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**HONG LIU:**

OK, let us start. So first, let me remind you what we did at the end of the last lecture. So we can see there's a large  $N$  matrix field theory. And then we saw that when you write down your Feynman diagrams to calculate things, then there's a difference now between the order of contractions. Which is now refracted whether you have a Planar diagram or non-planar diagram, et cetera. And that in turn, also affects your end counting. So at the end, we discussed two observations.

One observation we said, is for the example to be considered is the non-planar diagram, even though it cannot be drawn on the plane without crossing lines, can actually be drawn on the torus without crossing lines. So the diagram can actually be straightened out on the torus. And another observation, is that the power of  $N$  is related to the number of faces you have in your diagram after you have straightened it out. So now, I'm going to generalize these two observations.

So first, I will tell you a fact. Many of you may already know this. So I'll first tell you a fact that any orientable two dimensional surface is classified topologically by integer, which I will call  $h$ . So this  $h$  is called a genus. So heuristically, this genus is equal to the number of holes a surface has. So for example, if you have a plane, the plane topologically is actually equivalent to a sphere, so when I draw a sphere, that means a plane, because if you identify the point on the plane of the infinity, then it becomes a sphere. So topologically, a plane is no different from a sphere. So this genus 0. So this is plane. So  $h$  is equal to 0. And then for torus, then there's one hole, and this is just genus 1. So this is torus.

You can also draw surfaces with as many holes as you want. So this is surface with two holes. A surface with two holes, and so this is genus 2. So the remarkable thing is that this actually classifies the topology of all two-dimensional surfaces. And this so-called topological invariant, there's a topological number a so-called Euler number, which is defined to be  $\chi$  is equal to  $2$  minus  $2h$ . So if  $h$  labels the topology, and so the  $\chi$  is related to  $h$  in this way. So any two surfaces with the same topology will have the same  $\chi$ , because if they have the same  $h$ ,

they have the same Chi. So Chi is what we call the topological invariants. So this is just a mathematical fact.

So now I'm going to make two claims. So these two claims are in some sense, still evident after you have studied a little bit. I'm not going to prove it here, will just make the claim, and then I will leave the task to familiarize these two claims to yourself. But as I said, these two claims are actually self-evident, if you just do a little bit of studying.

So claim one-- so this is regarding the structural Feynman diagrams. So remember, the example theory we conceded last time was something like this, which  $\phi$  is [INAUDIBLE] matrix. So if you can look at the Feynman diagrams of the theories, the claim is the following. for any non-planar diagram, there exists integer  $h$  so that the diagram can be straightened out. Straightened out just means non-crossing on a genus  $h$  surface, but not on a surface with a smaller genus. So actually, you can classify all the non-planar diagrams by drawing it on a genus  $h$  surface on two-dimensional surfaces with non-trivial topology. And then you should be able to find the number  $h$ , which is the lowest genus you need to make this diagram to be non-crossing, instead of crossing.

Last time, we saw an example which you can stretch it out on the torus, but it was a more complicated diagram and torus would not be enough. You would need, say, a genus 2 surface or higher genus surface. So you can easily convince yourself that this is doable just by some practice. Also, this is very reasonable. Do you have any questions regarding this claim? Good. So this was claim one. The claim two-- and this is a generalization of the second observation. It's a claim of the generalization of the first observation of last time, and claim two will be the generalization of the second observation.

For any diagram, the power of  $n$  coming from contracting propagators is given by the number of faces on such a genus  $h$  surface. As we explained last time, the number of faces just means the number of these connected regions in the diagram. But if you are stretching it out, then there's an ambiguous way you can count the number of disconnected regions in the diagram. And that number is the power of  $n$  coming from the contracting propagators. So this claim is also self-evident because each power of  $n$  comes from a single connected line. And essentially, a single connected line will be circles some face and so the number of independents inside the face of course, are just the number of power  $n$ . Any questions regarding the second claim? Good.

So based on these two claims, now immediately write down the independents. So from this, we can find a vacuum diagram. Again, we have been only talking about the vacuum diagram. Vacuum diagram means the diagram has no external legs, has the full independence. So from here, we immediately conclude that the vacuum diagram has the following,  $g^2$  and dependence. So  $g^2$  is just the coupling constant. So remember, each propagator is proportional to  $g^2$  and that each vertex is one over  $g^2$ . Just from there. So let's just take an arbitrary vacuum diagram. Let me call this amplitude to be  $A$ . And  $A$  should be proportional to  $g^2$  to the power  $E$ .  $E$  should be the number of propagators, because each propagator gives you a factor of  $g^2$ . And also minus  $V$ . And  $V$  is the number of vertices below to the number of vertices. And each vertex gives you one over  $g^2$ , so that should give you  $E - V$ . And then we just multiply  $N$  to the power of  $F$ , and  $F$  is the number of faces. So without doing any calculation, so this has essentially characterized the  $N$  and the  $g$  and the accompanying dependents of any diagrams.

Now, if you look at this expression, you say we are doomed, because this kind of expression does not have a sensible [INAUDIBLE] So remember our goal, the original goal of the 't Hooft is that you treat this  $N$  as a parameter and then you want to take it  $N$  to be large and then to its ranking, one over  $N$ . Doing expansion in one over  $N$  means you are expanding around the including infinity. You want to expand around the including infinity. But if you look at this expression and then go to infinity, it's not a well-defined limit, because I can draw a [INAUDIBLE] Feynman diagram, which  $F$  can be as large as possible. So you just have sufficiently many vertices, and sufficient to be mainly propagators,  $F$  is unbounded. So then, there's another well-defined angle to infinity limits. So this expression is not a sensible  $N$  of infinity limit. If you don't have a sensible  $N$  infinity limit, then you cannot talk about doing a longer expansion, and you cannot even define the limit we just finished. Yes?

**AUDIENCE:** I remember people do partial sum to-- can they do the same thing here? Put the large in the denominator pattern from series of singular bubbles and--

**HONG LIU:** Yeah.

**AUDIENCE:** Like a random face approximation--

**HONG LIU:** Yes.

**AUDIENCE:** Somehow they can infinite a sum over an infinite number of  $A$ , and put the margin into like  $1 - N$  something for the denominator--

**HONG LIU:**

Yeah. So people certainly have been talking about expansion. And certainly, if this just failed here, we will not be talking about this. I'm just setting up a target, which I'm going to shoot it down. Right. Yeah, I would just say if you do the margin expansion, this probably would be the place which would turn you back. You say, ah, there's no margin limits, and that's from another problem. But hopefully, that's no ordinary person.

And so remember, there are several nice mathematical tricks you have to go through. First, you have to invent the stop line notation so that you can count the end very easily. Yeah, first you have to come up with this margin idea. I think maybe not due to him, maybe other people already considered similar things like right here. But first you have to think about the double line notation to make your computation easier, and then even after you reach the staff and then you need to know this kind of topology, et cetera. After you reach this staff, still, there is a roadblock here. But 't Hooft found a very simple way to go around it. Because when I say this limit it is not well-defined, I made the assumption that when I go to infinity limit, the  $g$  is kept fixed. So that's the reason that this will blow up.

But then 't Hooft came up with a different image. He said when you can see that  $N$  goes infinity, but at the same time, you can see that  $g$  squared goes to zero. At the same time, you can see that  $g$  squared goes to zero. Because the problem with this, is that if you have a diagram with lots of faces and if you multiply by a finite number, if you take  $g$  equal to 0, then you have infinity multiply something potentially goes to 0, maybe you will get something finite. One second. So you said let's consider this limit, so that the product of them is kept finite.

Let's consider this limit. So in this limit, the end counting will be different. So this  $A$  will be  $g$  squared  $N$ ,  $E$  minus  $I$ , because we want to keep this fixed. So we put an  $N$  factor here, then we take the  $N$  factor out.  $N$  plus  $F$  plus  $V$  minus  $E$ . And then let me just write this slightly differently. So this is  $\Lambda$ . If I use this notation,  $\Lambda$  now is finite. So this is  $E$  minus  $V$ . So let me just write it as  $\Lambda L$  minus 1  $N$  to the power of  $\chi$ .

Now, let me explain [INAUDIBLE]. First,  $L$  is equal to  $E$  minus  $V$  plus 1, is the number of loops in the diagram. So do you guys remember this formula, why this is true? The reason this is true is very easy to understand. So if you look at the Feynman diagram, the number of loops is exactly the same as the number of undetermined momentum. And each propagator will carry momentum, and at each vertex, you have momentum conservation. But then this overall momentum conservation, which is guaranteed, so the number of independent momentum is  $E$

minus  $V$  and then plus 1. Yeah, it's just  $E$  minus  $V$  minus 1, because there's only  $V$  minus 1 independent momentum constraints. So this is the number of independent momentums, and so this is the number of loops.

And the Chi is just defined to be the number of faces plus the number of vertices minus the number of edge. Yes?

**AUDIENCE:** Is there any-- in general, is there a relation between  $E$ ,  $V$ , and  $F$ , like are they completely independent to each other?

**HONG LIU:**  $E$ ,  $V$ , and  $F$ ?

**AUDIENCE:** Yeah.

**HONG LIU:** Yeah, that I'm going to explain. Any other questions?

**AUDIENCE:** So the reason that you can take the  $F$  to infinity limits without taking it to 0, is that why we can't apply this to QCD to get a--

**HONG LIU:** No, because we apply this.

**AUDIENCE:** But if  $g$  is not small?

**HONG LIU:** It doesn't matter. So this is an expansion scheme. And you can apply it to QCD, then you can ask whether this particular expansion scheme is a good approximation or not. That's a separate question. And the focus is pretty good, it's not that bad. Yeah, focus of each squaring is finite.

**AUDIENCE:** But it's [INAUDIBLE].

**HONG LIU:** No, here it does not tell you if  $\lambda$  has to be small.  $\lambda$  can be very big here. It just tells you the  $\lambda$  in this limit has to be finite.  $\lambda$  can be convening and it can be very big. And actually, in the future, we will take  $\lambda$  to go to infinity. So now, in order to have a well-defined limit still depends on this chi. So now, you ask why this works because we still have this chi, but now again, we need another piece of mathematics. First, when you draw, let me remind you some diagram we had last time, so this is the simplest diagram. And there's a lot of non-planar diagrams which you can draw on the torus, which is the non-planar version of this, et cetera. And you can also have more complicated diagrams. So suppose you are on the torus, I can consider more complicated diagrams, like that. For example, such diagrams.

So if you think about such diagrams, then in a sense, each Feynman diagram can be considered as a partition of the surface. So if you draw a diagram of the surface into polygons. Yeah, it's very clear here. I drew this now with wavy lines, just make them straight. And this one topologically, I can just draw like this, and then I have one, one, and the other part, similarly here. So each Feynman diagram can be concealed if you just partition whatever surface you draw the diagram. Is this clear to you? So this is a very important point. Yes?

**AUDIENCE:** Can you repeat that?

**HONG LIU:** Yes. Look at this diagram. So this is a Feynman diagram drawn on the torus. Does this look like a partition of torus into polygons? Yes, so that's what I'm talking about. And the statement that this would apply for any Feynman diagrams. Yes, so this is like a partition of a sphere. You separate a sphere into three regions, one, two, three. Any questions about this thing? Yes, do you have a question?

**AUDIENCE:** Yeah, in that case the third one isn't abandoned, is it? The third--

**HONG LIU:** So this one I am going to show you, you should view the plane as a sphere.

**AUDIENCE:** Oh, OK.

**HONG LIU:** Topologically, it's the same as a sphere. So now, once you recognize this, now you can use a famous theorem due to Euler. Some of you could have learned it in junior in high school, because the-- so this theorem, given a surface composed of polygons-- so if you're not familiar with this, you can go to Wikipedia-- with  $F$  faces,  $E$  edges, and  $V$  vertices. Suppose you have a surface like this, and then this particular combination  $\chi$ , which is defined to be  $F$  plus  $V$  minus  $E$  is precisely equal to  $2$  minus  $h$ . So this is the-- this is why this is called Euler's number.

And they only depend on the topology. So this combination only depends on the topology of the surface and nothing else. So we can imagine 't Hooft a very good high school student. So he already knew this. So now, we can rewrite this thing as  $A$ , reading script  $A$  lambda to the power  $L$  minus  $1$   $N$  to the power  $2$  minus  $2h$ . So remember  $h$  is greater than zero. So now this expression has a well-defined limit. So now this has a well-defined analogy and limit. So to the leading order, and in particular, this  $N$  dependence only depends on the topology of the diagram. Only depend on the topology of the diagram.

For example, to leading order in large  $N$ , then the leading order will be given by the planar diagrams. Because that should be the diagrams with  $h$  equal zero because  $h$  is not negative. So the leading term is given by  $h$  zero. So all of the planar diagrams in this limit, in this so-called 't Hooft limit, will have  $N$  dependence which is in  $N$  squared.  $N$  squared. So now if you go back to your loads, above the fourth diagram we started last time, above the two planar diagrams we started last time, you can immediately tell that indeed it is  $N$  squared in this limit.

Then you still haven't explained in terms of  $\lambda$ . So you can do Feynman diagrams, et cetera. Say  $\lambda$  just depends on the number of loops. So if you have one loop, then start it with a  $\lambda$  equal to zero, start it with some constant. If it's two loops, then  $\lambda$ , three loops,  $\lambda$  squared, et cetera. So the sum of four-- sum of all planar diagrams, we have this structure.

So you can imagine you can sum all this, if you're powerful enough. And then you can write it as  $N$  squared, then some function of  $\lambda$ . So the planar diagram would be just  $N$  squared time some function of  $\lambda$ . So if you are powerful enough to compute this  $f_0(\lambda)$  exactly, then you can say you have solved the planar. You have solved the large  $N$  limit of this series. Unfortunately for  $\lambda$  being in gauge theory we cannot do that. We don't know how to compute this. We can only compute perturbatively which actually does not work for QCD. Yes?

**AUDIENCE:** But  $N$  going to be infinite. It just seems like-- why does this overcome the same problem that we had before?

**HONG LIU:** Because this is a specific power. Here there's no specific power. That  $F$  can be as large as you want.

**AUDIENCE:** But wasn't the other problem also that  $N$  was going to infinity?

**HONG LIU:** No that's another problem. Because here, there is a specification limit, when you take  $N$  going to infinity. In there, there's no specific limit. There's no limit when you take  $N$  go to infinity. No, the limit is  $N$  squared. I can say-- this is your essentially, your vacuum energy, right? I can say  $E_0$ , divided by  $N$  squared has a well defined limit. So the key is that this is a specific dependence on  $N$ . But there it's unbounded.

**AUDIENCE:** I guess another question is, we didn't assume that  $\lambda$  is a small prohibiter.

**HONG LIU:** No we did not assume  $\lambda$ .

**AUDIENCE:** So isn't this kind of perturbative analysis, saying that that series converges to some function, is that OK?

**HONG LIU:** It's a very good question, and it is OK. But the reasoning is more complicated. The reasoning is more complicated. The keys are falling. What I'm writing here you can understand using two perspectives. Just one is heuristic. Just one say, suppose you can only do perturbation series, and that would be the thing you compute. And then say, if you are powerful enough to solve the theory with perturbability, then you're guaranteed just to find some general functions. Yeah but whether this series actually converges is an important mathematical question, of course, one has to answer.

It turns out, actually this is convergent. You can mathematically prove this is convergent, for very simple reasons. Yeah this is a side remark, but let me mention it because this is a cool fact. So if you do just  $\lambda$  of  $g^5$ , 4 series, just do the standard Feynman diagram calculation, input the basic series, the series is divergent. The series is divergent, no matter how  $g$  is small, just as the radius of convergence is 0 for any non-zero value. Yeah the radius of convergence is 0 in terms of  $g$ , no matter how small  $g$  is.

The reason that it's divergent is because the number of diagrams. So when you go to  $N$ th order, it increases exponentially in  $N$ . So the coefficient of  $g^N$  -- say to the power  $N$  -- the coefficient can become huge. Because of the number of diagrams to increase exponentially in  $N$ . So you think, I have  $g$  to the power  $N$  times  $N$  factorial, something like that. And that series is not convergent. But the thing that is convergent here, because can see the planar diagram, the number of planar diagram is very, very small. It only increases with  $N$  as a power of  $N$ , rather than as a factorial of  $N$ . And so you can make  $\lambda$  small enough, then this can be convergent. Yeah.

**AUDIENCE:** Didn't we already see at the end of the last lecture that only planar diagrams keep the biggest contribution, so why is that equal to--

**HONG LIU:** Sorry?

**AUDIENCE:** So at the end of the last lecture, we saw that--

**HONG LIU:** No not necessarily. That is only for the two diagrams. If you look at this thing, I can draw arbitrary complicated non-planar diagrams, with a very large  $F$ . They only depend on  $F$ . It

does not depend on anything else, if you do this. OK so in general, then of course if you include the higher non-planar diagram, et cetera-- so in general the vacuum energy, which you would normally call  $\log z$ . OK  $\log z$  is, essentially, the sum of all vacuum diagrams. So this is the partition functions, so it's a path integral. So  $\log z$ , then we'll have the expansion from  $h$  equal to 0 to infinity,  $N^2$  the power 2 minus  $2h$ ,  $F h \lambda$ . OK so at each genus level, you will have some function of  $\lambda$ .

Yeah the leading, order we just showed is  $f_0$ . And then if you add the Torus now-- we'll add to the torus now our on non-planar diagram. It's order one. And then, for the two genus, it's  $f$  over  $n$  squared, et cetera.

So let me just write down  $z$  explicitly.  $Z$  is the partition function,  $i S \phi$ . So if you compute this, with the right boundary condition-- if you compute this path integral, with the proper boundary conditions, then that gives you the vacuum diagrams. The  $\log z$  is the sum of all connected vacuum diagrams. I should say, the sum of all connected vacuum diagrams. Any questions regarding this? Good.

Now there is a heuristic way we can understand this expansion. So it's actually a heuristic way to understand this one-way expansion. So let's just look at this path integral. So let's look at the Lagrangian. So the Lagrangian I wrote there. In this 't Hooft limit, I can write it as  $\text{minus } N$  divided by  $\lambda$ . So I want to write things in terms of  $\lambda$ . So I multiply the pre-factor  $N$  downstairs, and then upstairs. So  $g^2 N$  give me  $\lambda$ . And then I have a trace, et cetera.

So now it's easy to see that you're leading order, this things should give you order  $N^2$ . Because there is already a factor of  $N$  here, and the trace is the sum of  $N$  things. So generically, this should be of the order  $N^2$ . So now we-- a little bit leap of faith-- say supposing the large  $N$  limit, since the leading order is order  $N^2$ , you can argue that actually all the  $N^2$  is the expansion parameter, if you do scatter point approximation. And then naturally, you will see the power will be given by  $1$  over  $N^2$ . any questions regarding this? Yes?

**AUDIENCE:** What does  $N$  over  $\lambda$  factor to?

**HONG LIU:** It's--

**AUDIENCE:** OK.

**HONG LIU:** So clearly this discussion actually does not-- so when I say clearly here, it requires a little bit of practice. But clearly, our discussion only depend on the matrix nature of the Lagrangian and the fields. So I'm going to make a claim. It says for any Lagrangian, of matrix valued fields of the form  $L$ , which is  $N$  divided by some coupling constant, times a trace of something. Doesn't matter what you put inside the trace here, as far as you have a single trace. Such a series will always have the expansion like this. It will always have an expansion like this.

So let me just summarize. In the 't Hooft limit-- by 't Hooft limit, I always mean this form-- the coupling constant is defined such that you have some coupling concept here, and then you have over  $N$  factor. And then of course, you can also have some coupling constant inside here. It doesn't matter. As far as those coupling constants are independent of  $N$ . And as far as things inside the trace are independent of  $N$ . So for such a kind of series, the  $1$  over  $N$  expansion is equal to the topological expansion. It's the expansion in terms of topology of Feynman diagrams.

So this is a very, very beautiful-- and as I said, it's [INAUDIBLE], because in principle, it puts a very simple structure into something that's, in principle, very complicated. Yeah, any questions regarding this? Yes?

**AUDIENCE:** Maybe it's not can I just understand it in such a way that he kind of treats the Feynman diagram as a triangulation of different spaces.

**HONG LIU:** Yeah for example, you can think of from that point. Yeah, so we use that to derive, to use this formula. Yeah and that picture will be very useful in a little bit from now on. Keep that picture in mind. The Feynman diagram is like the partition of some surfaces. And that will be very useful later.

**AUDIENCE:** Does there ever arise a situation in which you care not about two surfaces, but Feynman diagrams on three surfaces or something like that? Because this asks the question, you don't necessarily have to consider the topology of two surfaces. Are there any situations in which it's more complicated?

**HONG LIU:** Yeah but we always draw Feynman diagrams on the paper, which is two dimensional. Yeah it's enough. Two dimensions is enough. You don't need to go to three dimensions. Yeah and this structure only comes when you go to two dimensions, because if you go to three dimension, of course, they don't cross. In three dimension, you can no longer distinguish

planar or non-planar diagram.

**AUDIENCE:** Well if you did-- OK.

**HONG LIU:** Good any other questions?

**AUDIENCE:** Why is it always orientable surfaces?

**HONG LIU:** That's a good question. It's because those lines are orientable because when we draw the double rotation, so you have this two, double rotation. So essentially those lines are orientable. They have a direction. And essentially, this is, of course, one and two. Yeah so those Feynman diagrams actually have a direction, have a sense of orientation. So I'm going to mention, by passing in nature, but let me just mention also now. So if it's not [INAUDIBLE] matrix, say if it's a real symmetric matrix, then then there's no difference between two index. And then there's no orientation. And that would be related to m orientable surfaces. And then you need to slightly generalize this. Any other questions?

**AUDIENCE:** Now you mentioned that it has something to do with string theory, but does that it has anything to do with scatter particles--

**HONG LIU:** Yeah, so we're going to talk about that. No, we're going to talk about that. Good? So now let me talk about general observables. I think we're a little bit short on time, if we want to reach the punchline today. So right now we only have looked at the vacuum diagrams. So now let's look at the general observables. Before talking about general observables, let me just again make a side remark, Which is the gauge versus global symmetries.

So in the example we talk about here-- let me call this a, equation a. Then let me write down another equation b, which is a Yang-Mills theory. So the difference between a and b-- so a is this guy. So the difference between a and b, is that a, as we discussed last time, is invariant on the global symmetry, is a  $U(N)$  global symmetry. It's that  $\phi$  is invariant under the acting of a unitary matrix. But this  $U$  must be constant. Only for a constant  $U$  is this a symmetry. But if you have studied a little bit of gauge theory, or if you have not studied gauge theory, just take my word for it, the b is invariant under a local symmetry, a local  $U(N)$  symmetry. He said  $A_{\mu}$  so  $A_{\mu}$  is what make up the  $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ . It doesn't matter. The only thing I want to say is that this  $U$  is arbitrary. It can have arbitrary space time dependence. Just like of the generalization of the QED, it's the gauge symmetry.

So the key difference between the two. the key difference between this local and the global

symmetry, are manifested in what kind of observables we can see there. For example, for a--

**AUDIENCE:** I think that should be  $U$  dagger, partial  $\mu$  and then  $U$  dagger.

**HONG LIU:** So this difference, you can say what's the big deal? in one case, this is constant, and this is dependent on space time. So the key difference between them is that in the case for a, operator like this  $\phi^2$ ,  $\phi$  is a matrix. So  $\phi^2$  is a matrix.  $\phi^2 \times$  is an allowed operator. So this operator is not invariant under this global  $U(N)$  symmetry. But it doesn't matter because this is a global symmetry. So this is an allowed operator. But if you have gauge symmetry, all operators must be gauge invariant. That means that all operator must be invariant under this kind of transformations. So the analog of this is not allows operators.

So observables in the gauge theory are much more limited. So we will be interested gauge theories. We will be interested in gauge theories. So that means we are always interested in observables, which are invariant under the symmetry. So we're interested in gauge theories. So that means we're always interested in gauge invariant operators. So the kind of Lagrangian does not matter. So you can have the gauge fields. You can also have some other field, say some matrix  $\phi$ , et cetera. As  $\phi$  is the Lagrangian of this form, it's OK. We always only consider the Lagrangian of that form. But it doesn't have an arbitrary number of fields, and with arbitrary kind of interactions.

So you start your theory. So let's for simplicity, this can see the local operators. In this kind of theory, then allowed, say local operators, must always have some kind of trace in it. Say you must have some form trace,  $F \mu U$ ,  $F \mu U$ , a trace  $\phi^2$ , et cetera. A trace  $\phi$  to some power  $F N$ ,  $\phi$  to some power  $k$ , et cetera. You can also have operators with more than one trace, say trace  $\phi^2$ , trace  $F \mu U$ ,  $F \mu U$ . So we are going to make a distinction because the operator with only a single trace, we call them single trace operator. And the operator with more than one trace, we call them multiple trace operators. OK so the reason for this distinction will be clear soon.

So multiple trace operators-- it's self-evident again-- that multiple trace operators can always be written as products of single trace operators.

**AUDIENCE:** I have a question. Is it a possible case to have a local gauge invariant operator, the  $F \mu U$  times  $F \mu U$ ?

**HONG LIU:** Yeah I always can see the local gauge invariant operators. We can see the gauge theories.

**AUDIENCE:** So this combination,  $F \mu U$  times  $F \mu U$ , it's only--

**HONG LIU:** No this is gauge invariant.

**AUDIENCE:** Is that the only gauge invariant component?

**HONG LIU:** Sorry?

**AUDIENCE:** Is this is the only gauge invariant component that we can use to construct the gauge invariant local operators?

**HONG LIU:** Sorry, I don't quite understand what you mean. No this is just one specific operator. No, you can take  $F$  to the power  $N$ , an arbitrary number of-- as far as they're inside the trace, it's always gauge invariant.

**AUDIENCE:** I see.

**HONG LIU:** I'm just writing down a particular example. So just for notational simplicity, I will just write-- so from now on I will just denote the single trace operators collectively just as  $O$  with some script  $n$ , which denotes the different operators. so  $n$  denotes different operators. And then for the multiple trace operator, then you have  $O_1, O_2$ . That would be a double trace operator. And  $O_1, O_2, O_3$ , say times  $O_3$  would be a triple trace operator.

$n$  just labels different operators. I'm just using abstract notation. And the reason for this-- a distinction will be clear soon. Then for such gauge theories, the general observables, in the quantum field theory is just correlation functions of gauge invariant operators. So by gauge invariant operators, you can have local operators, non-local operators, et cetera. So for simplicity, I will restrict my discussion only on the local operators. But local operators means that the fields evaluated at a single point.

So the typical correlation functions, then the typical observables will have this form, will be just a product of some correlation functions, a product of some operators, and I say their correlation functions. By  $c$  I mean the connected correlation functions. So you can see immediately, these multi trace operators are just the product of a single trace operator. And the correlation function of a multiple trace operator can be obtained from those of a single trace one. You just identify some of the acts. Then that will be enough. So we only need to talk

about the correlation function of single trace operators.

So now the question-- let me call this equation one. So now the question follows what we discussed before, is how do we decide the  $N$  dependence of the guy? Previously we determined the  $N$  dependence of this vacuum diagrams of this partition function. But now what should be the  $N$  dependence of general correlation functions? One way to do it, you just start immediately calculating. And then you can find some root, et cetera. But actually there's a very simple trick. There's a very simple trick to determine the  $N$  dependence of this. So now I will explain. Now I will tell you.

So I will not give you any examples because this trick is so nice, and it just works very easy. And so the question, what is  $N$  dependence of one? So this is the question we want to address. so here is a very simple, beautiful, trick. So let's consider the following generating functional. So in quantum field theory, when we talk about correlation functions, it's always convenient to talk about the generating functional. So whatever is your field, you do the path integral of all fields. And then you look at the action. So you have your regional action, which I call  $S_0$ . And then that's add those operators.  $\int dx, O_i(x)$ .

Yes so this is a standard story. When you take the derivative over  $J_i$ , then you will bring down a factor of  $O_i$ . Then that essentially give you the correlation function. You have, for example, a correlation function, the connected correlation function.  $O_n$ , the connected correlation function would be just you take the derivative of  $\log z$ . And then  $\delta J_1 \times \delta J_n \times z^n$ . And then you set  $J$  equal to zero. You set all the  $J$  equal to zero. And that gives you the end point function. I should write  $i$  here, so  $i$  to the power  $n$ .

So now here is the beautiful trick. You can determine this in a single shot, the  $N$  dependence of this guy. And this simple trick is just to add  $N$  here. You add  $N$  here. In order to get the correlation function, we need to divide it by  $N$  to the power  $N$ . Now you wanted to get  $O_i$ , you need to divide-- take the root of  $N$  times  $J_i$ , so you also need to divide it by  $1$  over  $N$ . But why does this help? Why does this help? It helps for the following reason. Let me call this whole thing  $iS$  effective.  $iS$  effective, so the key thing is that this  $O_i$ , single trace operators, then this  $S$  effective then has the form  $N$  times the trace something. Because you already know  $S_0$ , which is our starting point, has this form. And now the term you added in, precisely, also has this form, is the  $N$  factor times something single trace.

So that means the whole thing still has this form. Then now we can immediately conclude this

$\log z$ ,  $J_1$ ,  $J_n$ , must have this expansion.  $h$  from 0 to infinity, and to the  $2 - 2h$ ,  $f h \lambda$   $J_1$ , et cetera. So adding that  $N$  is a powerful, powerful trick. So now you can immediately, just from here, we can immediately find out that for endpoint function, connected endpoint function, the leading order is  $2 - n$ , because the leading order is  $n^2$ . Yeah it's  $2 - n$ . And then suppressed by  $1/N^2$ , et cetera. Good?

So for example, if you look at a 0 point function, which is essentially the partition function, so this is order  $N^2$ . So this is what we found before, to leading order. And if you look at the one point function, some operator, then, will be order  $N$ . If you look at the two point function, connected two point function of some operator, so it would be order one. And the three point function of some operator will be order  $1/N$ , as the leading order. And then all higher order just down by  $1/N^2$ , compared to the leading order.

And again, the leading order is given by the planar diagrams. Because of the leading order contribution to here, in terms of this  $S$  effective is planar diagram. And then they must be, under those things, just obtained by through [INAUDIBLE], so they must be planar diagrams. so again this comes from planar diagrams. Good?

So now let's talk about the physical implications of this. So what does this mean? So we have found out, this is our  $N$  dependence for our gauge invariant operators. So what does this mean? Now let me talk about physical implications.

**AUDIENCE:** What did one of those [INAUDIBLE] have with--

**HONG LIU:** Yeah, I defined them without  $\chi$ .

**AUDIENCE:** Then something like--

**HONG LIU:** You define something. So when you write down your theory, you define this 't Hooft limit. Then everything is already in terms of  $\lambda$  or some other order one number. So that can depend on those numbers in an arbitrary way. It doesn't matter can depend on coupling constants in an arbitrary, but it cannot have  $N$  dependence defined inside the operator. Once we introduce this 't Hooft limit, then the operator can depend on the coupling in the 't Hooft limit in an arbitrary way. Because they're all just all the one constant.

Let's talk about physical implications of this. It turns out, these simple and [INAUDIBLE] behavior actually has a very simple physical picture behind it. So first he said, in the large  $N$  limit-- so let's just look at leading order behavior. So in a large  $N$  limit, if we consider this state

of  $O_i$  acting on the vacuum. So some single trace operator acting on the vacuum. So  $i$ , again, just labels different state, different operators. These can be interpreted as creating a "single particle" state.

I'm first describing the conclusion. Then I will explain why. So you can see that the state obtained by adding a single trace operator on the vacuum, then this can be interpreted as a single particle state. So if you add on the double trace operator on the vacuum, then this can be interpreted as two particle state. Similarly, say  $O_n$  acting on the vacuum would be  $N$  particle states. So why is this so? Why this is so. So I will support this statement using three arguments. First remember  $O_i O_j$ , the connecting Green function of any two operators of order one. So we can actually just diagonalize them. If you can just diagonalize them, so that  $O_i O_j$  are proportional to  $\delta_{ij}$ . So in some sense, a two point function, those operators can be considered as independent.

And now the second statement. So if you want to call this single particle, this two particle, multi particle, then they should be that they don't overlap. Because a single particle cannot overlap with two particle, et cetera. And then to see the overlap between a single particle with a multi particle, you look at these correlation functions. So if you look at the overlap of a single particle state, with a two particle state, say some double trace operator. Let me just, to avoid confusion, let me just use the inside-- use this notation to see this as a single operator. So you can see that the overlap we saw of the single trace operator with this double trace operator. And they start off-- this whole thing is like a three point function. Just put these two over the same point. So from our discussion here, is how you connect the Green functions of order 1 over  $N$ .

So this goes to 0, compared to this overlap with itself. So that means in the  $N$  goes to infinity limit, there is no mixing between what we called single particle state, single trace, and the multiple trace states.

So the third thing is that now let's look at the two point function of two double trace operator. So  $O_1 O_2$  x, say  $O_1 O_2$  and  $y$ . OK double trace operators. So let's look at the two point function with double trace operators. So there's an even contribution to this. So let's include all diagrams. Also these can all be connected So leading order is a disconnected diagram, which is essentially  $O_1 \times O_1$  y and  $O_2$  x and  $O_2$  y. Because I have a diagonal next to them. have And then plus the connected Green functions, which is order 1 over  $N$  squared. So if you see this leading order contribution, it's just like two independent particle propagating. Just like the

product of two independent propagators. So it's sensible to interpret this as just the propagating of two particles. So it's sensible to interpret this two particle state, this double trace operator as just creating some two particle state.

Yeah, again this goes to zero, in the large  $N$  limit. So I should emphasize when we call this a single particle state, it's not necessary they really exist on a shell particle corresponding to this state. We're just saying that the behavior of these states can be interpreted as some kind of single particles. The behavior you can just interpret as single particles. It's not really necessary they exist as stable on shell particles. In certain cases, there might be. There might be actual particles, actual stable, on shell particles associated with these kind of states. But for this interpretation to be true, it does not have to be.

So in QCD actually sometimes-- so I just say they exist. So we call them "glueball" state. For example, in QCD, the analog of this kind of operator can create some state, which they are short-lived. They are not long-lived. They quickly decay. And so they're typically called the glueball state. So from now on we just call them glueballs. Call them glueballs. So this is one of the first implication-- the first indication is that they're just a single particles. A single trace operator can be interpreted as creating single particle states, and the multiple trace operator can be interpreted as creating multi particle states.

Another second, he said if this glueball operators, so fluctuations of glueballs suppressed. So let me explain what this means. So let me suppose some single trace operator,  $O$ , which has a non-zero expectation value, suppose some state has a non-zero expectation value. And then let's look at the variance of this operator, the variance of the expectation value, which is given by  $O$  squared, minus  $O$  squared for the fluctuation. And this is, by definition, just a collected Green function of  $O$  squared. This is the full  $O$  squared. This is the disconnected one. So this is just a collected part of the  $O$  squared. And this one,  $N$  dependence, we know this is order one, order  $N$  to the power 0.

So that means that these to below this order  $N$ . So the variance of this, compared to the expectation value of this operator itself is surprising, the large  $N$  limit. So essentially, in the large  $N$  limit, so that means the variance provided by the operator itself is  $1$  over  $N$  because of the  $0$  in the large  $N$  limit. So assuming that if you have a two point function-- so suppose each two point function, so each  $O^2$  have a non-zero expectation value, then you can factorize this  $O_1 O_2$ , plus  $O_1 O_2$ . So this is disconnected part. And again this will be of order one. But this

part is of order  $N^2$ .

So essentially the disconnected part is always factorized. So the disconnected part is always factorized. Yeah this connected part is always small, compared to the disconnected part. So this is like a classical theory.

**AUDIENCE:** We have just five minutes. We can proceed before it's 5:00

**HONG LIU:** Yeah but it may go to 10 minutes. It's just hard to say. It's just hard to say. Yeah, let me do it next time.