi}{\partial r} \quad . \quad (1.139) \end{aligned}$$

This may seem daunting (especially when one considers that the polar and azimuthal equations are similarly lengthy), but in practice one typically makes a number of simplifying assumptions before invoking the radial Jeans equation:

- a) Steady-state hydrodynamic equilibrium implies $\frac{\partial}{\partial t} = 0$ and $\bar{v}_r = 0$.
- b) Spherical symmetry implies $\bar{v}_\theta = \bar{v}_\phi = 0$, $\sigma_{r\theta}^2 = \sigma_{r\phi}^2 = \sigma_{\theta\phi}^2 = 0$, and a single tangential velocity dispersion (“temperature”) $\sigma_{t1}^2 = \sigma_{\theta\theta}^2 = \sigma_{\phi\phi}^2$.

With these assumptions the radial Jeans Equation becomes considerably more manageable,

$$\frac{1}{\nu} \frac{\partial}{\partial r} (\nu \sigma_{rr}^2) + 2 \frac{(\sigma_{rr}^2 - \sigma_{t1}^2)}{r} = -\frac{\partial \Phi}{\partial r} = -\frac{GM(r)}{r^2}, \quad (1.140)$$

which reduces to spherical hydrostatic equilibrium when the velocity dispersion tensor is isotropic and $\sigma_{rr}^2 = \sigma_{t1}^2 = \sigma^2$,

$$\frac{1}{\nu} \frac{\partial \nu \sigma^2}{\partial r} = -\frac{\partial \Phi}{\partial r} = -\frac{GM_r}{r^2} \quad . \quad (1.141)$$

To measure the departure from this condition, we define the *anisotropy parameter* β as

$$\beta \equiv 1 - \frac{\sigma_{t1}^2}{\sigma_{rr}^2} \quad (1.142)$$

which can take values in the range $-\infty < \beta < 1$, with the extremes corresponding, respectively, to purely circular and purely radial orbits. The spherical Jeans equation (1.140) can be rewritten to give an expression for the mass $M(r)$ within a radius r

$$M(r) = -\frac{r\sigma_{rr}^2}{G} \left[\underbrace{\frac{d \ln \nu}{d \ln r}}_{\approx -3} + \underbrace{\frac{d \ln \sigma_{rr}^2}{d \ln r}}_{\approx -0.2} + \underbrace{2\beta(r)}_{?} \right]. \quad (1.143)$$

The difficulty with determining $\beta(r)$ from observations, is that one can only measure the line-of-sight velocity dispersion σ_{los}^2 . It is not in general possible to make independent measurements of σ_{rr}^2 and σ_{t1}^2 (see Figure 1.22).

The above table gives “typical” values of the velocity dispersion and scale length for a giant elliptical galaxy and a very rich cluster of galaxies. The cluster of galaxies will have of order 100 bright galaxies (and a great many very much fainter galaxies). We can use these values and equation (1.143)

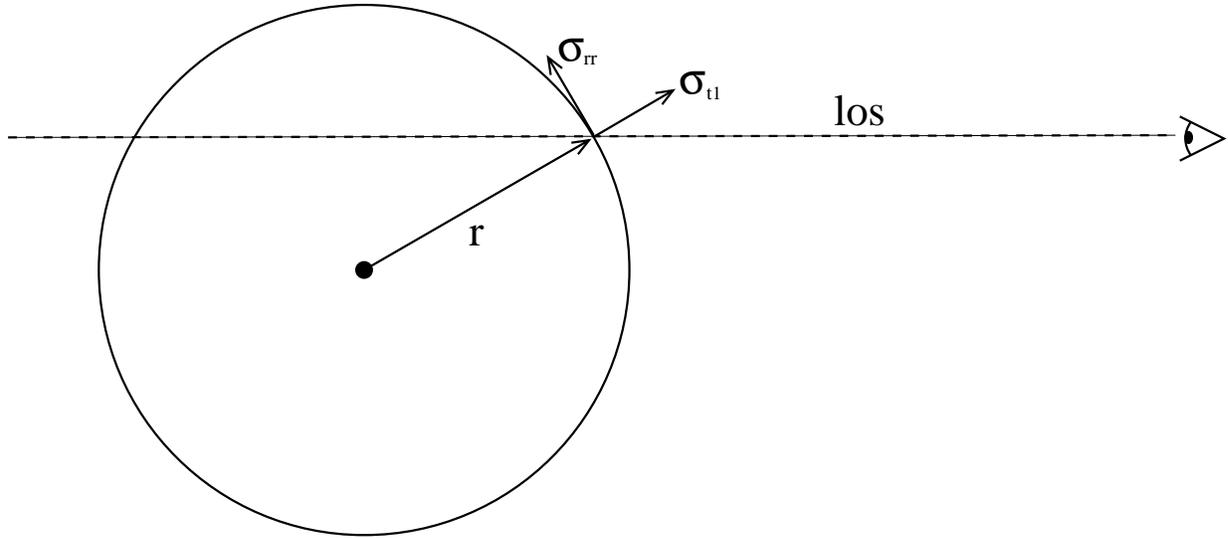


Figure 1.22: An observer measuring the Doppler broadening of lines in a galaxy spectrum sees some combination of the radial and tangential velocity dispersion, averaged over the line of sight.

system	σ_{rr}^2	N	R
galaxy	$(300 \text{ km/s})^2$	10^{11}	10 kpc
cluster	$(1000 \text{ km/s})^2$	10^2	1000 kpc

above to calculate a dynamical mass for a rich cluster. This can be compared to the dynamical masses computed for the constituent galaxies. The ratio of these two is

$$\frac{M_{\text{cluster}}}{N_{\text{gal}} M_{\text{gal}}} \approx \frac{1}{N_{\text{gal}}} \frac{\sigma_{\text{clus}}^2}{\sigma_{\text{gal}}^2} \frac{R_{\text{clus}}}{R_{\text{gal}}} \approx \frac{1}{100} \cdot 10 \cdot 100 = 10. \quad (1.144)$$

This very large discrepancy was first noted by Zwicky in the mid-thirties. It was at first known as the “missing mass” problem, but “missing light” would have been more correct, as the mass was surely present. For the next 40 years this problem was given scant attention, or dismissed as the result of some combination of bad data and bad modeling. When the first X-ray observations of clusters were made in the 1970s, very different observations and modeling led to the same conclusion. The missing mass problem became part of the larger “dark matter” problem that we first encountered within the Milky Way. Thirty years of effort have failed to produce a non-gravitational detection of this dark matter. In the meantime evidence (to be described later) has mounted that this matter must be non-baryonic.

Starting with the hydrostatic equation (1.141) and making the additional assumption of *isothermality*, taking the velocity dispersion (σ^2) to be independent of radius, allows us to move the velocity dispersion outside the derivative. Taking all particles to have the same mass m the density is then $\rho = m\nu$ and the hydrostatic equation reduces to

$$\frac{\sigma^2}{\rho} \frac{\partial \rho}{\partial r} = -\frac{\partial \Phi}{\partial r} = -\frac{GM_r}{r^2}. \quad (1.145)$$

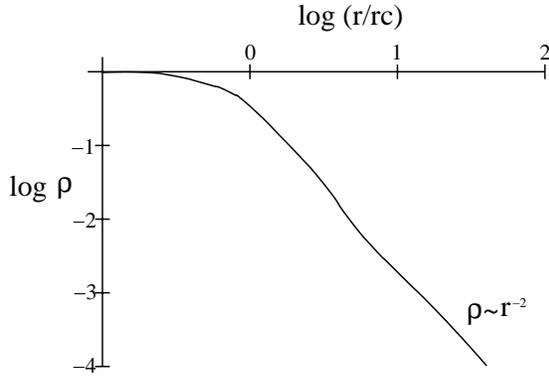


Figure 1.23: A schematic representation of the density profile of an isothermal sphere.

Multiplying this by r^2 , and differentiating with respect to r and then multiply by $1/r^2$ we get

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \rho \right) = -\frac{4\pi G \rho}{\sigma^2} . \quad (1.146)$$

Substituting $\rho \equiv \lambda e^\psi$ (with the central density given by $\rho_c = \lambda$), we get

$$\left(\frac{\sigma^2}{4\pi G \lambda} \right) \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \psi \right) = -e^\psi . \quad (1.147)$$

Making a change of variables $\xi \equiv \frac{r}{\alpha}$ with $\alpha^2 = \left(\frac{\sigma^2}{4\pi G \lambda} \right)$, we get

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \psi \right) = -e^\psi , \quad (1.148)$$

which looks much like the Lane-Emden equation for polytropic stellar models, except the right-hand-side is an exponential instead of a power law. Indeed it is the limiting case of the Lane-Emden analysis for infinite polytropic index. The solution to this equation is called the *isothermal sphere*. We are interested in solutions with boundary conditions $\psi(0) = \psi'(0) = 0$. Figures 1.23 and 1.24 show the density and circular velocity of the isothermal sphere as a function of radius. Note that for large r , they approach the asymptotic values first mentioned in Sec. (1.5):

$$\frac{v_c^2}{\sigma^2} = -\frac{d \ln \rho}{d \ln r} \rightarrow 2. \quad (1.149)$$

Note also that wiggles persist at large radii, though with smaller and smaller amplitudes.